Saint-Venant’s Principle

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Synonyms

Linear theory; Spatial behavior; Thermoelasticity

Overview

Saint-Venant’s principle in linear elasticity was established by Toupin [20] and Knowles [15], who gave an inequality describing exponential spatial decay of effects with distance from the excited end of the right cylinder. A comprehensive review of the results concerning Saint-Venant’s principle was given in [11, 13, 14]. In elastodynamics, the study of spatial decay of solutions was first suggested by Boley [2]. Boley [3–5] was also the first who studied the validity of Saint-Venant’s principle for transient heat conduction.

In the dynamical theory of linear thermoelasticity, Chiriță [7] established a spatial decay estimate of Saint-Venant type for a certain energetic measure. Then, Chiriță and Ciarletta [8] introduced a new method, called the time-weighted surface power method, which was applied to study the spatial behavior of elastic, viscoelastic, and thermoelastic processes outside of the support of the external given data. Within the framework of thermoelasticity theory, Chiriță and Ciarletta [8] studied the spatial behavior of dynamic processes in terms of two appropriate time-weighted surface power functions. Using the properties of these time-weighted surface power functions, for bounded domains, they established decay estimates of Saint-Venant type, while for unbounded domains, the authors established alternatives of Phragmén–Lindelöf type with both time-dependent and time-independent decay and growth rates.

On the other hand, following the research line initiated by Horgan et al. [12] (for parabolic equations), Quintanilla [17, 18] (for coupled systems of parabolic and hyperbolic equations) has shown that further exponential estimates may be established for semi-infinite cylinders. For a fixed time, the results in question prove that the mechanical and thermal effects are controlled by an exponential decay estimate in terms of the square of the distance from the support of the external given data.

The present work gives a short description of the results established in [7, 8, 17, 18].

Formulation of the Problem

Throughout this entry we shall consider a thermoelastic homogeneous material which occupies the region $B$ of the three-dimensional Euclidian space $E^3$ whose boundary is the smooth surface $\partial B$. 

We refer the motion of the body to a rectangular Cartesian system of axes $Ox_i$. Throughout this entry, Latin subscripts take the values $1, 2, 3$; summation is carried out over repeated indices. Typical conventions for differential operations are implied such as a superposed dot or comma followed by a subscript to denote the partial derivative with respect to time or to the corresponding Cartesian coordinate, respectively.

Let us consider a given time $t_1 > 0$. In the absence of body forces and heat supplies, the equations of the linear theory of thermoelasticity consist of (see [6, 10])

- The equations of motion

$$
\sigma_{ij,j} = \rho_0 \ddot{u}_i \tag{1}
$$

- The energy equation

$$
-q_{ij,i} - T_0 M_{ij} \dot{u}_{i,j} = c \dot{\theta} \tag{2}
$$

- The stress–strain–temperature relation and the heat conduction equation

$$
\sigma_{ij} = C_{ijkl} e_{kl} - M_{ij} \theta \tag{3}
$$

- The strain–displacement relations and the thermal gradient–temperature relation

$$
e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \tag{4}
$$

$$
\theta_i = \theta_{,i}
$$

Here we have used notation: $u_i$ are the components of the displacement vector, $\theta$ is the variation of temperature from the reference configuration, $\rho_0$ is the mass density of the continuum at the initial time, $\sigma_{ij}$ is the stress tensor, $T_0$ is the absolute temperature in the reference configuration, $e_{ij}$ is the infinitesimal strain tensor, $q_{ij}$ is the heat flux vector, and $C_{ijkl}, M_{ij}, k_{ij}$, and $c$ are constitutive coefficients satisfying the following properties of symmetry:

$$
M_{ij} = M_{ji} \tag{5}
$$

$$
C_{ijkl} = C_{klij} = C_{ijkl}
$$

In this work we suppose that $C_{ijkl}$ is positive definite. It follows that there exist the minimum and maximum elastic moduli for $C_{ijkl}$, denoted by $\mu_m > 0$, $\mu_M > 0$, such that

$$
\mu_m \xi_{ij} \xi_{ij} \leq C_{ijkl} \xi_{ij} \xi_{kl} \leq \mu_M \xi_{ij} \xi_{ij}, \text{ for all symmetric tensors } \xi_{ij} \tag{6}
$$

We assume that the conductivity tensor $k_{ij}$ is symmetric and positive definite. Thus,

$$
k_{ij} = k_{ji} \tag{7}
$$

and

$$
k_m \xi_{ij} \xi_{ij} \leq k_{ij} \xi_{ij} \xi_{ij} \leq k_M \xi_{ij} \xi_{ij}, \text{ for all vectors } \xi_{ij} \tag{8}
$$

where $k_m > 0$ and $k_M > 0$ denote the minimum and maximum conductivity moduli for $k_{ij}$.

Since we consider the linear theory of coupled thermoelasticity, we can assume that

$$
m = (M_{ij} M_{ij})^{1/2} > 0 \tag{9}
$$

In view of the constitutive equations (3), we have [7, 8]

$$
\sigma_{ij} \sigma_{ij} = C_{ijkl} e_{kl} \sigma_{ij} - \theta M_{ij} \sigma_{ij}
$$

$$
\leq (C_{rmmm} e_{rs} e_{mn})^{1/2} (C_{ijkl} \sigma_{ij} \sigma_{kl})^{1/2}
$$

$$
+ \theta (M_{rs} M_{rs})^{1/2} (\sigma_{ij} \sigma_{ij})^{1/2}
$$

$$
\leq (\sigma_{ij} \sigma_{ij})^{1/2} \left[ (\mu_M C_{rmmm} e_{rs} e_{mn})^{1/2} + m |\theta| \right] \tag{10}
$$

Thus, by means of the arithmetic–geometric mean inequality, we get

$$
\sigma_{ij} \sigma_{ij} \leq (1 + \varepsilon) \mu_M C_{ijkl} e_{ij} e_{kl} + \left( 1 + \frac{1}{\varepsilon} \right) m^2 \theta^2,
$$

for all $\varepsilon \in \mathbb{R} \tag{11}$

Moreover, we have that

$$
q_{ij} q_{ij} \leq k_M k_{ij} \theta_{,i} \theta_{,m} \tag{12}$$
Throughout this entry, by an admissible process, we mean an ordered array \([u_i, \dot{u}_i, \ddot{u}_i, (u_{ij} + u_{ji})]\) with the following properties:

1. \(u_i, \dot{u}_i, \ddot{u}_i, (u_{ij} + u_{ji})\) are continuous on \(\overline{B} \times [0, t_1]\).
2. \(e_{ij}\) is continuous symmetric tensor field on \(\overline{B} \times [0, t_1]\).
3. \(\sigma_{ij}\) and \(\sigma_{ij} - \sigma_{ji}\) are continuous on \(\overline{B} \times [0, t_1]\).
4. \(\theta, \dot{\theta}\) and \(\theta\) are continuous on \(\overline{B} \times [0, t_1]\).
5. \(g_i\) and \(g_{i,j}\) are continuous on \(\overline{B} \times [0, t_1]\).
6. \(q_i\) and \(q_{i} - q_{j}\) are continuous on \(\overline{B} \times [0, t_1]\).

By a thermoelastic process we mean an admissible process \([u_i, e_{ij}, \sigma_{ij}, g_i, q_i]\) that meets the fundamental system of field equations (1)–(4), listed at the beginning of this section.

The corresponding surface traction \(s_i\) and heat flux \(q\) are defined at every regular point of the boundary surface \(\partial B\) by

\[
s_i = \sigma_{ij} n_j, \quad q = q_i n_i
\]

where \(n_i\) are the components of outward unit normal to the boundary surface \(\partial B\).

Let \(S_z\) be the intersection with \(B\) of the plane \(x_3 = z\); let

\[
B_z = \{x \in B | x_3 > z\}
\]

and let \(L = \max\{x_3 \in [0, \infty) | x \in B\}\).

We associate with the thermoelastic process \([u_i, e_{ij}, \sigma_{ij}, g_i, q_i]\) the kinetic energy on \(B_z\)

\[
K(z, t) = \int_{B_z} K \, dv
\]

the strain energy on \(B_z\)

\[
U(z, t) = \int_{B_z} U \, dv
\]

the thermal energy on \(B_z\)

\[
S(z, t) = \int_{B_z} S \, dv
\]

the dissipation energy on \(B_z\), on the time interval \([0, t]\)

\[
D(z, t) = \int_{0}^{t} \int_{B_z} D \, dv dt
\]

and the total energy on \(B_z\)

\[
E(z, t) = K(z, t) + U(z, t) + S(z, t) + D(z, t)
\]

We define the energy stored in the portion \(B_z\) of \(B\) in the time interval \([0, t]\) by

\[
E^*(z, t) = \int_{0}^{t} E(z, \tau) d\tau
\]

Let the body be subject to null initial conditions so that

\[
u_i(x, 0) = 0, \quad \dot{u}_i(x, 0) = 0, \quad \theta(x, 0) = 0, \quad \text{on } \overline{B}
\]
In this section, we suppose that the body is deformed under the action of nonzero boundary conditions only on the plane end face $S_0$. So, we assume that
\begin{align}
\sigma_{ij}(x, t)n_j &= 0 \text{ on } (\partial B \setminus S_0) \times (0, t_1) \tag{26} \\
q_i(x, t)n_i &= 0 \text{ on } (\partial B \setminus S_0) \times (0, t_1) \tag{27}
\end{align}

The boundary conditions on the end face $S_0$ are
\begin{align}
s_i(x, t) &\equiv \sigma_{ij}(x, t)n_j = \hat{s}_i(x, t) \text{ on } S_0 \times (0, t_1) \tag{28} \\
q(x, t) &\equiv q_i(x, t)n_i = \hat{q}(x, t) \text{ on } S_0 \times (0, t_1) \tag{29}
\end{align}

where $\hat{s}_i$ and $\hat{q}$ are prescribed continuous functions on $S_0 \times (0, t_1)$. Further, we assume that the traction $\hat{s}_i$ is self-equilibrated at each instant, i.e.,
\begin{align}
\int_{S_0} \hat{s}_i \, da &= 0, \quad \int_{S_0} \epsilon_{ijk} x_j \hat{s}_k \, da = 0 \tag{30}
\end{align}

Moreover, we assume that $\hat{s}_i$ is continuous on $S_0 \times [0, t_1)$.

In this entry [7], it is shown in the following result:

**Theorem 1.** Let $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ be a thermoelastic process on $\overline{B}$ satisfying the initial conditions (25) and the boundary conditions (26) and (27). Then we have
\begin{align}
E^*(z, t) &\leq E^*(0, t) \exp \left\{ -\frac{z}{v(t)\sqrt{t}} \right\} \tag{31}
\end{align}

for $0 \leq z \leq L$, $t \geq 0$, where
\begin{align}
v(t) &= \max \left\{ 2\sqrt{\frac{\mu t}{\rho_0}}, \frac{T_0}{2\mu_M} \sqrt{\kappa_M c}, 2 \frac{T_0 \rho_0}{c} \sqrt{\frac{t}{\rho_0 \mu_M}} + \frac{\mu_M}{T_0 c} \sqrt{\kappa_M c} \right\} \tag{32}
\end{align}

**Proof.** We note that the function $E^*$ is continuous differentiable with respect to $z$, and, moreover, we have
\begin{align}
\frac{\partial E^*(z, t)}{\partial z} &= -\frac{1}{2} \int_0^t \int_{S_1} \left\{ \rho_0 \ddot{u}_i \ddot{u}_i + C_{ijkl} \epsilon_{ij} \epsilon_{kl} + \frac{1}{T_0} c \theta^2 + \frac{2}{T_0} \int_0^t k_{ij} \theta_i \theta_j \, dt \right\} \, da \, ds \tag{33}
\end{align}

Further, on the basis of (1), (3), and (4) and the symmetry of $C_{ijkl}$ and $M_{ij}$, we get
\begin{align}
\frac{1}{2} \frac{\partial}{\partial t} \left( \rho_0 u_i \ddot{u}_i \right) &= (\sigma_{ij} \dot{u}_j)_i - C_{ijkl} \epsilon_{ij} \dot{e}_{kl} + M_{ij} \dot{e}_j \dot{\theta} \tag{34}
\end{align}

We next eliminate the last term on the right-hand side of the above equation by means of (2). Then we obtain
\begin{align}
\frac{1}{2} \frac{\partial}{\partial t} \left( \rho_0 u_i \ddot{u}_i + C_{ijkl} \epsilon_{ij} \epsilon_{kl} \right) &= \left( \sigma_{ij} \dot{u}_j - \frac{1}{T_0} \theta q_i \right)_i \\
&\quad - \frac{1}{T_0} c \theta \dot{\theta} + \frac{1}{T_0} q_i \theta_i \tag{35}
\end{align}

Finally, we integrate relation (35) over $B \times [0, t]$ and use the divergence theorem, the initial conditions, and the boundary conditions in order to obtain
\begin{align}
\frac{1}{2} \int_B \left\{ \rho_0 u_i \ddot{u}_i + C_{ijkl} \epsilon_{ij} \epsilon_{kl} + \frac{1}{T_0} c \theta^2 + \frac{2}{T_0} \int_0^t k_{ij} \theta_i \theta_j \right\} \, dv \\
&= \int_0^t \int_{S_1} \left\{ \dot{u}_i s_i - \frac{1}{T_0} \theta q \right\} \, da \, ds \tag{36}
\end{align}

Thus, the (36) and (24) lead to the identity
\begin{align}
E^*(z, t) &= \int_0^t \int_{S_1} \left\{ \dot{u}_i s_i - \frac{1}{T_0} \theta q \right\} \, da \, ds \tag{37}
\end{align}

Using the Schwarz inequality and the relation (8), we get
\begin{align}
E^*(z, t) &\leq t \left( \int_0^t \int_{S_1} \dot{u}_i \dot{u}_i \, da \right)^{1/2} \\
&\times \left( \int_0^t \int_{S_1} s_i s_i \, da \right)^{1/2} \tag{38}
\end{align}
Then, we have the inequalities \([1, 7, 9]\)

\[
\begin{align*}
\int_{S_0} u_i u_i \, da &\leq b^{-1} \int_B C_{ijkl} e_{ij} e_{kl} \, dv \\
\int_{\partial B} \theta^2 \, d\sigma &\leq \frac{C}{2} \int_B \left( \frac{1}{T_0} \sigma^2 + \frac{k_m \rho_0}{\mu M c} \frac{2}{T_0} k_{ij} \theta_i \theta_j \right) \, dv 
\end{align*}
\] (43)

\[
\begin{align*}
\int_0^t \int_{S_0} u_i u_i \, d\tau \, ds &\leq 2 b^{-1} E^*(0, t) \\
\int_0^t \int_{S_0} \theta^2 \, d\tau \, ds &\leq C \max \left( t, \frac{k_m \rho_0}{\mu M c} \right) E(0, t)
\end{align*}
\] (44)

where \(b\) and \(C\) are positive constants depending on \(B, S_0, T_0\), and the thermoelastic coefficients. It follows that

\[
\begin{align*}
\int_0^t \int_{S_0} u_i u_i \, d\tau \, ds &\leq 2 b^{-1} E^*(0, t) \\
\int_0^t \int_{S_0} \theta^2 \, d\tau \, ds &\leq C \max \left( t, \frac{k_m \rho_0}{\mu M c} \right) E(0, t)
\end{align*}
\] (45)

On the other hand, from the relation (8) and using the initial conditions, we obtain

\[
E^*(0, t) = \int_0^t \int_{S_0} u_i \dot{s}_i \, d\tau \, ds - \int_0^t \int_{S_0} u_i \dot{s}_i \, d\tau \, ds - \int_0^t \int_{S_0} \frac{1}{T_0} \theta \dot{q} \, d\tau \, ds
\] (46)

Moreover, due to the compatibility between the initial conditions (25) and the end conditions (28), we have the inequality

\[
\frac{d}{dt} \left( \int_0^t \int_{S_0} \dot{s}_i \dot{s}_i \, d\tau \, ds \right)^{1/2} \leq 2 \left( \int_0^t \int_{S_0} \dot{s}_i \dot{s}_i \, d\tau \, ds \right)^{1/2}
\] (47)

which implies that

\[
\left( \int_0^t \int_{S_0} \dot{s}_i \dot{s}_i \, d\tau \, ds \right)^{1/2} \leq \int_0^t \left( \int_{S_0} \dot{s}_i \dot{s}_i \, d\tau \, ds \right)^{1/2} \, ds
\] (48)

\[
\int_B u_i \, dv = 0, \quad \int_B \varepsilon_{ijkl} u_k \, dv = 0
\] (49)

\[
\begin{align*}
\frac{\partial}{\partial z} (T) &\leq 0
\end{align*}
\] (50)

By integrating this first-order differential inequality, we obtain the estimate (2) and the proof is complete.

In the following we give an explicit upper bound for \(E^*(0, t)\) in terms of the given data [7]

**Theorem 2.** Let \([u_i, \varepsilon_{ij}, \sigma_{ij}, \theta, g_i, q_i]\) be a thermoelastic process on \(B\) satisfying the initial conditions (25) and the boundary conditions (26)—(29). We assume that \(\dot{s}_i, \ddot{s}_i\) and \(\dot{q}\) are continuous on \(S_0 \times [0, t_1]\) and \(\ddot{s}_i\) is self-equilibrated at each instant. Then we have

\[
E^*(0, t) \leq 16 b^{-1} \left[ \int_0^t \left( \int_{S_0} \dot{s}_i \ddot{s}_i \, d\tau \right)^{1/2} \, ds \right]^2 + \frac{2}{T_0} C \max \left( t, \frac{k_m \rho_0}{\mu M c} \right) \int_0^t \int_{S_0} \dot{q}^2 \, d\tau \, ds
\]

(41)

where \(b\) and \(C\) are positive constants depending on the domain \(B\), on \(S_0, T_0\), and the thermoelastic moduli.

**Proof.** Using the integration over \(B\) for (1) and by means of the boundary conditions (26) and (28) and the initial conditions (25), we get

\[
\int_B u_i \, dv = 0, \quad \int_B \varepsilon_{ijkl} u_k \, dv = 0
\]

(42)
Now, we use the relations (45), (46), (47), and the Schwarz inequality in order to obtain

\[ E^2(0,t) \leq \left\{ 16b^{-1} \left( \int_0^t \int_{S_0} \dot{s}_i \dot{s}_j \text{d}a \text{d}t \right)^{1/2} \right\}^2 + \frac{2}{T_0} C \max \left( \begin{array}{c} k_{MP} \\ \mu_{MC} \end{array} \right) \int_0^t \int_{S_0} \dot{q}^2 \text{d}a \text{d}t (50) \]

which leads to the estimate (12), and the proof is complete.

**Estimates for Arbitrary Domains**

In this section, we consider a domain \( B \) with an arbitrary shape. Given the thermoelastic process \([u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]\) on \( \overline{B} \), we define \( \hat{D}_{t_i} \) to be the set of all \( \mathbf{x} \in \overline{B} \) such that:

1. If \( \mathbf{x} \in B, \) then
   \[ u_i(\mathbf{x},0) \neq 0 \text{ or } \dot{u}_i(\mathbf{x},0) \neq 0 \text{ or } \theta(\mathbf{x},0) \neq 0 \] (51)

   or

2. If \( \mathbf{x} \in \partial B, \) then
   \[ s_i(\mathbf{x},\tau) \dot{u}_i(\mathbf{x},\tau) \neq 0 \text{ or } q(\mathbf{x},\tau) \theta(\mathbf{x},\tau) \neq 0 \text{ for some } \tau \in [0,t_1] \] (52)

   Roughly speaking, \( \hat{D}_{t_i} \) represents the support of the initial and boundary data on the time interval \([0,t_1]\). In what follows we shall assume that \( \hat{D}_{t_i} \) is a bounded set. We consider next a nonempty set \( \hat{D}_{t_i}^* \) so that \( \hat{D}_{t_i} \subseteq \hat{D}_{t_i}^* \subseteq \overline{B} \) and such that:

   1. If \( \hat{D}_{t_i} \cap B \neq \emptyset \), then we choose \( \hat{D}_{t_i}^* \) to be the smallest bounded regular region in \( \overline{B} \) that includes \( \hat{D}_{t_i} \); in particular, we set \( \hat{D}_{t_i}^* = \hat{D}_{t_i} \) if \( \hat{D}_{t_i} \) also happens to be a regular region.

   2. If \( \emptyset \neq \hat{D}_{t_i} \subseteq \partial B \), then we choose \( \hat{D}_{t_i}^* \) to be the smallest regular subsurface of \( \partial B \) that includes \( \hat{D}_{t_i} \); in particular, we set \( \hat{D}_{t_i}^* = \hat{D}_{t_i} \) if \( \hat{D}_{t_i} \) is a regular subsurface of \( \partial B \).

   3. If \( \hat{D}_{t_i} = \emptyset \), then we choose \( \hat{D}_{t_i} \) to be an arbitrary nonempty regular subsurface of \( \partial B \).

   On this basis, we define the set \( D_r, \ r \geq 0 \), by

\[ D_r = \left\{ \mathbf{x} \in \overline{B} : \hat{D}_{t_i}^* \cap \Sigma(\mathbf{x},r) \neq \emptyset \right\} \]

where \( \Sigma(\mathbf{x},r) \) is the open ball with radius \( r \) and center at \( \mathbf{x} \). Further, we shall use the notation \( B_r \), for the part of \( B \) contained in \( B \setminus D_r \), and we set \( B(r_1,r_2) = B_{r_1} \setminus B_{r_2}, \ r_1 > r_2 \). Moreover, we shall denote by \( S \), the subsurface of \( \partial B \), contained inside of \( B \) and whose outward unit normal vector is forwarded to the exterior of \( D_r \).

We associate with the thermoelastic process \([u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i] \) the following time-weighted surface power function [8]:

\[ Q(r,t) = -\int_0^t \int_{S_0} e^{-i\lambda s} \left[ s_i(s) \dot{u}_i(s) - \frac{1}{T_0} q(s) \theta(s) \right] \text{d}a \text{d}s \]

for all \( r \geq 0 \) and \( 0 \leq t \leq t_1 \), where \( \lambda \) is a prescribed strictly positive parameter at our disposal. Moreover, we consider the function

\[ \hat{Q}(r,t) = \int_0^t Q(r,s) \text{d}s \]

In the paper [8], the following result has been established:

**Theorem 3.** Let \([u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i] \) be a thermoelastic process on \( \overline{B} \) corresponding to given data with the bounded support \( \hat{D}_{t_i} \) on the time interval \([0,t_1]\). Then the corresponding time-weighted surface power function \( Q(r,t) \) has the following properties:

(Q1) For each \( t \in [0,t_1] \)

\[ Q(r_1,t) - Q(r_2,t) = -\frac{1}{2} \int_{\partial(r_1,r_2)} e^{-i\lambda s} \left[ \rho_0 \ddot{u}_i \dot{u}_i + C_{ijkl} \dot{e}_{ij} \dot{e}_{kl} \right] \text{d}v + \int_0^t \int_{\partial(r_1,r_2)} e^{-i\lambda s} \left[ \frac{1}{2} \left( \rho_0 \ddot{u}_i \dot{u}_i + C_{ijkl} \dot{e}_{ij} \dot{e}_{kl} \right) + \frac{c}{T_0} \theta^2 \right] \text{d}v \text{d}s, \ 0 \leq r_2 \leq r_1 \] (56)
(Q2) \( Q(r,t) \) is a continuous differentiable function on \( r \geq 0, 0 \leq t \leq t_1 \), and

\[
\frac{\partial}{\partial r} Q(r,t) = -\frac{1}{2} \int_{S_t} e^{-j\lambda} \left[ \rho_0 u_0 \dot{u}_i + C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{c}{T_0} \theta^2 \right] dv \\
- \int_0^t \int_{S_t} e^{-i\alpha} \left\{ \frac{\lambda}{2} \left[ \rho_0 u_0 \dot{u}_i + C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \right] + \frac{c}{T_0} \theta^2 \right\} dads \leq 0
\]

(57)

(Q3) For each fixed \( t \in [0, t_1] \), \( Q(r,t) \) is a nonincreasing function with respect to \( r \).

(Q4) \( Q(r,t) \) satisfies the following first-order differential inequality

\[
\frac{\lambda}{\kappa} |Q(r,t)| + \frac{\partial}{\partial r} Q(r,t) \leq 0, \quad r \geq 0, \quad 0 \leq t \leq t_1
\]

where

\[
\kappa = \sqrt{\frac{(1 + \varepsilon)\mu_M}{\rho_0}}
\]

(59)

and \( \varepsilon \) is the positive root of the algebraic equation

\[
\varepsilon^2 + \varepsilon \left( 1 - \frac{m T_0}{c \mu_M} - \frac{\lambda \rho_0 k_M}{2c \mu_M} - \frac{m T_0}{c \mu_M} \right) = 0
\]

(60)

(Q5) \( \tilde{Q}(r,t) \) satisfies the following first-order differential inequality

\[
t_\chi(t) \frac{\partial}{\partial r} \tilde{Q}(r,t) + |\tilde{Q}(r,t)| \leq 0, \quad r \geq 0, \quad 0 \leq t \leq t_1
\]

where

\[
\chi(t) = \sqrt{\frac{(1 + \delta(t))\mu_M}{\rho_0}}
\]

(62)

and \( \delta(t) \) is the positive root of the algebraic equation

\[
\delta^2 + \delta \left( 1 - \frac{m T_0}{c \mu_M} - \frac{\rho_0 k_M}{2c \mu_M} - \frac{m T_0}{c \mu_M} \right) = 0
\]

(63)

(Q6) For a bounded body, \( Q(r,t) \) and \( \tilde{Q}(r,t) \) are positive functions.

The analysis of spatial behavior of thermoelastic processes requests a separate discussion for bounded and unbounded domains. Using the above theorem, in [8] there were established two spatial decay estimates of Saint-Venant type, one of which predicts a time-independent decay rate, while the other predicts a time-dependent decay rate.

We first consider the case of a bounded domain, that is, \( r \) ranges on \( [0, l] \), \( l < \infty \), where

\[
l = \max \left\{ \min \{ |(x_i - y_i)(x_i - y_i)|^{1/2} | y \in \tilde{D}_h \} | x \in \tilde{B} \} \right\}
\]

(64)

**Theorem 4.** (Spatial behavior for bounded domains). Let \( [u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i] \) be a thermoelastic process on the bounded regular region \( \tilde{B} \), corresponding to given data with the bounded support \( \tilde{D}_{t_1} \) on the time interval \( [0, t_1] \), and let \( Q(r,t) \) be the corresponding time-weighted surface power function. Then, for each fixed \( t \in [0, t_1] \), either

\[
Q(r,t) \leq Q(0,t) \exp \left( -\frac{\lambda}{\kappa} r \right), \quad 0 \leq r \leq l
\]

(65)

where \( \kappa \) is given by (59) or

\[
\tilde{Q}(r,t) \leq \tilde{Q}(0,t) \exp \left( -\frac{1}{t_\chi(t)} r \right), \quad 0 \leq r \leq l
\]

(66)

where \( \chi(t) \) is given by (62).

Let us discuss the spatial behavior of thermoelastic processes on unbounded domains. For such bodies, a Phragmêè–Lindelôf-type result has been established in [8].

**Theorem 5.** (Spatial behavior for unbounded domains). Let \( [u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i] \) be a thermoelastic process on the unbounded regular region \( \tilde{B} \), corresponding to given data with the bounded support \( \tilde{D}_{t_1} \) on the time
interval $[0, t_1]$, and let $Q(r, t)$ be the corresponding time-weighted surface power function. Then, for each fixed $t \in [0, t_1]$, the following alternative holds:

1. Either $Q(r, t) \geq 0$ for all $r \geq 0$ and then
   \begin{equation}
   Q(r, t) \leq Q(0, t) \exp \left( -\frac{\lambda}{\kappa} r \right), \quad r \geq 0
   \end{equation}
   or

2. There exists a value $r_t \geq 0$ so that $Q(r, t) < 0$ and then $Q(r, t) < 0$ for all $r \geq r_t$ and
   \begin{equation}
   -Q(r, t) \geq -Q(r_t, t) \exp \left( \frac{\lambda}{\kappa} (r - r_t) \right), \quad r \geq r_t
   \end{equation}

Estimates with time-dependent rates are given by the following theorem:

**Theorem 6.** Assume the hypotheses of Theorem 3 hold. Then, for each fixed $t \in [0, t_1]$, the following alternative holds:

1. Either $\tilde{Q}(r, t) \geq 0$ for all $r \geq 0$ and then
   \begin{equation}
   \tilde{Q}(r, t) \leq \tilde{Q}(0, t) \exp \left( -\frac{1}{T_0(t)} r \right), \quad r \geq 0
   \end{equation}
   or

2. There exists a value $r^*_t \geq 0$ so that $\tilde{Q}(r^*_t, t) < 0$ and then $\tilde{Q}(r, t) < 0$ for all $r \geq r^*_t$ and
   \begin{equation}
   -\tilde{Q}(r, t) \geq -\tilde{Q}(r^*_t, t) \exp \left( \frac{1}{T_0(t)} (r - r^*_t) \right), \quad r \geq r^*_t
   \end{equation}

**Further Estimates for Unbounded Domains**

In this section, we consider an unbounded domain, and we present an estimate in the sense described by Horgan, Payne, and Wheeler [12] and by Quintanilla [17, 18].

To this end, we assume that the support $\hat{D}_{t_1}$ of the initial and boundary data, defined in the previous section, is enclosed in the half-space $x_3 < 0$. We introduce the notation $S_z$ for the open cross section of $B$ for which $x_3 = z$, $z \geq 0$ and whose unit normal vector is $(0, 0, 1)$. We assume that the unbounded set $B$ is so that $S_z$ is bounded for all finite $z \in [0, \infty)$. We denote by $B_z$ that portion of $B$ for which $x_3 > z$.

In the works by Quintanilla [17, 18], with the $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$, the following function was associated:

\begin{equation}
H(z, t) = -\int_0^t \int_{S_z} \left[ \sigma_{3i} \hat{u}_i - \frac{1}{T_0} q_3 \hat{\theta} \right] da d\tau
\end{equation}

We assume further the following asymptotic behavior for $H(z, t)$ (see [17, 18]):

\begin{equation}
H(z, t) \rightarrow 0 \quad \text{uniformly in } t, \text{ as } z \rightarrow \infty.
\end{equation}

Using (1), the divergence theorem and having in mind the definition of $S_z$ and $B_z$, we deduce that the function $H(z, t)$ can be written in the form

\begin{equation}
H(z, t) = \int_0^t \int_{B_z} \left[ \sigma_{3i} \hat{u}_i - \frac{1}{T_0} q_3 \hat{\theta} \right] da d\tau
\end{equation}

and note that

\begin{equation}
\frac{\partial E}{\partial z} = -H(z, t)
\end{equation}

and

\begin{equation}
\frac{\partial^2 E}{\partial z^2} = \int_{S_z} [K(t) + S(t) + U(t)] da + \int_0^t \int_{S_z} D(\tau) d\tau d\tau
\end{equation}
On the other hand, from the constitutive equations (3) and the relations (71) and (74), we have

$$\frac{\partial E}{\partial t} = - \int_{B_t} \left( \sigma_{13} \nu_i - \frac{1}{T_0} q_3 \theta \right) dv$$  \hspace{1cm} (77)

We remark that

$$\int_{B_t} q_3 \theta dv = - \int_{z}^{\infty} \int_{S_z} k_{33} \theta \partial z \, da \, dz = \frac{1}{2} \int_{S_z} k_{33} \theta^2 \, da$$  \hspace{1cm} (78)

The next step is to estimate the time derivative of $E$ in terms of the two first spatial derivatives of $E$. In view of the relations (75), (76), (77), and (78), we can conclude that there exists the positive constants $A_1$ and $A_2$ such that

$$\frac{\partial E}{\partial t} \leq -A_1 \frac{\partial E}{\partial z} + A_2 \frac{\partial^2 E}{\partial z^2}$$  \hspace{1cm} (79)

Following Horgan et al. [12], we use the change of variable

$$w(z, t) = \exp(b_2 t - b_1 z) E(z, t)$$  \hspace{1cm} (80)

with

$$b_1 = \frac{A_1}{2A_2}, \quad b_2 = b_1^2 A_2$$  \hspace{1cm} (81)

Thus, the inequality (79) becomes

$$\frac{\partial w}{\partial t} \leq A_2 \frac{\partial^2 w}{\partial z^2}$$  \hspace{1cm} (82)

We have that $w(z, t)$ satisfies the relations

- $A_2 \frac{\partial^2 w}{\partial z^2} (z, t) - \frac{\partial w}{\partial t} (z, t) \geq 0$, $z \in [0, \infty), t \in [0, t_1)$
- $w(z, 0) = 0$, $z \in [0, \infty)$
- $w(0, t) = \exp(b_2 t) E(0, t) \geq 0$, $t \in [0, t_1)$
- $w(z, t) \to 0$ (uniformly in $t$) for $z \to \infty$

Using the maximum principle for parabolic partial differential equations, we obtain

$$w(z, t) \leq J(z, t), \quad z \in [0, \infty), \quad t \in [0, t_1)$$

where $J(z, t)$ is solution of the following problem

$$A_2 \frac{\partial^2 J}{\partial z^2} (z, t) - \frac{\partial J}{\partial t} (z, t) = 0, \quad z \in [0, \infty), \quad t \in [0, t_1)$$
$$J(z, 0) = 0, \quad z \in [0, \infty)$$
$$J(0, t) = \exp(b_2 t) E(0, t) \geq 0 \quad t \in [0, t_1)$$
$$J(z, t) \to 0 \quad \text{(uniformly in} \ t) \quad \text{for} \ z \to \infty$$  \hspace{1cm} (83)

The solution of the above problem is (c.f. Tikhonov and Samarskii [19])

$$J(z, t) = \frac{A_2}{2 \sqrt{\pi}} \int_0^t \frac{z}{|A_2(t - \tau)|^{3/2}} \exp \left( - \frac{z^2}{4A_2(t - \tau)} \right) \times \exp(b_2 t) E(0, \tau) d\tau$$  \hspace{1cm} (84)

From (80) and the above relation, we deduce that

$$E(z, t) \leq \left( \max_{\tau \in [0, t]} E(0, \tau) \right) \exp \left( \frac{A_1 z}{2A_2} \right) G(z, t)$$  \hspace{1cm} (85)

where

$$G(z, t) = \frac{1}{2 \sqrt{A_3 \pi}} \times \int_0^t z \tau^{-3/2} \exp \left( - \frac{z^2}{4A_2 \tau} - \frac{A_1^2}{4A_2} \right) d\tau$$  \hspace{1cm} (86)

Using the estimate discussed by Pompei and Scalia [16], we have

$$G(z, t) \leq \frac{2z(A_2 t / \pi)^{1/2} \exp \left( - \frac{A_1^2 t}{4A_2} \right)}{z^2 - A_1^2 t^2} \times \exp \left( - \frac{z^2}{4A_2 t} \right)$$  \hspace{1cm} (87)

for $z > A_1 t$.
Thus, we find the estimate

$$E(z, t) \leq \left( \max_{\tau \in [0, t]} E(0, \tau) \right) \frac{1}{z^2 - A_1^2 t^2} \frac{A_2 t}{\pi} \exp \left( -\frac{A_1^2}{4A_2^2} t \right) \exp \left( \frac{A_1^2 z}{2A_2} - \frac{z^2}{4A_2^2} \right)$$

(88)

for $z > A_1 t$.

The estimate (88) proves that, for a fixed time, at large distance to the support $D_n$ of the given data, the dominant term is $\exp \left( -\frac{z^2}{4A_2^2} \right)$. Moreover, $A_2$ depends only on the thermal coefficients (see [17, 18]). We can conclude that at large distance from the support of the external given data, the spatial decay of processes is influenced only by the thermal effect.

References


Saint-Venant’s Problem

▶ Saint-Venant’s Problem for Cosserat Elastic Shells

Saint-Venant’s Problem for Cosserat Elastic Shells

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Synonyms

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