On some fundamental misunderstandings in the indeterminate couple stress model. A comment on recent papers of A.R. Hadjesfandiari and G.F. Dargush

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Abstract

In a series of papers which are either published [13, 14] or available as preprints [12, 16, 15, 17, 18] Hadjesfandiari and Dargush have reconsidered the linear indeterminate couple stress model. They are postulating a certain physically plausible split in the virtual work principle. Based on this postulate they claim that the second-order couple stress tensor must always be skew-symmetric. Since they do not consider that the set of boundary conditions intervening in the virtual work principle is not unique, their statement is not tenable and leads to some misunderstandings in the indeterminate couple stress model. This is shown by specifying their development to the isotropic case. However, their choice of constitutive parameters is mathematically possible and we show that it still yields a well-posed boundary value problem.

Key words: modified couple stress model, symmetric Cauchy stresses, Boltzman axiom, symmetry of couple stress tensor, generalized continua, microstructure, size effects, strain gradient elasticity, conformal invariance, gradient elasticity, consistent traction boundary conditions.

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1 Introduction

Among higher gradient elasticity models [27, 28, 3, 25, 21, 26] one of the very first models considered in
the literature is the so called indeterminate couple stress model [9, 29, 44, 20] in which the higher gradient
contributions only enter through gradients of the continuum rotation, i.e. the total elastic energy can be
written as

$$W(\nabla u, \nabla(\nabla u)) = W_{\mathrm{lin}}(\mathrm{sym}\nabla u) + W_{\mathrm{curv}}(\nabla \mathrm{curl}\, u).$$

In general, higher gradient elasticity models are used to describe mechanical structures at the micro- and
nano-scale or to regularize certain ill-posed problems by means of these higher gradient contributions.

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jesfandiari and Dargush have reconsidered the linear indeterminate couple stress model. They are postulating

a certain physically plausible split in the virtual work principle. Based on this postulate they claim that the second-order couple stress tensor must be always skew-symmetric. Since their development has spread considerable confusion in the field of higher gradient elasticity, we were prompted to carefully re-examine their claim. In doing so we hope to contribute an important clarification in the field and to put an end to the above mentioned confusion.

In the course of our re-examination it turned out that the boundary conditions in the classical indeterminate couple stress theory are not uniquely defined and it seems that the set of boundary conditions which are directly derived from general strain gradient elasticity have never been clearly presented before. In [24, 7] we provide this kind of approach and we propose a new set of boundary conditions for the classical indeterminate couple stress model. In doing so, we find the underlying problems in the argument by Hadjesfandiari and Dargush [12, 13, 14, 15, 17]. While Hadjesfandiari and Dargush start with the linear general anisotropic couple stress response and only later specify to isotropy, for definiteness, we consider from the outset the linear isotropic indeterminate couple stress case as done in Koiter [20, p. 26]. Nevertheless, our development is essentially independent of any isotropy assumption. It is clear that exhibiting the gaps in their development for the simpler case of isotropy is sufficient for invalidating their claim.

Before discussing the papers by Hadjesfandiari and Dargush [13, 14, 17, 15] we will first recall the indeterminate couple stress model in its accepted format as far as kinematics and equilibrium equations are concerned. We also need to introduce the new, alternative, set of traction boundary conditions which rectify the shortcomings in all previous papers. For comparison, the up to now accepted traction boundary conditions, which are also consistent, are presented as well.

In the light of this new framework, we try to follow the argument given by Hadjesfandiari and Dargush [13, 14, 17, 15] as closely as possible. We will show that their implicitly formulated requirement, recast as a physically plausible postulate by us, leads to the skew-symmetry of the couple stress tensor if and only if the classical traction boundary conditions are assumed. However, within the new format of traction boundary conditions no similar conclusion is possible.

Despite the finally erroneous claim by Hadjesfandiari and Dargush we recognize their work in being the reason to reconsider the indeterminate couple stress model and to find a new variant of the accepted boundary conditions which do not appear in Hadjesfandiari and Dargush's work.

2 Notational agreements

First of all, let us point out that we do not completely adopt the notations from the papers by Hadjesfandiari and Dargush. However, in order to help the reader to compare and follow both approaches, all expressions

which are important for our main argument are rewritten in our notation.

In this paper, we denote by $\mathbb{R}^{3\times3}$ the set of real 3×3 second order tensors, written with capital letters. For $a, b \in \mathbb{R}^3$ we let $\langle a, b \rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|_{\mathbb{R}^3}^2 = \langle a, a \rangle_{\mathbb{R}^3}$. The standard Euclidean scalar product on $\mathbb{R}^{3\times3}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{3\times3}} = \mathrm{tr}(XY^T)$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{3\times3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{R}^{3\times3}$. The identity tensor on $\mathbb{R}^{3\times3}$ will be denoted by $\mathbb{1}$, so that $\mathrm{tr}(X) = \langle X, \mathbb{1} \rangle$. For all $X \in \mathbb{R}^{3\times3}$ we set $\mathrm{sym}\, X = \frac{1}{2}(X^T + X) \in \mathrm{Sym}(3)$, $\mathrm{skew}\, X = \frac{1}{2}(X - X^T) \in \mathfrak{so}(3)$ and the deviatoric part $\mathrm{dev}\, X = X - \frac{1}{3}\,\mathrm{tr}(X)\mathbb{1}$, where $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3\times3}\,|\,X^T = -X\}$ denotes the set of skew-symmetric matrices and $\mathrm{Sym}(3) = \{X \in \mathbb{R}^{3\times3}\,|\,X^T = X\}$ denotes the set of symmetric matrices.

Throughout this paper (when we do not specify else) Latin subscripts take the values $1, 2, 3$. Typical conventions for differential operations are implied such as comma followed by a subscript to denote the partial derivative with respect to the corresponding cartesian coordinate. We also use the Einstein notation of the sum over repeated indices if not differently specified. Here, we consider the operators $\mathrm{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ and $\mathrm{anti} : \mathbb{R}^3 \to \mathfrak{so}(3)$ through

$$(\mathrm{axl}\,\overline{A})_k = -\frac{1}{2}\,\epsilon_{ijk}\overline{A}_{ij}, \qquad \overline{A}.\,v = (\mathrm{axl}\,\overline{A}) \times v, \qquad (\mathrm{anti}(v))_{ij} = -\varepsilon_{ijk}v_k, \qquad \overline{A}_{ij} = \mathrm{anti}(\mathrm{axl}\,\overline{A})_{ij}, \qquad (2.1)$$

for all $v \in \mathbb{R}^3$ and $\overline{A} \in \mathfrak{so}(3)$, where ϵ_{ijk} is the totally antisymmetric third order permutation tensor. We recall that for a third order tensor $\mathbb{E}$ and $X \in \mathbb{R}^{3\times3}$, $v \in \mathbb{R}^3$ we have the contraction operations $\mathbb{E} : X \in \mathbb{R}^3$, $\mathbb{E}.\,v \in \mathbb{R}^{3\times3}$ and $X.\,v \in \mathbb{R}^3$, with the components

$$(\mathbb{E} :\, X)_i = \mathbb{E}_{ijk}\,X_{kj}, \qquad (\mathbb{E}.\,v)_{ij} = \mathbb{E}_{ijk}\,v_k, \qquad (X.\,v)_i = X_{ij}\,v_j. \qquad (2.2)$$

For multiplication of two matrices we will not use other specific notations, this means that for $A, B \in \mathbb{R}^{3\times3}$ we are setting $(AB)_{ij} = A_{ik}B_{kj}$.

We consider a body which occupies a bounded open set Ω of the three-dimensional Euclidian space $\mathbb{R}^3$ and assume that its boundary $\partial\Omega$ is a piecewise smooth surface. An elastic material fills the domain $\Omega \subseteq \mathbb{R}^3$ and we refer the motion of the body to rectangular axes Ox_i. Let us consider an open subset Γ of $\partial\Omega$. Here, ν is a vector tangential to the surface Γ and which is orthogonal to its boundary $\partial\Gamma$, $\tau = n \times \nu$ is the tangent to the curve $\partial\Gamma$ with respect to the orientation on Γ. We assume that $\partial\Omega$ is a smooth surface. Hence, there are no singularities of the boundary and the jump $[\![a \cdot \nu]\!] := [a \cdot \nu]^+ + [a \cdot \nu]^- = ([a]^+ - [a]^-) \cdot \nu$ of a across the joining curve $\partial\Gamma$ arises only as consequence of possible discontinuities of the corresponding quantities which follows

from the prescribed boundary conditions on Γ and $\partial\Omega \setminus \overline{\Gamma}$, where

$$[\,\cdot\,]^- := \lim_{\substack{x \in \partial\Omega \setminus \overline{\Gamma} \\ x \to \partial\Gamma}} [\,\cdot\,], \qquad [\,\cdot\,]^+ := \lim_{\substack{x \in \Gamma \\ x \to \partial\Gamma}} [\,\cdot\,].$$

For smooth enough vector fields v we define $\nabla v = \left((\nabla v_1)^T, (\nabla v_2)^T, (\nabla v_3)^T\right)^T$, while for tensor fields P with smooth enough rows we define $\mathrm{Div}\, P = (\mathrm{div}\, P_1, \mathrm{div}\, P_2, \mathrm{div}\, P_3)^T$.

3 The classical indeterminate couple stress model

We are now shortly re-deriving the classical equations based on the $\nabla[\mathrm{axl}(\mathrm{skew}\,\nabla u)]$-formulation of the indeterminate couple stress model. This part does not contain new results, see, e.g., [24] for further details, but is included for setting the stage for this contribution.

The linear isotropic indeterminate couple stress problem can be viewed as a minimization problem

$$\int_\Omega \left[\mu \,\|\,\mathrm{sym}\,\nabla u\|^2 + \frac{\lambda}{2}\,[\mathrm{tr}(\nabla u)]^2 + W_{\mathrm{curv}}(\nabla \mathrm{curl}\, u) - \langle f, u\rangle\right] dv \quad \to \min. \text{ w.r.t. } u, \tag{3.1}$$

subjected to geometric and mechanical boundary conditions, in part depending on the form of $W_{\mathrm{curv}}(\nabla\mathrm{curl}\,u)$, which will be specified later on.

In the following, in order to place the subject in the literature, we outline some curvature energies proposed in different isotropic second gradient elasticity properly models:

- **the indeterminate couple stress model** (Grioli-Koiter-Mindlin-Toupin model) [9, 2, 20, 29, 44, 42, 10] in which the higher derivatives (apparently) appear only through derivatives of the infinitesimal continuum rotation $\mathrm{curl}\,u$. Hence, the curvature energy has the equivalent forms

$$\begin{aligned} W_{\mathrm{curv}}(\nabla \mathrm{curl}\, u) &= \mu\, L_c^2 \left[\frac{\alpha_1}{4}\,\|\,\mathrm{sym}\,\nabla\mathrm{curl}\,u\|^2 + \frac{\alpha_2}{4}\,\|\,\mathrm{skew}\,\nabla\mathrm{curl}\,u\|^2\right] \\ &= \mu\, L_c^2 \left[\alpha_1\,\|\,\mathrm{sym}\,\nabla[\mathrm{axl}(\mathrm{skew}\,\nabla u)]\|^2 + \alpha_2\,\|\,\mathrm{skew}\,\nabla[\mathrm{axl}(\mathrm{skew}\,\nabla u)]\|^2\right] \\ &= \mu\, L_c^2 \left[\frac{\alpha_1}{4}\,\|\,\mathrm{dev}\,\mathrm{sym}\,\nabla\mathrm{curl}\,u\|^2 + \frac{\alpha_2}{4}\,\|\,\mathrm{skew}\,\nabla\mathrm{curl}\,u\|^2\right] \\ &= \frac{\mu\, L_c^2}{8}\,(\alpha_1 + \alpha_2) \left[\|\nabla\mathrm{curl}\,u\|^2 + \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\,\langle\nabla\mathrm{curl}\,u, (\nabla\mathrm{curl}\,u)^T\rangle\right]. \end{aligned} \tag{3.2}$$

We remark that the spherical part of the couple stress tensor remainsindeterminate since $\mathrm{tr}(\nabla\mathrm{curl}\,u) = \mathrm{div}(\mathrm{curl}\,u) = 0$. In order to prove the pointwise uniform positive definiteness it is assumed following [20], that $\alpha_1 > 0, \alpha_2 > 0$. Note that pointwise uniform positivity is often assumed, $-1 < \eta := \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} < 1$ [20,

Eq. (4.6)], when deriving analytical solutions for simple boundary value problems because it allows to invert the couple stress-curvature relation. Let us also note that the modified symmetric couple stress model corresponds to the limit case $\eta = 1$, while the skew-symmetric couple stress model corresponds to the limit case $\eta = -1$. It is clear that pointwise positive definiteness is not necessary for well-posedness (see [19] and the well-posedness from Section 6). Mindlin [29, p. 425] explained the relations between Toupin's constitutive equations [43] and Grioli's [9] constitutive equations and concluded that the obtained equations in the linearized theory are identical, since the extra constitutive parameter η of Grioli's model (see $(3.2)_4$, Grioli considers $\eta = 0$) does not explicitly appear in the equations of motion but enters only the boundary conditions[1] (see [29]). The same extra constitutive coefficient appears in Mindlin and Eshel's version [28].

- **the modified - symmetric couple stress model - the conformal model**. On the other hand, in the conformal case [37, 36] one may consider that $\alpha_2 = 0$, which makes the couple stress tensor $\widetilde{m}$ symmetric and trace free. This conformal curvature case has been considered by Neff in [37], the curvature energy having the form

$$W_{\mathrm{curv}}(\nabla \operatorname{curl} u) = \mu\, L_c^2\, \frac{\alpha_1}{4}\, \| \operatorname{sym} \nabla \operatorname{curl} u \|^2 = \mu\, L_c^2\, \alpha_1\, \| \operatorname{dev} \operatorname{sym} \nabla [\operatorname{axl}(\operatorname{skew} \nabla u)] \|^2. \qquad (3.3)$$

Indeed, there are two major reasons uncovered in [37] for using the modified couple stress model. First, in order to avoid singular stiffening behaviour for smaller and smaller samples in bending [35] one has to take $\alpha_2 = 0$. Second, based on a homogenization procedure invoking an intuitively appealing natural "micro-randomness" assumption (a strong statement of microstructural isotropy) requires conformal invariance, which is again equivalent to $\alpha_2 = 0$. Such a model is still well-posed [19] leading to existence and uniqueness results with only one additional material length scale parameter, while it is **not** pointwise uniformly positive definite.

- **the skew-symmetric couple stress model - the non-conformal model (and the object of this discussion)**. Hadjesfandiari and Dargush strongly advocate [13, 14, 15] the opposite extreme case, $\alpha_1 = 0$ and $\alpha_2 > 0$, i.e. they used the curvature energy

$$W_{\mathrm{curv}}(\nabla \operatorname{curl} u) = \mu\, L_c^2\, \frac{\alpha_2}{4}\, \| \operatorname{skew} \nabla(\operatorname{curl} u) \|^2 = \mu\, L_c^2\, \alpha_2\, \| \operatorname{axl} \operatorname{skew} \nabla(\operatorname{curl} u) \|^2 = 4\, \mu\, L_c^2\, \alpha_2\, \| \operatorname{curl}(\operatorname{curl} u) \|^2.$$

In that model the nonlocal force stresses and the couple stresses are both assumed to be skew-symmetric.

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[1] $\langle \nabla \operatorname{curl} u, (\nabla \operatorname{curl} u)^T \rangle = \operatorname{tr}[(\nabla \operatorname{curl} u)^2] = [\operatorname{tr}(\nabla \operatorname{curl} u)]^2 - 2\, \operatorname{tr}[\operatorname{Cof}(\nabla \operatorname{curl} u)] = [\operatorname{div}(\operatorname{curl} u)]^2 - 2\, \operatorname{tr}[\operatorname{Cof}(\nabla \operatorname{curl} u)] = -2\, \operatorname{tr}[\operatorname{Cof}(\nabla \operatorname{curl} u)]$, which is a boundary term.

Their reasoning, based in fact on an incomplete understanding of boundary conditions is critically discussed in this paper and generally refuted, while mathematically it is well-posed, which will also be shown.

3.1 Equilibrium and constitutive equations

Taking free variations $\delta u \in C^\infty(\Omega)$ in the energy $W(\operatorname{sym}\nabla u, \nabla\operatorname{curl} u) = W_{\mathrm{lin}}(\operatorname{sym}\nabla u) + W_{\mathrm{curv}}(\nabla\operatorname{curl} u)$, where

$$W_{\mathrm{lin}}(\operatorname{sym}\nabla u) = \mu \,\|\operatorname{sym}\nabla u\|^2 + \frac{\lambda}{2}\,[\operatorname{tr}(\nabla u)]^2 = \mu\,\|\operatorname{dev}\operatorname{sym}\nabla u\|^2 + \frac{2\,\mu + 3\,\lambda}{6}\,[\operatorname{tr}(\nabla u)]^2,$$

$$W_{\mathrm{curv}}(\nabla\operatorname{curl} u) = \mu\,L_c^2\,[\alpha_1\,\|\operatorname{dev}\operatorname{sym}\nabla[\operatorname{axl}(\operatorname{skew}\nabla u)]\|^2 + \alpha_2\,\|\operatorname{skew}\nabla[\operatorname{axl}(\operatorname{skew}\nabla u)]\|^2], \qquad (3.4)$$

we obtain the virtual work principle

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega W(\nabla u + t\,\nabla\delta u)\,dv\Big|_{t=0} = \int_\Omega \Big[2\mu\,\langle\operatorname{sym}\nabla u, \operatorname{sym}\nabla\delta u\rangle + \lambda\,\operatorname{tr}(\nabla u)\,\operatorname{tr}(\nabla\delta u)$$

$$+ \mu\,L_c^2\,[2\,\alpha_1\,\langle\operatorname{dev}\operatorname{sym}\nabla[\operatorname{axl}(\operatorname{skew}\nabla u)], \operatorname{dev}\operatorname{sym}\nabla[\operatorname{axl}(\operatorname{skew}\nabla\delta u)]\rangle \qquad (3.5)$$

$$+ 2\,\alpha_2\,\langle\operatorname{skew}\nabla[\operatorname{axl}(\operatorname{skew}\nabla u)], \operatorname{skew}\nabla[\operatorname{axl}(\operatorname{skew}\nabla\delta u)]\rangle] + \langle f, \delta u\rangle \Big]\,dv = 0,$$

where f denotes the body force density.

The classical divergence theorem leads to

$$\int_\Omega \langle\operatorname{Div}(\sigma - \widetilde{\tau}) + f, \delta u\rangle\,dv \underbrace{- \int_{\partial\Omega} \langle(\sigma - \widetilde{\tau}).\,n, \delta u\rangle\,dv - \int_{\partial\Omega} \langle\widetilde{m}.\,n, \operatorname{axl}(\operatorname{skew}\nabla\delta u)\rangle\,da}_{\text{the virtual power work of the surface forces}} = 0, \qquad (3.6)$$

where

$$\widetilde{\sigma}_{\mathrm{total}} = \sigma - \widetilde{\tau} \notin \operatorname{Sym}(3) \qquad\qquad\qquad \text{total force-stress tensor}$$

$$\sigma = 2\,\mu\,\operatorname{sym}\nabla u + \lambda\,\operatorname{tr}(\nabla u)\,\mathbb{1} \in \operatorname{Sym}(3) \qquad\qquad \text{local force-stress tensor}$$

$$\widetilde{\tau} = \frac{1}{2}\,\operatorname{anti}\operatorname{Div}[\widetilde{m}] \in \mathfrak{so}(3), \qquad\qquad\qquad \text{nonlocal force-stress tensor} \qquad (3.7)$$

$$\widetilde{m} = \mu\,L_c^2\,[\alpha_1\,\operatorname{sym}\nabla(\operatorname{curl} u) + \alpha_2\,\operatorname{skew}\nabla(\operatorname{curl} u)] \qquad \text{couple stress tensor}$$

$$= \mu\,L_c^2\,[\alpha_1\,\operatorname{dev}\operatorname{sym}\nabla(\operatorname{curl} u) + \alpha_2\,\operatorname{skew}\nabla(\operatorname{curl} u)]$$

$$= \mu\,L_c^2\,[2\,\alpha_1\,\operatorname{dev}\operatorname{sym}\nabla[\operatorname{axl}(\operatorname{skew}\nabla u)] + 2\,\alpha_2\,\operatorname{skew}\nabla[\operatorname{axl}(\operatorname{skew}\nabla u)]],$$

and n is the unit outward normal vector at the surface $\partial\Omega$. The equilibrium equation are therefore

$$\operatorname{Div}\widetilde{\sigma}_{\mathrm{total}} + f = 0. \qquad (3.8)$$

Note that the local force-stress tensor σ is always symmetric, the nonlocal force-stress tensor $\widetilde{\tau}$ is automatically skew-symmetric, while the second order hyperstress tensor (the couple stress tensor) $\widetilde{m}$ may or may not be symmetric, depending on the material parameters. The asymmetry of force stress is a hidden constitutive assumption, compare to [7].

3.2 Boundary conditions

3.2.1 The classical Grioli-Koiter-Mindlin-Tiersten variant of the boundary conditions

Since the variations at the boundary and the interior can be assigned independently, see (3.6), we must have:

$$-\int_{\partial\Omega} \langle (\sigma - \widetilde{\tau}).\, n, \delta u \rangle \, da - \int_{\partial\Omega} \langle \widetilde{m}.\, n, \mathrm{axl}(\mathrm{skew}\,\nabla \delta u) \rangle \, da = 0 \qquad \text{or equivalently} \qquad (3.9)$$

$$-\int_{\partial\Omega} \langle (\sigma - \widetilde{\tau}).\, n, \delta u \rangle \, da - 2\int_{\partial\Omega} \langle \widetilde{m}.\, n, \mathrm{curl}\, \delta u \rangle \, da = 0.$$

This suggests 6 possible independent prescriptions of mechanical boundary conditions; three for the normal components of the total force stress $(\sigma - \widetilde{\tau}).n$ and three for the normal components of the couple stress tensor. The possible Dirichlet boundary conditions on $\Gamma \subset \partial\Omega$ seem to be the 6 conditions[2]

$$u = \widetilde{u}^{\mathrm{ext}}, \qquad \mathrm{axl}(\mathrm{skew}\,\nabla u) = \widetilde{w}^{\mathrm{ext}} \quad (\text{or equivalently } \mathrm{curl}\, u = 2\,\widetilde{w}^{\mathrm{ext}}), \qquad (3.10)$$

for two given functions $\widetilde{u}, \widetilde{w} : \mathbb{R}^3 \to \mathbb{R}^3$ at the open subset $\Gamma \subset \partial\Omega$ of the boundary (3+3 boundary conditions).

However, following Koiter we note

Remark 3.1. [independent variations and curl] *Assume $u \in C^\infty(\overline{\Omega})$ and $u\big|_\Gamma$ is known. Then $\mathrm{curl}\big|_\Gamma$ exists and for any open subset $\Gamma \subset \partial\Omega$ the integral $\int_\Gamma \langle \mathrm{curl}\, u, n \rangle \, da$ is already known by Stokes theorem, while $\int_\Gamma \langle \mathrm{curl}\, u, \tau \rangle \, da$ is still free, where τ is any tangential vector field on the open set $\Gamma \subset \partial\Omega$. Only the two tangential components of $\mathrm{curl}\, u$ may be independently prescribed on an open subset of the boundary.*

Already, Mindlin and Tiersten [29] have rightly remarked that also in this formulation only 5 mechanical boundary conditions can be prescribed. They rewrote (3.9) in a further separated form

$$-\int_{\partial\Omega} \langle (\sigma - \widetilde{\tau}).\, n - \underbrace{\frac{1}{2} n \times \nabla[\langle (\mathrm{sym}\,\widetilde{m}).n, n \rangle]}_{\substack{\text{performs already} \\ \text{work only against } \delta u}}, \delta u \rangle \, da - \int_{\partial\Omega} \langle (\mathbb{1} - n \otimes n)\, \widetilde{m}.n.\underbrace{(\mathbb{1} - n \otimes n)\,[\mathrm{axl}(\mathrm{skew}\,\nabla \delta u)]}_{\substack{\text{still contains certain} \\ \text{"tangential" derivatives of } \delta u \in C^\infty(\Omega), \\ \text{therefore it is affected by variations of } \delta u,}} da$$

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[2]as indeed proposed by Grioli [9] in concordance with the Cosserat kinematics for independent fields of displacements and microrotation.

$$(3.11)$$

$$= -\int_{\partial\Omega} \langle (\sigma - \widetilde{\tau}). \, n - \frac{1}{2} n \times \nabla [\langle (\operatorname{sym} \widetilde{m}). n, n \rangle], \delta u \rangle \, da - 2 \int_{\partial\Omega} \langle (\mathbb{1} - n \otimes n) \, \widetilde{m}. n, (\mathbb{1} - n \otimes n) \, [\operatorname{curl} \delta u] \rangle \, da = 0,$$

which already shows that the second term in (3.9) still contains contributions which perform work against δu, namely $\frac{1}{2} n \times \nabla [\langle (\operatorname{sym} \widetilde{m}). n, n \rangle]$, while the remaining higher order term $(\mathbb{1} - n \otimes n) \, \widetilde{m}. n$ performs work against combinations of second derivatives. Mindlin and Tiersten [29] concluded that 3 boundary conditions derive from the first integral (correctly) and two other from the second integral, since [29, p. 432] "the normal component of the couple stress vector $[\langle \widetilde{m}. n, n \rangle = \langle \operatorname{sym} \widetilde{m}. n, n \rangle]$ on" $\partial\Omega$ "enters only in the combination with the force-stress vector shown in the coefficient of" δu "in the surface integral (our first term on the right hand side of (3.11)).

Therefore, Mindlin and Tiersten [29] concluded that the boundary conditions consists in:

- **Geometric boundary conditions** on $\Gamma \subset \partial\Omega$:

$$u = \widetilde{u}^{\text{ext}}, \qquad\qquad\qquad\qquad (3 \text{ bc})$$

$$\begin{cases} (\mathbb{1} - n \otimes n). \operatorname{axl}(\operatorname{skew} \nabla u) &=& (\mathbb{1} - n \otimes n). \widetilde{w}^{\text{ext}}, \\[2mm] \text{or} \quad (\mathbb{1} - n \otimes n). \operatorname{curl} u &=& 2 \, (\mathbb{1} - n \otimes n). \widetilde{w}^{\text{ext}}, \qquad (2 \text{ bc}) \end{cases} \qquad (3.12)$$

for given functions $\widetilde{u}^{\text{ext}}$, $\widetilde{w}^{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ at the boundary. The latter condition prescribes only the tangential component of $\operatorname{axl}(\operatorname{skew} \nabla u)$. Therefore, we may prescribe only 3+2 independent boundary conditions.

- **Traction boundary conditions** on $\partial\Omega \setminus \overline{\Gamma}$:

$$(\sigma - \widetilde{\tau}). \, n - \frac{1}{2} n \times \nabla [\langle (\operatorname{sym} \widetilde{m}). n, n \rangle] = \widetilde{t}^{\text{ext}}, \qquad\qquad \text{traction} \qquad\qquad (3 \text{ bc})$$

$$(\mathbb{1} - n \otimes n) \, \widetilde{m}. n = (\mathbb{1} - n \otimes n) \, \widetilde{g}^{\text{ext}}, \quad \text{"double force normal traction"} \ (2 \text{ bc}) \ \ (3.13)$$

for prescribed functions $\widetilde{t}^{\text{ext}}$, $\widetilde{g}^{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ at the boundary.

However, while (3.12) and (3.13) correctly describe the maximal number of independent boundary conditions in the indeterminate couple stress model and these conditions have been rederived again and again by Yang et al. [45], Park and Gao [39], [23], etc. among others they are not unique. This is explained in the following paragraphs.

3.2.2 A new set of traction boundary conditions

The indeterminate couple stress model is not simply obtained as a constraint Cosserat model [42], i.e. assuming that $\overline{A} = \text{axl}(\text{skew}\nabla u)$. In the indeterminate couple stress model the only independent kinematical degree of freedom is u. We understand that the indeterminate couple stress model constructed as a constraint Cosserat model represents at most an approximation of the indeterminate couple stress model, in the sense that the boundary conditions are not completely considered. We remark that the quantity $\langle \widetilde{m}.n, (\mathbb{1} - n \otimes n)[\text{axl}(\text{skew}\,\nabla\delta u)]\rangle$ does still contain contributions performing work against δu alone (even though there is a projection $\mathbb{1} - n \otimes n$ involved), which can be assigned arbitrarily and are therefore somehow related to independent variation δu. This case is not similar to the Cosserat theory in which one assumes a priori that displacement u and microrotation $\overline{A} \in \mathfrak{so}(3)$ are independent kinematical degrees of freedom.

At this point, it must also be considered that the tangential trace of the gradient of virtual displacement can be integrated by parts once again and that the surface divergence theorem can be applied to this tangential part of $\nabla\delta u$. As it is well known from differential geometry, the projectors $\mathbb{1} - n \otimes n$ and $n \otimes n$ allow to split a given vector or tensor field in a part projected on the plane tangent to the considered surface and in another part projected on the normal to such surface (see also [5, 41, 4] for details)). For the sake of simplicity we assume that $\partial\Omega$ is a smooth surface of class C^2. As in the notation section, we consider the curve $\partial\Gamma$ which joins the open subsets Γ and $\partial\Omega \setminus \Gamma$ of the boundary. Therefore, in our case and in our abbreviations, the surface divergence theorem [11, p. 58, ex. 7] reads:

$$\int_{\partial\Omega} \text{Div}^S(v)\,da := \int_{\partial\Omega} \langle \mathbb{1} - n \otimes n, \nabla\left((\mathbb{1} - n \otimes n)\cdot v)\right]\rangle\,da = \int_{\partial\Gamma} [\![\,\langle v, \nu \rangle\,]\!]\,ds. \tag{3.14}$$

for any field $v \in \mathbb{R}^3$. Indeed, using the surface divergence theorem, we have obtained in [24] that

$$
\begin{aligned}
-\int_{\partial\Omega} &\langle (\sigma - \widetilde{\tau}).\,n, \delta u \rangle\,da - \int_{\partial\Omega} \langle \widetilde{m}.\,n, \text{axl}(\text{skew}\,\nabla\delta u)\rangle\,da \\
&= -\int_{\partial\Omega} \langle (\sigma - \tfrac{1}{2}\,\text{anti}\,\text{Div}[\widetilde{m}]).\,n - \tfrac{1}{2}n \times \nabla[\langle (\text{sym}\,\widetilde{m}).n, n \rangle] \\
&\qquad - \tfrac{1}{2}\{\nabla[(\text{anti}[(\mathbb{1} - n \otimes n)\widetilde{m}.\,n])\,(\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n), \delta u \rangle\,da \\
&\quad - \tfrac{1}{2}\int_{\partial\Omega} \underbrace{\langle (\mathbb{1} - n \otimes n)\,\text{anti}[(\mathbb{1} - n \otimes n)\widetilde{m}.\,n].n, \nabla\delta u.n\rangle}_{\substack{\text{completely } \delta u\text{-independent second} \\ \text{order normal variation of the gradient,} \\ \text{i.e. not affected by variations of } \delta u}}\,da - \tfrac{1}{2}\int_{\partial\Gamma} \langle [\![\text{anti}[(\mathbb{1} - n \otimes n)\widetilde{m}.\,n].\nu]\!], \delta u\rangle ds.
\end{aligned}
\tag{3.15}
$$

Hence, there are indeed two terms

$$(\sigma - \tfrac{1}{2}\,\text{anti}\,\text{Div}[\widetilde{m}]).\,n - \tfrac{1}{2}n \times \nabla[\langle (\text{sym}\,\widetilde{m}).n, n\rangle] - \tfrac{1}{2}\nabla[(\text{anti}[(\mathbb{1} - n \otimes n)\widetilde{m}.\,n])(\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n)$$

and

$$\llbracket \mathrm{anti}[(\mathbb{1} - n \otimes n)\widetilde{m}.\,n].\,\nu \rrbracket$$

which perform work against δu, while only the term

$$(\mathbb{1} - n \otimes n)\,\mathrm{anti}[(\mathbb{1} - n \otimes n)\widetilde{m}.\,n].n$$

is related solely to the independent second order normal variation of the gradient $\nabla \delta u.n$. This split of the boundary condition is not the one as obtained e.g. by Gao and Park [40] and seems to be entirely new in the context of the indeterminate couple stress model.

Therefore, it is also possible to adjoin on $\partial\Omega$ the following complete and consistent set of boundary conditions (moreover, these boundary conditions coincide with those which arise directly from general strain gradient elasticity, see [24]):

- Geometric (essential) boundary conditions on $\Gamma \subset \partial\Omega$:

$$u = u^{\mathrm{ext}}, \qquad\qquad (3 \text{ bc}) \qquad\qquad (3.16)$$

$$(\mathbb{1} - n \otimes n)(\nabla u).n = (\mathbb{1} - n \otimes n)w^{\mathrm{ext}}, \qquad\qquad (2 \text{ bc})$$

    where $u^{\mathrm{ext}}, w^{\mathrm{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ are prescribed functions (i.e. 3+2=5 boundary conditions), as in [29].

- Traction boundary conditions on $\partial\Omega \setminus \overline{\Gamma}$:

$$(\sigma - \widetilde{\tau}).\,n - \tfrac{1}{2}n \times \nabla[\langle (\mathrm{sym}\,\widetilde{m}).n, n \rangle]$$

$$-\tfrac{1}{2}\{\nabla[(\mathrm{anti}[(\mathbb{1} - n \otimes n)\widetilde{m}.\,n])(\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n) \;\; = \;\; t^{\mathrm{ext}}, \qquad (3 \text{ bc}) \qquad (3.17)$$

$$(\mathbb{1} - n \otimes n)\,\mathrm{anti}[(\mathbb{1} - n \otimes n)\widetilde{m}.\,n].n \;\; = \;\; (\mathbb{1} - n \otimes n)\,g^{\mathrm{ext}} \quad (2 \text{ bc})$$

    where $t^{\mathrm{ext}}, g^{\mathrm{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ are prescribed functions on $\partial\Omega \setminus \overline{\Gamma}$.

- Jump boundary conditions on $\partial\Gamma \subset \partial\Omega$:

$$\llbracket \mathrm{anti}[(\mathbb{1} - n \otimes n)\widetilde{m}.\,n].\,\nu \rrbracket \;\; = \;\; \pi^{\mathrm{ext}}, \qquad\qquad (3 \text{ bc}) \qquad\qquad (3.18)$$

    where π^{ext} is prescribed on $\partial\Gamma$ and leads to 3 boundary conditions.

Remark 3.2. *The difference between the two sets of boundary conditions is similar to the distinctions between two linear independent vectors (the classical set of boundary conditions), and two orthogonal vectors (the new*

fully independent set of boundary conditions).

4   The Hadjesfandiari and Dargush's postulate

Let us now turn to Hadjesfandiari and Dargush's far reaching claims. In the abstract of their paper [13] Hadjesfandiari and Dargush write:

> *"By relying on the definition of admissible boundary conditions, the principle of virtual work and some kinematical considerations, we establish the skew-symmetric character of the couple-stress tensor* $[\widetilde{m}]$ *in size-dependent continuum representations of matter. This fundamental result, which is independent of the material behavior* [e.g. isotropy]*, resolves all difficulties in developing a consistent couple stress theory."*

In their appendix of [14, p. 1263-1264] they add:

> *"Therefore, the present determinate theory is not mathematically a special case of* [the] *indeterminate* [couple stress] *theory obtained by letting* $[\alpha_1 = 0]$*. This is just a coincidence for the linear isotropic case where equations in both theories have some similarities. It should be realized that the determinate theory is not simply about fixing the constitutive equations for* [the] *linear isotropic couple stress theory of elasticity. The present size-dependent couple stress theory is the consistent couple stress theory in continuum mechanics. This has been achieved by discovering the skew-symmetric character of the couple-stress tensor* $[\widetilde{m}]$*. Mindlin, Tiersten and Koiter did not recognize the mean curvature tensor* $[\mathrm{skew}[\nabla \mathrm{curl}\, u]]$ *as the consistent measure of deformation in continuum mechanics."*[3]

In our understanding this claim is completely unfounded. Assuming isotropic response, we treat their "consistent and determinate" couple stress theory as the linear and isotropic accepted indeterminate couple stress model with constitutive parameters $\alpha_1 = 0$, $\alpha_2 > 0$ in (3.7).

Turning to their most important claim regarding the skew-symmetry of the coule stress tensor $\widetilde{m}$ we will exhibit their line of thought. Their reasoning is based on their fundamental hypothesis that the normal component of the couple stress traction vector $\langle \widetilde{m}.n, n \rangle$ should vanish on any bounding surface of an arbitrary volume[4]. In the case $\alpha_1 = 0$ we have $\widetilde{m} \in \mathfrak{so}(3)$ and $\langle \widetilde{m}.n, n \rangle = 0$ would be satisfied automatically.

In our notation the argument of Hadjesfandiari and Dargush is given as follows [12, p. 12-13]:

---

[3]In [14, p. 1283, A.16] Hadjesfandiari and Dargush erroneously take the strict positivity of curvature parameters $\alpha_1, \alpha_2 > 0$, used initially by Koiter, Mindlin and others, as belonging to the definition of the indeterminate couple stress model. This is clearly not the case. It can be shown that both limit cases $\alpha_1 > 0, \alpha_2 = 0$ (modified couple stress theory) and $\alpha_1 = 0, \alpha_2 > 0$ (Hadjesfandiari and Dargush choice) are mathematically admitted with the provision of using the attendant correct boundary conditions being defined by the solution space depending on the values of α_1, α_2, see Section 6.

[4]They also wrote [17, p.13]: *"...the corresponding generalized force must be zero and, for the normal component of the surface moment-traction vector* $[\widetilde{m}.n]$*, we must enforce the condition* $[\langle \widetilde{m}.n, n \rangle = 0]$*."*.

"From kinematics, since $[\omega^{nn} := \langle \mathrm{curl}\, u, n \rangle]$ is not an independent generalized degree of freedom, its apparent corresponding generalized force must be zero. Thus, for the normal component of the surface couple vector $[\widetilde{m}.n]$, we must enforce the condition

$$[\langle \widetilde{m}.n, n \rangle = 0].$$

[...] we notice that the energy equation can be written for any arbitrary volume with arbitrary surface within the body. Therefore, for any point on any arbitrary surface with unit normal n, we must have

$$[\langle \widetilde{m}.n, n \rangle = 0].$$

Since $n_i n_j$ is symmetric and arbitrary in [..], $[\widetilde{m}]$ must be skew-symmetric. Thus, $[\widetilde{m}^T = -\widetilde{m}.]$ This is the fundamental property of the couple-stress tensor in polar continuum mechanics, which has not been recognized previously." (see also [13, p. 2500]).

Let us interpret this statement. We rephrase it for our purpose through the formulation of an implicit additional

Postulate 4.1. [Hadjesfandiari and Dargush as we understand it] *In any extended continuum model only the total force stress traction vector $(\sigma - \widetilde{\tau}).n$ should perform work against the independent virtual displacement δu at the part $\partial\Omega \setminus \overline{\Gamma}$ of the boundary $\partial\Omega$, where traction boundary conditions are applied.*

Remark 4.2. *Incidentally, this postulate is automatically satisfied in classical elasticity, Cosserat and micromorphic models [32], since the possible variations of the field variables are independent anyway. Whether such a postulate can be satisfied in a higher gradient continuum is the concern of Hadjesfandiari and Dargush. We will see that this is not always possible.*

Hadjesfandiari and Dargush apply this postulate to the classical Mindlin and Tiersten's format of the boundary conditions, namely (3.12) and (3.13). Inspection of the indeterminate couple stress model within the framework of these classical variant of the boundary conditions, see e.g. (5.2) and (5.3), shows that choosing sym $\widetilde{m} = 0$ is indeed sufficient for this postulate to be satisfied.

Hadjesfandiari and Dargush accept

$$\sigma.n = \sigma_n, \qquad \widetilde{m}.n = m_n, \tag{4.1}$$

and they also realize correctly that the number of geometric or mechanic boundary conditions is 5 since the

tangential component of the test function δu cannot independently be varied, which is by now well established. To this aim, they spilt the term

$$\int_{\partial V} \langle \widetilde{m}.n, \operatorname{axl} \operatorname{skew} \nabla \delta u \rangle \, da = 2 \int_{\partial V} \langle \widetilde{m}.n, \operatorname{curl} \delta u \rangle \, da \tag{4.2}$$

$$= 2 \int_{\partial V} \langle \underbrace{(n \otimes n)\, \widetilde{m}.n}_{\text{normal part}}, \operatorname{curl} \delta u \rangle \, da + 2 \int_{\partial V} \langle \underbrace{(\mathbb{1} - n \otimes n)\, \widetilde{m}.n}_{\text{tangential part}}, \operatorname{curl} \delta u \rangle \, da$$

$$= 2 \int_{\partial V} \langle \widetilde{m}.n, \underbrace{(n \otimes n).\operatorname{curl} \delta u}_{\text{normal part}} \rangle \, da + 2 \int_{\partial V} \langle \widetilde{m}.n, \underbrace{(\mathbb{1} - n \otimes n).\operatorname{curl} \delta u}_{\text{tangential part}} \rangle \, da$$

on any arbitrary subdomain $V \subset \Omega$, into its tangential and normal part. They observe that the normal part

$$\int_{\partial V} \langle (n \otimes n)\, \widetilde{m}.n, \operatorname{curl} \delta u \rangle \, da = \int_{\partial V} \underbrace{\langle (\operatorname{sym} \widetilde{m}).n, n \rangle}_{\text{generalized force}} \cdot \underbrace{\langle \operatorname{curl} \delta u, n \rangle}_{\substack{\text{is affected} \\ \text{by variantions of } \delta u \ (\checkmark)}} \, da \tag{4.3}$$

cannot be prescribed independently of δu, since indeed their $\omega^{nn} := \langle \operatorname{curl} \delta u, n \rangle$ is affected by variations of δu due to Stokes theorem. To avoid a somehow felt inconsistency, they state that the corresponding generalized force must be zero and, for the normal component of the surface moment-traction vector, they enforce accordingly the (misguided) condition

$$\underbrace{\langle (\operatorname{sym} \widetilde{m}).n, n \rangle}_{\text{generalized force}} = 0 \tag{4.4}$$

on any arbitrary subdomain $V \subset \Omega$ having the boundary ∂V.

The equilibrium equations considered by Hadjesfandiari and Dargush ($\alpha_1 = 0$)[13] read therefore

$$\operatorname{Div} \widetilde{\sigma}_{\text{total}} + f = 0, \tag{4.5}$$

where the total force stress is given by

$$
\begin{aligned}
\widetilde{\sigma}_{\text{total}} &= \sigma - \widetilde{\tau} \notin \operatorname{Sym}(3) && \text{total force-stress tensor} \\
\sigma &= 2\,\mu\, \operatorname{sym} \nabla u + \lambda \, \operatorname{tr}(\nabla u)\mathbb{1} \in \operatorname{Sym}(3) && \text{local force-stress tensor} \\
\widetilde{\tau} &= \frac{1}{2} \operatorname{anti} \operatorname{Div}[\widetilde{m}] \in \mathfrak{so}(3), && \text{nonlocal force-stress tensor} \quad (4.6) \\
\widetilde{m} &= \mu\, L_c^2\, \alpha_2\, \operatorname{skew} \nabla(\operatorname{curl} u) = 2\,\mu\, L_c^2\, \alpha_2\, \operatorname{skew} \nabla[\operatorname{axl}(\operatorname{skew} \nabla u)]] \in \mathfrak{so}(3) && \text{couple stress tensor.}
\end{aligned}
$$

To the equilibrium equation, Hadjesfandiari-Dargush adjoin on $\partial\Omega$ the following boundary conditions

- **Geometric boundary conditions** on $\Gamma \subset \partial\Omega$:

$$u = \widetilde{u}^{\text{ext}},$$

$$\begin{cases} (\mathbb{1} - n \otimes n).\,\mathrm{axl}(\mathrm{skew}\,\nabla u) & = & (\mathbb{1} - n \otimes n).\widetilde{w}^{\text{ext}}, \\ \text{or} \quad (\mathbb{1} - n \otimes n).\,\mathrm{curl}\,u & = & 2\,(\mathbb{1} - n \otimes n).\,\widetilde{w}^{\text{ext}}, \end{cases} \tag{4.7}$$

for given functions $\widetilde{u}^{\text{ext}}, \widetilde{w}^{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ at the boundary. The latter condition prescribes only the tangential component of $\mathrm{axl}(\mathrm{skew}\nabla u)$. Therefore, they prescribe(correctly) only 3+2 independent boundary conditions.

We may consider the equivalent geometric boundary conditions

$$\begin{aligned} u & = & u^{\text{ext}} & \quad \text{on} \quad \Gamma, \\ (\mathbb{1} - n \otimes n)\,\nabla u.\,n & = & (\mathbb{1} - n \otimes n).\,w^{\text{ext}} & \quad \text{on} \quad \Gamma, \end{aligned} \tag{4.8}$$

where $u^{\text{ext}}, w^{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ are given, i.e. 3+2 boundary conditions.

- **Traction boundary conditions** on $\partial\Omega \setminus \overline{\Gamma}$:

The Hadjesfandiari-Dargush-choice:
$$\begin{aligned} (\sigma + \widetilde{\tau}).\,n & = & \widetilde{t}^{\text{ext}} & \quad \text{on} \quad \partial\Omega \setminus \overline{\Gamma}, \\ (\mathbb{1} - n \otimes n).\,\widetilde{m}.n & = & (\mathbb{1} - n \otimes n).\,\widetilde{g}^{\text{ext}} & \quad \text{on} \quad \partial\Omega \setminus \overline{\Gamma}, \end{aligned} \tag{4.9}$$

where $\widetilde{t}^{\text{ext}}, \widetilde{g}^{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ are given.

5 No necessity for the Hadjesfandiari-Dargush formulation with skew-symmetric couple stress tensor

Let us try to follow the argument of Hadjesfandiari and Dargush (bona fide). It follows that the split (4.2) considered by Hadjesfandiari-Dargush is not fully complete, since they nowhere do use the surface divergence theorem and, therefore, they do not see the extra terms performing work against δu. We see, on the contrary, that, when integrated over $\partial\Omega$, even upon the Hadjesfandiari and Dargush restriction the remaining tangential part from (4.2), namely

$$\int_{\partial\Omega} \underbrace{\langle (\mathbb{1} - n \otimes n)\widetilde{m}.n, \mathrm{curl}\,\delta u \rangle}_{\text{tangential part}}\,da = \frac{1}{4}\int_{\partial\Omega} \Big\{ \langle \nabla[(\mathrm{anti}(\widetilde{m}.\,n))\,(\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n), \delta u \rangle \tag{5.1}$$

$$+ \underbrace{\langle (\mathbb{1} - n \otimes n) \operatorname{anti}(\widetilde{m}.\,n).n, \nabla \delta u.n \rangle}_{\substack{\text{completely } \delta u\text{-independent second order} \\ \text{normal variation of gradient}}} \Big\} \, da,$$

still contains some parts with independent degrees of freedom, performing work against the normal derivative $\nabla \delta u.n$.

Further on, we explain this in more detail. Looking at the Mindlin and Tiersten approach, see also [24], in the framework of Hadjesfandiari-Dargush's assumption that the couple stress tensor is skew-symmetric, $\widetilde{m} \in \mathfrak{so}(3)$, we get

$$\langle \widetilde{m}.\,n, \operatorname{axl}(\operatorname{skew} \nabla \delta u) \rangle = \langle (\mathbb{1} - n \otimes n) \, \widetilde{m}.n, \operatorname{axl}(\operatorname{skew} \nabla \delta u) \rangle$$
$$+ \frac{1}{2} \langle n, \operatorname{curl} [\underbrace{\langle (\operatorname{sym} \widetilde{m}).n, n \rangle}_{\substack{\text{zero generalized force} \\ \text{according to} \\ \text{Hadjesfandiari-Dargush's assumption}}} \delta u] \rangle - \frac{1}{2} \underbrace{\langle n \times \nabla [\langle (\operatorname{sym} \widetilde{m}).n, n \rangle]}_{\substack{\text{performs work against } \delta u \\ \text{and should therefore vanish} \\ \text{corresponding to their Postulate 4.1}}} , \delta u \rangle, \qquad (5.2)$$
$$= \langle (\mathbb{1} - n \otimes n) \, \widetilde{m}.n, \operatorname{axl}(\operatorname{skew} \nabla \delta u) \rangle = \frac{1}{2} \langle (\mathbb{1} - n \otimes n) \, \widetilde{m}.n, \operatorname{curl} \delta u \rangle.$$

Therefore (3.11) can immediately be written as

$$- \int_{\partial \Omega} \langle \Big\{ (\sigma - \widetilde{\tau}).\,n, \delta u \rangle \, da - \int_{\partial \Omega} \langle (\mathbb{1} - n \otimes n) \, \widetilde{m}.n, (\mathbb{1} - n \otimes n) \, [\operatorname{axl}(\operatorname{skew} \nabla \delta u)] \rangle \, da = 0, \qquad (5.3)$$

which is used and accepted by Hadjesfandiari and Dargush, see (4.2). We comprehend their curvature parameter choice (based on this format of the independent boundary conditions): normal tractions would be automatically completely separated into pure total force-stress tractions and pure couple stress tractions.

However, after integration, the second term from (5.3) $\langle (\mathbb{1} - n \otimes n) \, \widetilde{m}.n, (\mathbb{1} - n \otimes n) \, [\operatorname{axl}(\operatorname{skew} \nabla \delta u)] \rangle$ leads to two new quantities: one performing work against the normal derivative $\nabla \delta u.n$ and one still performing work against δu. This means that the assumption of Hadjesfandiari and Dargush does not remove all quantities which may also perform work against δu, besides the total force stress tensor $(\sigma - \widetilde{\tau}).n$. This fact follows as a direct consequence of (3.15), since, also in the Hadjesfandiari-Dargush formulation similar to [24], it is possible for us to use the surface divergence theorem and to obtain

$$- \int_{\partial \Omega} \langle (\sigma - \widetilde{\tau}).\,n, \delta u \rangle \, da - \int_{\partial \Omega} \langle \widetilde{m}.\,n, \operatorname{axl}(\operatorname{skew} \nabla \delta u) \rangle \, da$$
$$= - \int_{\partial \Omega} \langle (\sigma - \widetilde{\tau}).\,n - \frac{1}{2} \nabla [(\operatorname{anti}[(\mathbb{1} - n \otimes n) \widetilde{m}.\,n])(\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n), \delta u \rangle \, da$$

$$-\frac{1}{2}\int_{\partial\Omega}\langle(\mathbb{1}-n\otimes n)\operatorname{anti}[(\mathbb{1}-n\otimes n)\widetilde{m}.n].n,\nabla\delta u.n\rangle\,da \tag{5.4}$$

$$-\frac{1}{2}\int_{\partial\Gamma}\langle[\![\operatorname{anti}((\mathbb{1}-n\otimes n)\widetilde{m}.n)]\!].\nu,\delta u\rangle\,ds,$$

$$=-\int_{\partial\Omega}\langle(\sigma-\widetilde{\tau}).n-\frac{1}{2}\nabla[(\operatorname{anti}[\widetilde{m}.n])(\mathbb{1}-n\otimes n)]:(\mathbb{1}-n\otimes n),\delta u\rangle\,da$$

$$-\frac{1}{2}\int_{\partial\Omega}\langle(\mathbb{1}-n\otimes n)\operatorname{anti}[\widetilde{m}.n].n,\nabla\delta u.n\rangle\,da-\frac{1}{2}\int_{\partial\Gamma}\langle[\![\operatorname{anti}(\widetilde{m}.n)]\!].\nu,\delta u\rangle\,ds,$$

for all variations $\delta u\in C^{\infty}(\Omega)$, where we have used that $\langle\widetilde{m}.n,n\rangle=0$ implies

$$(\mathbb{1}-n\otimes n)\widetilde{m}.n=\widetilde{m}.n-n\langle\widetilde{m}.n,n\rangle=\widetilde{m}.n.$$

We consider this last representation as the only correct form following the Hadjesfandiari-Dargush assumption $\langle\widetilde{m}.n,n\rangle=0$ made in (4.3).

Similar to the new variant of boundary conditions proposed in [24] but with the specification that $\widetilde{m}\in\mathfrak{so}(3)$ (which is now seen to be not necessary but possible), we arrive rather, upon the Hadjesfandiari-Dargush assumption $\langle\widetilde{m}.n,n\rangle=0$, at the traction boundary condition

$$[(\sigma-\widetilde{\tau}).n-\tfrac{1}{2}\nabla[(\operatorname{anti}[(\mathbb{1}-n\otimes n)\widetilde{m}.n])](\mathbb{1}-n\otimes n)]:(\mathbb{1}-n\otimes n)\,(x)\quad=\quad t^{\operatorname{ext}}(x),\qquad\text{(3 bc)}$$

$$[(\mathbb{1}-n\otimes n)\operatorname{anti}[(\mathbb{1}-n\otimes n)\widetilde{m}.n].n]\,(x)\quad=\quad[(\mathbb{1}-n\otimes n)\,g^{\operatorname{ext}}]\,(x)\qquad\text{(2 bc)}$$

$$\tag{5.5}$$

on $\partial\Omega\setminus\Gamma$, while on $\partial\Gamma$ we have to prescribe the jump conditions

$$\{[\![\operatorname{anti}[(\mathbb{1}-n\otimes n)\widetilde{m}.n]\!].\nu\}(x)\quad=\quad\pi^{\operatorname{ext}}(x),\qquad\text{(3 bc)}$$

From (5.5), we see finally that their intended response is not satisfied:

Remark 5.1. *Assuming that $\langle(\operatorname{sym}\widetilde{m}).n,n\rangle=0$ (the Hadjesfandiari-Dargush assumption) implies for the new alternative boundary conditions only that*

$$\int_{\partial\Omega}\langle\nabla[(\operatorname{anti}[(n\otimes n)\widetilde{m}.n])(\mathbb{1}-n\otimes n)]:(\mathbb{1}-n\otimes n),\delta u\rangle\,da=0. \tag{5.6}$$

However, from

$$\nabla[(\operatorname{anti}[\widetilde{m}.n])(\mathbb{1}-n\otimes n)]:(\mathbb{1}-n\otimes n),$$

there remains another contribution

$$\nabla[(\mathrm{anti}[(\mathbb{1} - n \otimes n)\widetilde{m}.\,n])(\mathbb{1} - n \otimes n)] : (\mathbb{1} - n \otimes n)$$

which still performs work against δu. The raison d'être of the Hadjesfandiari-Dargush formulation, as we understand it, was to avoid that any parts related to "couple stress normal traction" $\widetilde{m}.n$, other than the contributions to the total force-stresses, i.e. other than $\sigma - \widetilde{\tau}$, would perform work against δu. Our calculation for our alternative boundary conditions shows that, in general, this requirement can never be satisfied in any higher gradient elasticity theory.

Remark 5.2. *Let us point out again, that Hadjesfandiari and Dargush's choice and postulate is only meaningful with an additionally assumed format of boundary conditions. This format, as we have seen, is not mandatory. Therefore, Hadjesfandiari and Dargush's choice is a restriction and not a necessity.*

Continuing, Hadjesfandiari tries to support the claim of a skew-symmetric couple stress tensor with some independent motivations in [15]. There, he defines the "torsion-tensor" $\mathcal{X}$ and the "mean curvature tensor" ω

$$\mathcal{X}(u) = \mathrm{sym}\,\nabla\mathrm{curl}\,u, \qquad \omega(u) = \mathrm{skew}\,\nabla\mathrm{curl}\,u, \tag{5.7}$$

respectively. In Section 3.2. of [15, eq. 32] Hadjesfandiari claims that an inhomogeneous state u of constant torsional deformation $\mathcal{X}(u) = 0$ cannot exist. However, the conformal mapping, see Appendix A

$$\phi_c(x) = \frac{1}{2}\left(2\langle \mathrm{axl}\,\widehat{W}, x\rangle\, x - \mathrm{axl}\,\widehat{W})\|x\|^2\right) + [\widehat{p}\,\mathbb{1} + \widehat{A}].x + \widehat{b}, \tag{5.8}$$

where $\widehat{W}, \widehat{A} \in \mathfrak{so}(3)$, $\widehat{b} \in \mathbb{R}^3$, $\widehat{p} \in \mathbb{R}$ are arbitrary but constant is on the one hand inhomogeneous in the displacement $u = \phi_c - x$ but on the other hand gives precisely

$$\mathcal{X}(u) = \mathcal{X}(\phi_c(x) - x) = \mathrm{sym}\,\nabla\mathrm{curl}\,\phi_c(x) = 0. \tag{5.9}$$

For instance, a simple conformal displacement field is given by

$$\phi_c(x) = \begin{pmatrix} 2\,x_1^2 - (x_1^2 + x_2^2 + x_3^2) \\ 2\,x_1\,x_2 \\ 2\,x_1\,x_3 \end{pmatrix} = \begin{pmatrix} x_1^2 - (x_2^2 + x_3^2) \\ 2\,x_1\,x_2 \\ 2\,x_1\,x_3 \end{pmatrix} \Rightarrow \nabla\phi_c(x) = \begin{pmatrix} 2\,x_1 & -2\,x_2 & -2\,x_3 \\ 2\,x_2 & 2\,x_1 & 0 \\ 2\,x_3 & 0 & 2\,x_1 \end{pmatrix}$$

$$\Rightarrow \quad \text{skew} \nabla \phi_c(x) = \begin{pmatrix} 0 & -2\,x_2 & -2\,x_3 \\ 2\,x_2 & 0 & 0 \\ 2\,x_3 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{axl} \, \text{skew} \nabla \phi_c(x) = \begin{pmatrix} 0 \\ -2\,x_3 \\ 2\,x_2 \end{pmatrix} \tag{5.10}$$

$$\Rightarrow \quad \text{curl} \, \phi_c(x) = \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix} \quad \Rightarrow \quad \nabla \text{curl} \, \phi_c(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{sym} \, \nabla \text{curl} \, \phi_c(x) = 0.$$

6 Existence and uniqueness of the solution in the Hadjesfandiari-Dargush formulation

In view of the previous discussion, we do not think that the Hadjesfandiari-Dargush formulation has a sound physical motivation. Nevertheless, mathematically it is possible to consider the parameter choice inherent in the Hadjesfandiari-Dargush formulation. Indeed, they have given some interesting analytical solutions for these models [13, 14].

The usual Lebesgue spaces of square integrable functions, vector or tensor fields on Ω with values in $\mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{R}^{3\times3}$, respectively will be denoted by $L^2(\Omega)$. Moreover, we introduce the standard Sobolev spaces [1, 8, 22]

$$\begin{aligned}
\text{H}^1(\Omega) &= \{u \in L^2(\Omega) \,|\, \text{grad}\, u \in L^2(\Omega)\}, & \|u\|^2_{\text{H}^1(\Omega)} &:= \|u\|^2_{L^2(\Omega)} + \|\text{grad}\, u\|^2_{L^2(\Omega)}, \\
\text{H}(\text{curl};\Omega) &= \{v \in L^2(\Omega) \,|\, \text{curl}\, v \in L^2(\Omega)\}, & \|v\|^2_{\text{H}(\text{curl};\Omega)} &:= \|v\|^2_{L^2(\Omega)} + \|\text{curl}\, v\|^2_{L^2(\Omega)}, \\
\text{H}(\text{div};\Omega) &= \{v \in L^2(\Omega) \,|\, \text{div}\, v \in L^2(\Omega)\}, & \|v\|^2_{\text{H}(\text{div};\Omega)} &:= \|v\|^2_{L^2(\Omega)} + \|\text{div}\, v\|^2_{L^2(\Omega)},
\end{aligned} \tag{6.1}$$

of functions u or vector fields v, respectively. Furthermore, we introduce their closed subspaces $H^1_0(\Omega)$, $H_0(\text{curl};\Omega)$ as completion under the respective graph norms of the scalar valued space $C^\infty_0(\Omega)$ (the set of infinitely differentiable functions with compact support in Ω).

Let us consider for simplicity null boundary conditions. Hence, in the following we study the existence of the solution in the space

$$\widetilde{\mathcal{X}}_0 = \left\{ u \in H^1_0(\Omega) \,|\, \text{curl}\, u \in H(\text{curl};\Omega), \ (\mathbb{1} - n \otimes n).\,\text{curl}\, u\big|_\Gamma = 0 \right\}. \tag{6.2}$$

On $\widetilde{\mathcal{X}}_0$ we define the norm

$$\|u\|_{\widetilde{\mathcal{X}}_0} = \left(\|\nabla u\|^2_{L^2(\Omega)} + \|\text{skew} \nabla \, \text{axl}(\text{skew} \nabla u)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} = \left(\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4} \|\text{curl} \, \text{curl}\, u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \tag{6.3}$$

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and the bilinear form

$$((u, v)) = \int_{\Omega} \left[2\mu \left\langle \operatorname{sym} \nabla u, \operatorname{sym} \nabla v \right\rangle + \lambda \operatorname{tr}(\nabla u) \operatorname{tr}(\nabla v) \right.$$

$$\left. + 2\,\mu\,L_c^2\,\alpha_3 \left\langle \operatorname{skew}[\nabla \operatorname{axl}(\operatorname{skew} \nabla u)], \operatorname{skew}[\nabla \operatorname{axl}(\operatorname{skew} \nabla v)] \right\rangle \right] dv \qquad (6.4)$$

$$= \int_{\Omega} \left[2\mu \left\langle \operatorname{sym} \nabla u, \operatorname{sym} \nabla v \right\rangle + \lambda \operatorname{tr}(\nabla u) \operatorname{tr}(\nabla v) + 2\,\mu\,L_c^2\,\alpha_3 \left\langle \operatorname{curl} \operatorname{curl} u, \operatorname{curl} \operatorname{curl} v \right\rangle \right] dv,$$

where $u, v \in \widetilde{\mathcal{X}}_0$. Let us define the linear operator $l : \widetilde{\mathcal{X}}_0 \to \mathbb{R}$, describing the influence of external loads, $l(v) = \int_{\Omega} \langle f, v \rangle \, dv$ for all $\widetilde{w} \in \mathcal{X}_0$. We say that w is a weak solution of the problem $(\mathcal{P})$ if and only if

$$((u, v)) = l(v) \quad \text{for all} \quad v \in \mathcal{X}_0. \qquad (6.5)$$

A classical solution $u \in \mathcal{X}_0$ of the problem $(\mathcal{P})$ is also a weak solution.

Theorem 6.1. *Assume that*

   i) the constitutive coefficients satisfy $\mu > 0$, $\quad 3\lambda + 2\mu > 0$, $\quad \alpha_3 \geq 0$;

   ii) the loads satisfy the regularity condition $f \in L^2(\Omega)$.

Then there exists one and only one solution of the problem (6.5).

Proof. In the case $\alpha_3 = 0$ we have the boundary value problem from classical elasticity. Further, we consider the case $\boldsymbol{\alpha_3 > 0}$. The Cauchy-Schwarz inequality, the inequalities $(a \pm b)^2 \leq 2(a^2 + b^2)$ and the assumption upon the constitutive coefficients lead to

$$((u, v)) \leq C \, \|w\|_{\widetilde{\mathcal{X}}_0} \, \|\widetilde{w}\|_{\widetilde{\mathcal{X}}_0}, \qquad (6.6)$$

which means that $((\cdot, \cdot))$ is bounded. On the other hand, we have

$$((u, u)) = \int_{\Omega} \left[2\mu \, \| \operatorname{sym} \nabla u \|^2 + \lambda \, [\operatorname{tr}(\nabla u)]^2 + 2\,\mu\,L_c^2\,\alpha_3 \, \| \operatorname{skew}[\nabla \operatorname{axl}(\operatorname{skew} \nabla u)] \|^2 \right] dv, \qquad (6.7)$$

for all $u \in \widetilde{\mathcal{X}}_0$. Moreover, as a consequence of the properties i) of the constitutive coefficients we have that there exist the positive constant c

$$((u, u)) \geq c \int_{\Omega} \left(\| \operatorname{sym} \nabla u \|^2 + \| \operatorname{skew}[\nabla \operatorname{axl}(\operatorname{skew} \nabla u)] \|^2 \right) dv. \qquad (6.8)$$

From linearized elasticity we have Korn's inequality [30], that is

$$\|\nabla u\|_{L^2(\Omega)} \leq C \|\operatorname{sym} \nabla u\|_{L^2(\Omega)}, \tag{6.9}$$

for all functions $u \in H_0^1(\Omega; \Gamma)$ with some constants $C > 0$, for bounding the deformation of an elastic medium in terms of the symmetric strains. Hence, using the Korn's inequality (6.9), it results that there is a positive constant C such that

$$(u, u) \geq c \int_\Omega \left(\|\nabla u\|^2 + \|\operatorname{skew}[\nabla \operatorname{axl}(\operatorname{skew} \nabla u)]\|^2 \right) dv = c \|u\|_{\widetilde{\mathcal{X}}_0}^2. \tag{6.10}$$

Hence our bilinear form $((\cdot, \cdot))$ is coercive. The Cauchy-Schwarz inequality and the Poincaré-inequality imply that the linear operator $l(\cdot)$ is bounded. By the Lax-Milgram theorem it follows that (6.5) has one and only one solution and the proof is complete. □

Remark 6.2. *The Lax-Milgram theorem used in the proof of the previous theorem also offers a continuous dependence result on the load f. Moreover, the weak solution u minimizes on $\mathcal{X}_0$ the energy functional*

$$I(u) = \int_\Omega \left[2\mu \|\operatorname{sym} \nabla u\|^2 + \lambda [\operatorname{tr}(\nabla u)]^2 + 2\mu L_c^2 \alpha_3 \|\operatorname{skew}(\operatorname{Curl}(\operatorname{sym} \nabla u))\|^2 - \langle f, u \rangle \right] dv.$$

7 The constrained Cosserat formulation of the Hadjesfandiari-Dargush model: well posedness of a degenerate Cosserat model

Similarly to the classical indeterminate couple stress model, also the Hadjesfandiari-Dargush formulation can be obtained as a constrained Cosserat model. We only need to adapt the curvature energy. We consider the replacement $\operatorname{skew} \nabla u \mapsto \overline{A} \in \mathfrak{so}(3)$, and we obtain the energy

$$\mathcal{W} = 2\mu \|\operatorname{sym}\nabla u\|^2 + \lambda [\operatorname{tr}(\nabla u)]^2 + \mu_c \|\operatorname{skew} \nabla u - \overline{A}\|^2 + 2\mu L_c^2 \|\operatorname{skew}\nabla \operatorname{axl}(\overline{A})\|^2 \tag{7.1}$$
$$= 2\mu \|\operatorname{sym}\nabla u\|^2 + \lambda [\operatorname{tr}(\nabla u)]^2 + \mu_c \|\operatorname{skew} \nabla u - \overline{A}\|^2 + 2\mu L_c^2 \|\operatorname{curl} \operatorname{axl}(\overline{A})\|^2$$

for the Cosserat model, where $\mu_c \geq 0$ is the (difficult to interpret [38]) Cosserat couple modulus [33].

Using the usual procedure, it follows that there exists a unique solution $(u, \overline{A})$ of the corresponding minimization problem, i.e. to find the minimum of the energy

$$I(u) = \int_\Omega \left[2\mu \|\operatorname{sym}\nabla u\|^2 + \lambda [\operatorname{tr}(\nabla u)]^2 + \mu_c \|\operatorname{skew} \nabla u - \overline{A}\|^2 + 2\mu L_c^2 \|\operatorname{curl} \operatorname{axl}(\overline{A})\|^2 \right. \tag{7.2}$$

Left box:

skew$\nabla u \mapsto \overline{A} \in \mathfrak{so}(3)$, degenerate Cosserat model

$\sigma \sim 2\,\mu\,\mathrm{sym}\,\nabla u + 2\,\mu_c\,\mathrm{skew}(\nabla u - \overline{A}) \notin \mathrm{Sym}(3)$

$m = 2\,\mu\,L_c^2\,\mathrm{skew}\nabla\,\mathrm{axl}(\overline{A}) \in \mathfrak{so}(3)$

$\mathcal{E} \sim \mu\,\|\mathrm{sym}\nabla u\|^2 + \mu_c\,\|\,\mathrm{skew}\,\nabla u - \overline{A}\|^2 + \mu\,L_c^2\,\|\mathrm{skew}\,\mathrm{axl}(\overline{A})\|^2$

invariant under: $u \mapsto u + \overline{W}.x + \overline{b},\ \overline{b} \in \mathbb{R}^3$

$\qquad\qquad A \mapsto \overline{A} + W,\ \ W \in \mathfrak{so}(3)$

well-posed: $u \in H^1(\Omega),\ \mathrm{axl}(\overline{A}) \in H(\mathrm{curl};\Omega),$

degenerate curvature energy,

3 + 2 = 5 geometric bc: $u\big|_\Gamma,\ \ \langle \mathrm{axl}(\overline{A}), \tau_\alpha\rangle\big|_\Gamma$

3 + 2 = 5 traction bc: $\sigma.n\big|_{\partial\Omega\setminus\Gamma},\ \ \ \langle m.n, \tau_\alpha\rangle\big|_{\partial\Omega\setminus\Gamma}$

Middle arrow:

$\nabla u = \overline{A}$
constrain
$\mu_c \to \infty$

Right box:

Hadjesfandiari-Dargush model

$\sigma \sim 2\,\mu\,\mathrm{sym}\,\nabla u \in \mathrm{Sym}(3)$

$\widetilde{m} = 2\,\mu\,L_c^2\,\mathrm{skew}\nabla\,\mathrm{axl}(\mathrm{skew}\nabla u) \in \mathfrak{so}(3)$

$\widetilde{\tau} = 2\,\mu\,L_c^2\,\mathrm{anti}[\mathrm{Div}\,(\mathrm{skew}\nabla\,\mathrm{axl}(\mathrm{skew}\nabla u))] \in \mathfrak{so}(3)$

$\mathcal{E} \sim \mu\,\|\mathrm{sym}\nabla u\|^2 + \mu\,L_c^2\,\|\mathrm{skew}\nabla\,\mathrm{axl}(\mathrm{skew}\nabla u)\|^2$

invariant under: $u \mapsto u + \overline{W}.x + \overline{b}, \overline{b} \in \mathbb{R}^3$

$\qquad\qquad \nabla u \mapsto \nabla u + \overline{W}, \overline{W} \in \mathfrak{so}(3)$

well-posed: $u \in H^1(\Omega),\ \mathrm{curl}\,u \in H(\mathrm{curl};\Omega)$

5 geometric bc: $u\big|_\Gamma,\ \langle\mathrm{curl}\,u, \tau_\alpha\rangle\big|_\Gamma$
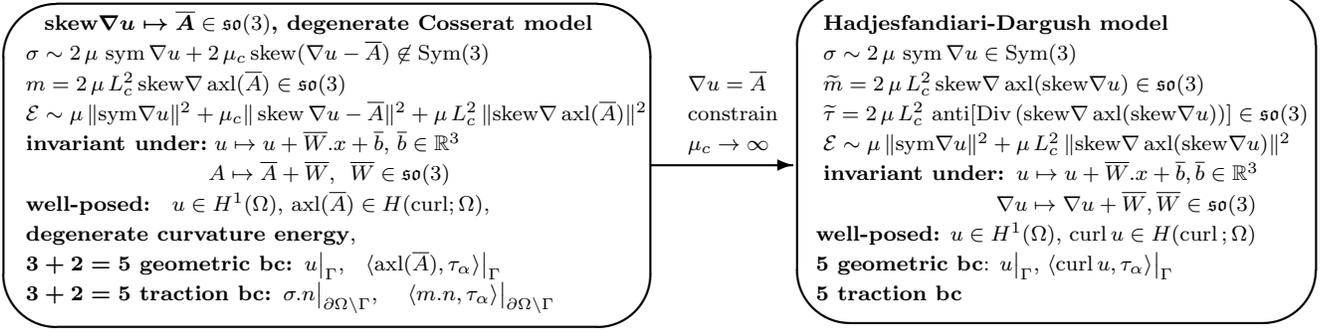
5 traction bc

Figure 1: A possibility of lifting the 4th.-order indeterminate couple stress model to a 2nd.-order micromorphic or Cosserat-type formulation formulation. Here τ_α, $\alpha = 1, 2$ denote two independent tangential vectors on the boundary.

$$-\,\langle f, u\rangle - \langle \mathrm{axl}(\mathfrak{M}), \mathrm{axl}(\overline{A})\rangle \Big]\, dv,$$

where $f : \Omega \to \mathbb{R}^3$ and $\mathfrak{M} : \Omega \to \mathbb{R}^{3\times3}$ are prescribed, such that $u \in H_0^1(\Omega)$ and $\mathrm{axl}(\overline{A}) \in H(\mathrm{Curl};\Omega)$, $\mathrm{axl}(\overline{A}) \times n\big|_\Gamma = 0$.

8   Conclusion

First, Hadjesfandiari and Dargush reject the notion of additional independent degrees of freedom, which is purely arbitrary [13, p. 2496]. The "consistent" theory proposed by Hadjesfandiari and Dargush "resolving all difficulties...", taking only the skew-symmetric part of the curvature tensor $\nabla\mathrm{curl}\,u$, is simply a special case corresponding to some vanishing moduli in the theory. As we have seen, this choice is a restriction, but not a necessity.

In summary, Hadjesfandiari and Dargush have raised [13, p. 17] three major concerns in the indeterminate couple stress model (see also [15]):

1) *The body-couple is present in the constitutive relations for the [total] force-stress tensor in the [indeterminate couple stress] theory.*

2) *The spherical part of the couple-stress tensor is indeterminate, because the curvature tensor $[\widetilde{k} = \nabla\mathrm{curl}\,u]$ is deviatoric.*

3) *The boundary conditions are inconsistent, because the normal component of moment traction $[\langle n, \widetilde{m}.n\rangle]$ appears in the formulation [of the force stress tensor].*

Regarding the above three "serious inconsistencies" presented in [13, p. 17] we may answer:

1) This is of course not inconsistent. It is well-known that the Cauchy-like total force stress tensor $\sigma - \widetilde{\tau}$ is not the constitutive stress. The constitutive stresses and couples are those arising from the virtual work principle (3.5) by fixing $\mathrm{sym}\,\nabla u$ and $\nabla\mathrm{curl}\,u$ as independent constitutive

quantities[5]. More precisely, the constitutively dependent quantities are the energetic conjugates of $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{sym}\,\nabla u$ and $\frac{\mathrm{d}}{\mathrm{d}t}\nabla\mathrm{curl}\,u$, respectively. We have to note that the constitutive dependent quantities are not the energetic conjugates of $\frac{\mathrm{d}}{\mathrm{d}t}u$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{curl}\,u$, respectively.

The relation between the Cauchy-like stress and constitutive stress indeed involve the volume simple double and triple forces (see [6] in French, first part on the classical theory). Note that in the Hadjesfandiari and Dargush-formulation (4.9), the boundary conditions would acquire the same form and meaning as in the classical format: the total force stress tensor $\sigma - \widetilde{\tau}$ would be the Cauchy stress tensor and the curvature of the surface normal would not intervene in the traction boundary conditions. However, the form of the boundary conditions used by them (4.9) are not any more in the completely orthogonally decomposed form (5.5) of the boundary conditions;

2) The indeterminacy of the spherical part of the couple stress is not inconsistent. Like the pressure in an incompressible body, it is indeterminate in the local constitutive law but can be found from the boundary conditions after solving the equilibrium equations. It is a reaction stress, as it is well-known in the theory of continua with internal constraints. Should we say that the theory of incompressible bodies is "inconsistent"? Surely not, it is mathematically clear even though it may induce computational difficulties.

3) As we have shown in this paper, see also [24], the boundary conditions used by Hadjesfandiari and Dargush [13] are not the unique possible format of boundary conditions, therefore no inconsistency occurs, see again (5.5) vs. (4.9). Moreover, our alternative format is consistent with the format one would obtain from more general gradient elasticity.

Finally, we like to mention that the extra constitutive parameter η' of Grioli's model does not intervene in the field partial differential equations. It will arise through the boundary conditions. We do not see which mechanical principle is violated by that. In summary, there is another fully consistent version of the indeterminate-couple stress model with only 3 constitutive parameters. This is the modified couple stress model, see [37, 7] and a novel variant of it is recently discussed in [7].

Acknowledgement

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[5]However, this does not imply that we may independently prescribe u and $\mathrm{curl}\,u$ on the boundary.

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A Conformal invariance of the curvature energy and group theoretic arguments in favor of the modified couple stress theory

This section is taken from [7] and included here for this contribution to be rather self-contained. An infinitesimal conformal mapping [34, 37] preserves (to first order) angles and shapes of infinitesimal figures. The included inhomogeneity is therefore only a global feature of the mapping. There is locally no shear-type deformation. Therefore it seems natural to require that the second gradient model should not ascribe energy to such deformation modes.

A map $\phi_c : \mathbb{R}^3 \to \mathbb{R}^3$ is infinitesimal conformal if and only if its Jacobian satisfies pointwise $\nabla\phi_c(x) \in \mathbb{R} \cdot \mathbb{1} + \mathfrak{so}(3)$, where $\mathbb{R} \cdot \mathbb{1} + \mathfrak{so}(3)$ is the conformal Lie-algebra. This implies [34, 37, 35] the representation

$$\phi_c(x) = \frac{1}{2} \left(2\langle \mathrm{axl}\,\overline{W}, x \rangle\, x - \mathrm{axl}\,\overline{W}\|x\|^2 \right) + [\widehat{p}\,\mathbb{1} + \widehat{A}].x + \widehat{b}\,, \tag{A.1}$$

where $\overline{W}, \widehat{A} \in \mathfrak{so}(3)$, $\widehat{b} \in \mathbb{R}^3$, $\widehat{p} \in \mathbb{R}$ are arbitrary given constants. For the infinitesimal conformal mapping ϕ_c we note

$$\begin{aligned}
&\nabla\phi_c(x) = [\langle \mathrm{axl}\,\overline{W}, x\rangle + \widehat{p}]\,\mathbb{1} + \mathrm{anti}(\overline{W}.x) + \widehat{A}, &\quad &\mathrm{div}\,\phi_c(x) = \mathrm{tr}[\nabla\phi_c(x)] = 3\,[\langle \mathrm{axl}\,\overline{W}, x\rangle + \widehat{p}], \\
&\mathrm{skew}\nabla\phi_c(x) = \mathrm{anti}(\overline{W}.x) + \widehat{A}, &\quad &\mathrm{sym}\,\nabla\phi_c(x) = [\langle \mathrm{axl}\,\overline{W}, x\rangle + \widehat{p}]\,\mathbb{1}, \\
&\mathrm{dev}\,\mathrm{sym}\,\nabla\phi_c(x) = 0, &\quad &\nabla\mathrm{curl}\,\phi_c(x) = 2\,\overline{W} \in \mathfrak{so}(3), \\
&\mathrm{sym}\,\nabla\mathrm{curl}\,\phi_c(x) = 0, &\quad &\mathrm{skew}\,\nabla\mathrm{curl}\,\phi_c(x) = 2\,\overline{W}.
\end{aligned} \tag{A.2}$$

These relations are easily established. By **conformal invariance** of the curvature energy term we mean that the curvature energy vanishes on infinitesimal conformal mappings. This is equivalent to

$$W_{\mathrm{curv}}(D^2\phi_c) = 0 \qquad \text{for all conformal maps} \quad \phi_c, \tag{A.3}$$

or in terms of the second order couple stress tensor $\widetilde{m} := D_{\nabla\mathrm{curl}\,u}W_{\mathrm{curv}}(\nabla\mathrm{curl}\,u)$,

$$\widetilde{m}(D^2\phi_c) = 0 \qquad \text{for all conformal maps} \quad \phi_c. \tag{A.4}$$

The classical linear elastic energy still ascribes energy to such a deformation mode, but only related to the bulk modulus, i.e.,

$$W_{\mathrm{lin}}(\nabla\phi_c) = \underbrace{\mu\,\|\,\mathrm{dev}\,\mathrm{sym}\,\nabla\phi_c\|^2}_{=0} + \frac{2\,\mu + 3\,\lambda}{2}\,[\mathrm{tr}(\nabla\phi_c)]^2 = \frac{2\,\mu + 3\,\lambda}{2}\,[\mathrm{tr}(\nabla\phi_c)]^2. \tag{A.5}$$

In case of a classical infinitesimal perfect plasticity formulation with von Mises deviatoric flow rule, conformal mappings are precisely those inhomogeneous mappings, that do not lead to plastic flow [31], since the deviatoric stresses remain zero: $\mathrm{dev}\,\mathrm{sym}\,\nabla\phi_c = 0$.

In that perspective

> *conformal mappings are ideally elastic transformations and should not lead to moment stresses.*

The underlying additional invariance property of the modified couple stress theory is precisely conformal invariance. In the modified couple stress model, these deformations are free of size-effects, while e.g. the Hadjesfandiari and Dargush choice would describe size-effects. In other words, the generated couple stress tensor $\widetilde{m}$ in the modified couple stress model is zero for this inhomogeneous deformation mode, while in the Hadjesfandiari and Dargush choice $\widetilde{m}$ is constant and skew-symmetric[6].

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[6]This observation is a further development in understanding why the Hadjesfandiari and Dargush [12, 13, 17] choice is rather meaningless, while mathematically not forbidden.