Rank-one convexity and polycovexity of Hencky-type energies

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We investigate a family of isotropic volumetric-isochorically decoupled strain energies based on the Hencky-logarithmic (true, natural) strain tensor $\log U$. The main result of this note is that for $n = 2$ the considered energies are rank-one convex for suitable values of two material parameters. We also conjecture that there are values of the material parameters such that the corresponding energies are polycovex.

1 Introduction

In a recent contribution [7], we have introduced a family of nonlinear elastic energies based on certain invariants of the Hencky tensor $\log U$, namely $\|\text{dev}_n \log U\|^2$ and $(\text{tr}(\log U))^2$, where $F = \nabla \varphi$ is the deformation gradient, $U = \sqrt{F^T F}$ is the right stretch tensor, $\log U$ is the referential (Lagrangian) logarithmic strain tensor, and $\text{dev}_n X = X - \frac{1}{n} \text{tr}(X) \cdot \mathbb{I}$ is the deviatoric part (the projection onto the traceless tensors) of the second order tensor $X \in \mathbb{R}^{n \times n}$ and $\| \cdot \|$ is the Frobenius tensor norm. The considered family of exponentiated Hencky-logarithmic strain type energies is given by

$$W_{eH}(F) := \begin{cases} \frac{\mu}{k} e^k \|\text{dev}_n \log U\|^2 + \frac{\kappa}{2} e^k (\text{tr}(\log U))^2 & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0, \end{cases}$$

where $\mu > 0$ is the shear (distortional) modulus, $\kappa = \frac{2\mu + 3\lambda}{3} > 0$ is the bulk modulus with the first Lamé constant $\lambda$, and $k, \hat{k}$ are dimensionless parameters. The immediate importance of the family (1) of free-energy functions is seen by looking at small (but not infinitesimally small) strains. For small elastic strains, $W_{eH}$ approximates the classical quadratic Hencky strain energy $W_H(F) := \mu \|\text{dev}_n \log U\|^2 + \frac{\kappa}{2} [\text{tr}(\log U)]^2$, which is not rank-one convex. The Hencky energy $W_H$ has been introduced by Heinrich Hencky starting from 1928 [4] and has since then acquired a unique status in finite elastostatics and especially in finite strain elasto-plasticity.

The next step is to prove the rank-one convexity of the isochoric exponentiated Hencky energy $W_{eH}(F)$ in plane elastostatics, i.e. $n = 2$, we prove the rank-one convexity of the proposed family $W_{eH}$ for $k \geq \frac{1}{2}$ and $\hat{k} \geq \frac{1}{4}$. Moreover, in [7] we show that the corresponding Cauchy (true) stress - true strain relation is invertible for $n = 2, 3$ and we discuss the monotonicity of the Cauchy stress tensor as a function of the true strain tensor. We also prove that the rank-one convexity of the energies belonging to this family is not preserved in dimension $n = 3$ and that the energy $W(F) := e^k \|\log U\|^2$, $F \in \text{GL}^+(n)$, $n \in \mathbb{N}$, $n \geq 2$, is not rank-one convex.

2 Rank-one convexity in plane elastostatics

In order to study the rank-one convexity of the family of energies $W_{eH}$, the first step is to check if the volumetric response $\det F \mapsto e^{\hat{k} [\log \det F]^m}$ is convex. In arbitrary dimensions, the following result holds:

**Proposition 2.1** The function $\det F \mapsto e^{\hat{k} [\log \det F]^m}$, $F \in \text{GL}^+(n)$, is convex in $\det F$ for $\hat{k} \geq \frac{1}{m(\ln m - 1)}$.

The next step is to prove the rank-one convexity of the isochoric exponentiated Hencky energy $F \mapsto e^k \|\text{dev}_2 \log U\|^2$ in plane elastostatics. We show that although $F \mapsto \|\text{dev}_2 \log U\|^2$ is not rank-one convex, the function $F \mapsto e^k \|\text{dev}_2 \log U\|^2$ is indeed rank-one convex.

In terms of the singular values $\lambda_1, \lambda_2$ of the right stretch tensor $U = \sqrt{F^T F}$, we have the following representation of the isochoric exponentiated Hencky energy:

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Lemma 2.2 Let $F \in \text{GL}^+(2)$ with singular values $\lambda_1, \lambda_2$. Then $e^k \|\text{dev} \log U\|^2 = e^k \|\text{dev} \log U \|^2 + g(\lambda_1, \lambda_2)$, where $g : \mathbb{R}^2 \to \mathbb{R}$, $g(\lambda_1, \lambda_2) := e^{\frac{k}{2} \left(\log \lambda_1 \right)^2}$.

Using this representation and the necessary and sufficient criteria for rank-one convexity given by Knowles and Sternberg [1] we establish the following result:

**Proposition 2.3** If $k \geq \frac{1}{2}$, then the function $F \mapsto e^k \|\text{dev} \log U\|^2$ is rank-one convex in $\text{GL}^+(2)$.

In view of the above auxiliary results we conclude:

**Theorem 2.4** (planar rank-one convexity) The functions $W_{eH} : \mathbb{R}^{n \times n} \to \mathbb{R}_+$ from the family of exponentiated Hencky type energies $W_{eH}(F)$ are rank-one convex for the two-dimensional situation $n = 2$, for $\mu > 0, \kappa > 0, k \geq \frac{1}{4}$ and $\tilde{k} \geq \frac{1}{8}$.

We give the conjecture:

**Conjecture 2.5** (planar polyconvexity) The functions $W_{eH} : \mathbb{R}^{n \times n} \to \mathbb{R}_+$ from the family of exponentiated Hencky type energies $W_{eH}(F)$ are polyconvex for the two-dimensional situation $n = 2$, for $\mu > 0, \kappa > 0, k \geq \frac{1}{4}$ and $\tilde{k} \geq \frac{1}{8}$.

In plane elasticity, the rank-one convex energy $W_{eH}(F)$ is applicable to the bending or shear of long strips and to all cases in which symmetry arguments can be applied to reduce the formulation to a planar deformation.  

### 3 Outlook for three dimensions and conclusions

In [7] it is also shown that for all parameters $k > 0$ the energy function $F \mapsto \frac{1}{k} \|\text{dev} \log U\|^2$, $F \in \text{GL}^+(3)$, is not rank-one convex. Numerical tests suggest that the polyconvexity domain of the distortional energy function $F \mapsto \frac{\mu}{k} e^k \|\text{dev} \log U\|^2$, $F \in \text{GL}^+(3)$, is a large cone $\{U \in \text{PSym}(3) \, | \, \|\text{dev} \log U\|^2 < 2\}$, while $k \geq \frac{3}{16}$ is the necessary condition for separate convexity of $e^k \|\text{dev} \log U\|^2$ in the three-dimensional situation $n = 3$.

By purely differential geometric reasoning, in forthcoming papers [5, 6, 10] it will be shown that

\[
\text{dist}^2_{\text{geod}} \left( (\text{det} F)^{1/n} \cdot \mathbb{I}, \text{SO}(n) \right) = \text{dist}^2_{\text{geod, \mathbb{R}_+}} \left( (\text{det} F)^{1/n} \cdot \mathbb{I}, \mathbb{I} \right) = \| \log \text{det} F \|^2,
\]

\[
\text{dist}^2_{\text{geod}} \left( \frac{F}{(\text{det} F)^{1/n}} , \text{SO}(n) \right) = \text{dist}^2_{\text{geod, \text{SL}(n)}} \left( \frac{F}{(\text{det} F)^{1/n}} , \text{SO}(n) \right) = \|\text{dev} \log U\|^2,
\]

where $\text{dist}^2_{\text{geod}}$ is the canonical left invariant geodesic distance on the Lie-groups $\text{GL}(n), \text{SL}(n)$ and $\mathbb{R}_+ \cdot \mathbb{I}$, see also [2, 10]. Hence, using this terminology, in the present note we have shown, for $\mu > 0, \kappa > 0$, $k \geq \frac{1}{3}$ and $\tilde{k} \geq \frac{1}{8}$, the rank-one convexity of $W_{eH}(F) := \frac{1}{k} e^{\frac{k}{2} \text{dist}^2_{\text{geod, \text{SL}(2)}} \left( \frac{F}{(\text{det} F)^{1/2}} , \text{SO}(2) \right)} + \frac{\kappa}{2k} e^{\frac{k}{2} \text{dist}^2_{\text{geod, \mathbb{R}_+}} \left( \frac{F}{(\text{det} F)^{1/2}} , \mathbb{I} , \text{SO}(2) \right)}$. In a forthcoming paper [8] we will provide a detailed study of the polyconvexity of the energies $W_{eH}$.

### References

[6] P. Neff, B. Eidel, F. Osterbrink, and R. Martin. Appropriate strain measures: The Hencky shear strain energy $\|\text{dev} \log \sqrt{FF^\top} \|^2$ measures the geodesic distance of the isochoric part of the deformation gradient $F \in \text{SL}(3)$ to $\text{SO}(3)$ in the canonical left invariant Riemannian metric on $\text{SL}(3)$, *in preparation*, 2014.