Loss of ellipticity in additive logarithmic finite strain plasticity

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Abstract

In this paper we consider the additive logarithmic finite strain plasticity formulation from the viewpoint of loss of ellipticity in elastic unloading. We prove that even if an elastic energy $F \mapsto W(F) = \hat{W}(\log U)$, where $U = \sqrt{F^T F}$, is everywhere rank-one convex as a function of $F$, the new function $F \mapsto \tilde{W}(\log U - \log U_p)$ need not remain rank-one convex at some given $U_p$ (viz. $E_{\log p} := \log U$). We show this unacceptable feature with the help of the recently considered family of exponentiated Hencky energies.

Key words: Hencky strain, logarithmic strain, natural strain, true strain, Hencky energy, multiplicative decomposition, elasto-plasticity, ellipticity domain, isotropic formulation, additive plasticity

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1 Introduction

Geometrically nonlinear plasticity is still a field of intensive ongoing research. There exist several fundamentally different approaches which reduce, more or less, to the additive model based on the additive split of the infinitesimal strain tensor $\varepsilon = \text{sym}\nabla u$ into elastic and plastic parts $\varepsilon_e = \varepsilon - \varepsilon_p$.

The involved nonlinearities make it difficult, both from an analysis and algorithmic point of view to obtain definitive results. There exists, however, one well known methodology to reduce the algorithmic complexity dramatically. It is based on the matrix-logarithm and the introduction of a plastic metric $C_p \in \text{PSym}(3)$ together with an additive ansatz for elastic strains $E_e = \log C - \log C_p$. We refer to these models as additive logarithmic. Within these, basically the small strain linearized framework is simply lifted to the geometrically nonlinear setting through the properties of the logarithm. Thus, frame-indifference, thermodynamical admissibility, plastic volume constraint, associative flow rule, principle of maximal dissipation etc. are all easily satisfied.

In this paper, however, we want to exhibit a major drawback of this approach which makes the additive logarithmic ansatz in our view inadmissible in those cases where large plastic deformations need to be considered,

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as are encountered e.g. in elastic spring back processes in the automobile industry. Without loss of generality, we will concentrate our exposition to the completely isotropic setting, in which most prominently the quadratic Hencky-logarithmic strain energy appears.

1.1 The Hencky energy and elasto-plasticity

As hinted at above, the logarithmic strain space description is arguably the simplest algorithmic approach to finite plasticity, suitable for the phenomenological description of isotropic polycrystalline metals if the structure of geometrically linear theories is used with respect to the Lagrangian logarithmic strain \( \log C \). As deduced by Itskov [27], the logarithmic strain yields the most appropriate results in case of rigid-plastic material subjected to simple shear.

In isotropic finite strain computational hyperelasto-plasticity [14, 19] the mostly used elastic energy is the quadratic Hencky logarithmic energy \( W \) (see also [22, 62, 53, 55, 18, 42]). The Hencky energy \( W \) is the energy considered by J.C. Simo (see Eq. (3.4), page 147, from [69] and also [1]) because

\[
W(F_e) = \mu \left| \varepsilon \right|^2 + \frac{\kappa}{2} \left( \text{tr} \left( \log \left( F_e^T F_e \right) \right) \right)^2
\]

\[
= \mu \left| \varepsilon \right|^2 + \frac{\kappa}{2} \left( \text{tr} \left( \log \left( F^T F \right) \right) \right)^2,
\]

where \( \mu > 0 \) is the infinitesimal shear modulus, \( \kappa > 0 \) is the infinitesimal bulk modulus, \( C_e := F_e^T F_e \) is the elastic strain tensor, \( U_e \) the right stretch tensor, i.e. the unique element of \( \text{PSym}(n) \) for which \( U_e^2 = C_e \) and

\[
F = F_e \cdot F_p
\]

is the multiplicative decomposition of the deformation gradient. Here we have used the Frobenius tensor norm \( \| X \|^2 = \langle X, X \rangle_{\mathbb{R}^{n \times n}} \), where \( \langle X, Y \rangle_{\mathbb{R}^{n \times n}} \) is the standard Euclidean scalar product on \( \mathbb{R}^{n \times n} \). The identity tensor on \( \mathbb{R}^{n \times n} \) will be denoted by \( \mathbb{1} \), so that \( \text{tr}(X) = \langle X, \mathbb{1} \rangle \). Here we adopt the usual abbreviations of Lie-group theory and we let \( \text{Sym}(n) \) and \( \text{PSym}(n) \) denote the symmetric and positive definite symmetric tensors respectively. We denote by \( C = F^T F \) the right Cauchy-Green tensor and by \( U \) the right stretch tensor.

The Hencky energy \( W \) has the correct behaviour for extreme strains in the sense that \( W(F_e) \to \infty \) as \( \det F_e \to 0 \) and, likewise \( W(F_e) \to \infty \) as \( \det F_e \to \infty \) [70] page 392], but it cannot be a polyconvex function of the deformation gradient. However, the model provides an excellent approximation for moderately large elastic strains, which is superior to the usual Saint-Venant-Kirchhoff model of finite elasticity. Several models of such a type have been considered in [53, 28]. The decisive advantage of using the energy \( W \) compared to other elastic energies stems from the fact that computational implementations of elasto-plasticity [18] based on the additive decomposition \( \varepsilon = \varepsilon_e + \varepsilon_p \) in infinitesimal model [14, 19, 24], can be used with nearly no changes also in isotropic finite strain problems [70] page 392]. The computation of the elastic equilibrium at given plastic distortion \( F_p \) suffers, however, under the well-known non-ellipticity of \( W_{hi} \) [14, 24, 28]. We know that \( W_{hi} \) is LH-elliptic in a large neighbourhood of the identity if \( \lambda, \mu > 0 \), \( \lambda_1 \in [0.21162..., 1.39561...] \) (see [14, 6]), therefore \( F \to W_{hi}(F) \) is LH-elliptic for small elastic strains.

Moreover, the elastic Hencky energy has been shown [35] to have a fundamental differential geometric meaning, not shared by any other elastic energy, i.e.

\[
\text{dist}^2_{\text{geod}} \left( \frac{F}{\det F)^{1/n}}, \text{SO}(n) \right) = \text{dist}^2_{\text{geod,SL}(n)} \left( \frac{F}{\det F)^{1/n}}, \text{SO}(n) \right) = \| \varepsilon_n \| U \|^2,
\]

\[
\text{dist}^2_{\text{geod}} \left( \frac{F}{\det F)^{1/n}}, \mathbb{1}, \text{SO}(n) \right) = \text{dist}^2_{\text{geod,SL}(n)} \left( \frac{F}{\det F)^{1/n}}, \mathbb{1}, \mathbb{1} \right) = | \log \det F |^2,
\]

where \( \text{dist}^2_{\text{geod,SL}(n)} \mathbb{1}, \mathbb{1} \) and \( \text{dist}^2_{\text{geod,SL}(n)} \mathbb{1}, \mathbb{1} \) are the canonical left invariant geodesic distances on the Lie-group \( \text{SL}(n) \) and on the group \( \mathbb{R}_+ \mathbb{1}, \mathbb{1} \) respectively (see [15, 17, 52]). For this investigation new mathematical tools had to be discovered [22, 29] also having consequences for the classical polar decomposition.
1.2 Additive metric plasticity

The Green-Naghdi additive plasticity models \([20, 21]\) are well known (see e.g. \([11, 12, 71]\)). They are based on the additive split of the total Green-Saint-Venant strain \(E = \frac{1}{2}(C - \mathbb{I})\) into

\[
E = E_c + E_p,
\]

where \(E_c = \frac{1}{2}(C_c - \mathbb{I})\) is the elastic Green-Saint-Venant strain and \(E_p = \frac{1}{2}(C_p - \mathbb{I})\) is the plastic strain. While \(E_c = E - E_p\) was identified with elastic strain in the original work of Green and Naghdi \([20]\), in later works \([21]\) this identification was dropped (see also \([34]\)). This decomposition is justified as an approximation that is valid \([34]\) when (i) small plastic deformations are accompanied by moderate elastic strains, (ii) small elastic strains are accompanied by moderate plastic deformations, or (iii) small strains are accompanied by moderate rotations. A formulation based on elastic strain-measures like

\[
W(E, E_p) := \frac{\mu}{4} ||E - E_p||^2 + \frac{\lambda}{8} \text{tr}(E - E_p)^2
\]

is not even rank-one convex for zero plastic strain (this is the well-known deficiency of the Saint-Venant-Kirchhoff model \([44, 59]\)).

Another finite plasticity model using logarithmic strains is taking the additive elastic Hencky energy \([38, 56, 33, 32, 65, 64]\) in the format

\[
\tilde{W}_u(\log U - \log U_p) = \mu \| \text{dev} \log U - \log U_p \|^2 + \frac{K}{2} \| \text{tr}(\log U) \|^2,
\]

as a starting point, in which plastic incompressibility \(\text{tr}(\log U_p) = 0 \) (\(\det U_p = 1\)) is already included. This approach is based on the ad hoc introduction of the symmetric elastic strain tensor

\[
E_c := E^\log - E_p^\log = \log U - \log U_p = \frac{1}{2} \log C - \frac{1}{2} \log C_p,
\]

in the spirit of \([20]\). An additive decomposition of logarithmic strain has been used in \([83]\) to construct a viscoplasticity theory. Note that the additive decomposition of the total strain is also considered in thermomechanics, see e.g. \([10, 31]\). Eisenberg et al. \([10]\) employ a additive decomposition of strain for thermo-plastic materials and provide experimental correlation to theory. For numerical results see also \([72, 74]\). We remark that this additive model has not much in common with models based on the multiplicative decomposition. Indeed, the independent plastic variable is, in fact, \(E_p^\log = \frac{1}{2} \log C_p = \log U_p\), which is determined to be trace free. Moreover, the definition of \(E_p^\log\) avoids any ambiguity of plastic rotation. Here, \(C_p = U_p^2 \in \text{PSym}(3)\) takes only formally the role of a local plastic metric. The flow rule will be an evolution equation in terms of the independent variable \(E_p^\log = \log U_p \in \mathfrak{sl}(3) \cap \text{Sym}(3)\)

\[
\frac{d}{dt}[\log U_p] \in \partial \chi(\text{dev} \Sigma) \quad \Leftrightarrow \quad \frac{d}{dt}[\log U_p] = \lambda_p^+ \frac{\text{dev} \Sigma}{\| \text{dev} \Sigma \|},
\]

where \(\Sigma = D \tilde{W}_u(\log U - \log U_p)\), \(\partial \chi(\text{dev} \Sigma)\) is the subdifferential of the indicator function \(\chi(\text{dev} \Sigma)\) of the convex elastic domain \(E_c(\Sigma, \frac{\sigma}{2} \Sigma^2) := \{\Sigma \in \text{Sym}(3) \big| \| \text{dev} \Sigma \|^2 \leq \frac{1}{2} \sigma^2\}\) and the plastic multiplier \(\lambda_p^+\) satisfies the Karush–Kuhn–Tucker (KKT)-optimality constraints

\[
\lambda_p^+ \geq 0, \quad \chi(\text{dev} \Sigma) \leq 0, \quad \lambda_p^+ \chi(\text{dev} \Sigma) = 0.
\]

Note that since \(U\) and \(U_p\) do not commute, \(\log U - \log U_p \neq \log(UU_p^{-1})\) in general. Having \(\text{tr}(\log U_p) = 0\) consistent with the flow rule, it follows \(\det U_p = 1\). Moreover, \(\log U_p \in \text{Sym}(3)\) implies \(U_p \in \text{PSym}(3)\). The advantage of this model is that its structure w.r.t. to plasticity is identical to the infinitesimal model, while all proper invariances (objectivity and isotropy) of the geometrically nonlinear theory are retained. In contrast to multiplicative approaches the thermodynamic driving force is not the Eshelby tensor \(\Sigma_c\) and this model gives decreasing shear stress in plastic simple shear which is physically unacceptable \([27]\).

Now, it could be argued that the reason for this deficiency is the dependence on the elastic strain \(E_c^\log = \log U - \log U_p\), together with using the (already) non-elliptic elastic Hencky energy giving rise to a non-elliptic
formulation. Another drawback of this model is that the constitutive equations depend on the choice of the reference configuration \[50\].

However, the additive logarithmic metric ansatz has another serious shortcoming which is the focus of this contribution. The main result of this paper is: even if the elastic energy is everywhere rank-one convex as a function of \( F \), i.e. \( F \mapsto \hat{W}(\log U) \) is rank-one convex, the new function \( F \mapsto \hat{W}(\log U - \log U_p) \) need not remain rank-one convex at some given \( U_p \) (viz. \( E_{\text{pol}}^{\text{iso}} \)). This type of loss of ellipticity is relevant in elastic unloading problems at given plastic deformation. It naturally appears in computation of the elastic spring back. In this paper we show this unacceptable feature with the help of the recently considered family \( W_{\text{eh}} \) of exponentiated Hencky energies.

### 1.3 The exponentiated Hencky energy

In a previous work \[50\] we have modified the quadratic Hencky energy and we considered

\[
W_{\text{eh}}(F) = W_{\text{iso}}^\text{iso} \left( \frac{F}{\det F} \right) + W_{\text{eh}}^\text{vol}(\det F^\frac{1}{2} \cdot \mathbb{I}) = \begin{cases} \frac{\mu}{k} e^{k \| \text{dev}_n \log U \|^2} + \frac{\kappa}{2k} e^{k (\log \det U)^2} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0. \end{cases} \tag{1.7}
\]

We have called this the **exponentiated Hencky energy**. For the two-dimensional situation \( n = 2 \) and for \( \mu > 0, \kappa > 0 \), we have established that the functions \( W_{\text{eh}} : \mathbb{R}^{n \times n} \to \mathbb{R}_+ \) from the family of exponentiated Hencky type energies are **rank-one convex** \[50\] for \( k \geq \frac{1}{4} \) and \( \hat{k} \geq \frac{1}{8} \). Moreover, these energies are **polyconvex** \[51\] and the corresponding minimization problem admits at least one solution.

## 2 Additive logarithmic plasticity does not preserve rank-one convexity

In this section we consider the isotropic exponentiated energy \( F \mapsto W_{\text{eh}}^\text{iso}(F) = e^{\| \text{dev}_n \log U \|^2} \) corresponding to the fitting parameter \( k = 1 \) and we prove that the new function \( F \mapsto W_{\text{eh}}^\text{iso}(\log U - \log U_p) \) need not remain rank-one convex at some given \( U_p \).

**Lemma 2.1.** The function \( F \mapsto W(F) = e^{\| \text{dev}_n \log U - \text{dev}_n \log U_p \|^2} \) is not rank-one convex for some given \( U_p \in \text{PSym}(2) \), while \( F \mapsto W(F) = e^{\| \text{dev}_n \log U \|^2} \) is rank-one convex.

**Proof.** The rank-one convexity of the isotropic exponentiated energy \( F \mapsto W_{\text{eh}}^\text{iso}(F) = e^{\| \text{dev}_n \log U \|^2} \) follows as a particular case of the result established in \[50\].

In order to prove that the function \( F \mapsto W(F) = e^{\| \text{dev}_n \log U - \text{dev}_n \log U_p \|^2} \) is not rank-one convex for some given \( U_p \in \text{PSym}(2) \), it suffice to consider the simple shear case. We choose the vectors \( \eta, \xi \in \mathbb{R}^3 \) so that

\[
\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta \otimes \xi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

and we find \( U_p \in \text{PSym}(2) \) such that the function \( h : \mathbb{R} \to \mathbb{R}, h(t) = W(\mathbb{I} + t(\eta \otimes \xi)) \) is not convex as function of \( t \). Thus, \( F = \mathbb{I} + t(\eta \otimes \xi) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \).

For this deformation we have that the polar decomposition \( F = R \cdot U = V \cdot R \) into the right Biot stretch tensor \( U = \sqrt{F^TF} \) of the deformation and the orthogonal polar factor \( R \) is given by

\[
U = \frac{1}{\sqrt{t^2 + 4}} \begin{pmatrix} 2 & t \\ t & t^2 + 2 \end{pmatrix}, \quad R = \frac{1}{\sqrt{t^2 + 4}} \begin{pmatrix} 2 & t \\ -t & 2 \end{pmatrix}. \tag{2.8}
\]

Further, \( U \) can be orthogonally diagonalized to

\[
U = Q \cdot \begin{pmatrix} 1 & \frac{1}{2}(\sqrt{t^2 + 4} + t) \\ 0 & \frac{1}{2}(\sqrt{t^2 + 4} - t) \end{pmatrix} \cdot Q^T = Q \cdot \begin{pmatrix} 1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \cdot Q^T, \tag{2.9}
\]

where

\[
Q = \begin{pmatrix} \sqrt{t^2 + 4} + t & -2 \\ \sqrt{t^2 + 4} - t & 2 \end{pmatrix} \tag{2.10}
\]
and $\lambda_1 = \frac{1}{2}\sqrt{t^2 + 4 + t}$ denotes the first eigenvalue of $U$. Hence, the principal logarithm of $U$ is

$$\log U = Q \cdot \begin{pmatrix} 1 & 0 \\ 0 & \log \lambda_1 \end{pmatrix} \cdot Q^T = \frac{1}{\sqrt{t^2 + 4}} \begin{pmatrix} -t \log \lambda_1 & 2 \log \lambda_1 \\ 2 \log \lambda_1 & t \log \lambda_1 \end{pmatrix}. \quad (2.11)$$

The general form of a matrix $U_p \in \text{PSym}(2)$ such that $\text{dev}_2 \log U_p \in \text{Sym}(2)$ and $\text{tr}(\log U_p) = 0$ is

$$\text{dev}_2 \log U_p = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \text{Sym}(2), \ a, b \in \mathbb{R} \Rightarrow \quad F_p = \log U_p = \text{dev}_2 \log U_p + \frac{1}{3} \text{tr}(\log U_p) = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

$$\Rightarrow \quad U_p = \exp \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \text{SL}(2). \quad (2.12)$$

We obtain that

$$\| \text{dev}_2 \log U - \text{dev}_2 \log U_p \|^2 = \| \text{dev}_2 \log U \|^2 - 2 \langle \text{dev}_2 \log U, \text{dev}_2 \log U_p \rangle + \| \text{dev}_2 \log U_p \|^2 \quad (2.13)$$

$$= 2 \log^2 \lambda_1 - \frac{2 \log \lambda_1}{t^2 + 4} (-2a + 4b) + 2a^2 + 2b^2.$$

With this representation we are able to disprove rank-one convexity of the function $F \mapsto e^\|\text{dev}_2 \log U - \text{dev}_2 \log U_p \|^2$. We have to check the convexity of the function

$$h(t) = W(I + t(\eta \otimes \xi)) = e^\|\text{dev}_2 \log C_{et} \|^2 \frac{3}{2} t \log \frac{3}{2} \left(\frac{1}{t^2 + 4 + t}\right) (-2a t + 4b) + 2a^2 + 2b^2, \quad t > 0.$$  

For $a = -0.56, b = -0.01$, i.e. $U_p \approx \begin{pmatrix} 0.5712 & 0.0105 \\ 0.0105 & 1.7507 \end{pmatrix}$, det $U_p \approx 1$, from Figure 1 we see that the function $h$ is not convex, which implies that for $U_p$ chosen as above the energy $F \mapsto e^\|\text{dev}_2 \log U - \text{dev}_2 \log U_p \|^2$ is not rank-one convex, while $F \mapsto e^\|\text{dev}_2 \log U \|^2$ is rank-one convex [50]. \hfill $\square$

3 \hspace{1mm} Final remarks

We have shown that the multiplicative plasticity models preserve ellipticity in purely elastic processes at frozen plastic variables provided that the initial elastic response is elliptic, see [29]. Preservation of LH-ellipticity is, in our view, a property which should be satisfied by any hyperelastic-plastic model since the elastically unloaded material specimen should respond reasonably under further purely elastic loading. However, the much used additive logarithmic model does not preserve LH-ellipticity in general. In Figure 2 we summarize some properties of such additive logarithmic model and we compare it with the small strain plasticity model. In contrast, the formulation based on $e^\|\text{dev}_2 \log C_{et} \|^2$ remains always rank-one convex. Moreover, the change of a given FEM-implementation of $W_{et}$ into $W_{eh}$ is nearly free of costs [72, 36, 43, 23]. For more constitutive issues regarding the interesting properties of $W_{eh}$ we refer to [50].
Additive logarithmic model

\[
W = \tilde{W} \left( \frac{1}{2} \log C - \frac{1}{2} \log C_p \right) = \tilde{W} \left( E^{\text{log}} - E_p^{\text{log}} \right), \quad E_p^{\text{log}} := \log C_p
\]

\[
\frac{1}{2} \partial \Theta \left[ \log C_p \right] \in \partial \chi \left( \text{dev}^3 \Sigma \right)
\]

\[
\Sigma = D \tilde{W} \left( \frac{1}{2} \log C - \frac{1}{2} \log C_p \right) \in \text{Sym}(3)
\]

thermodynamically correct

\[
\text{tr}(E_p^{\text{log}}) = \text{tr}(\log C_p) = 0, \quad \text{det} C_p(t) = 1, \quad C_p(t) \in \text{PSym}(3)
\]

convex elastic domain \( \mathcal{E}_c \left( \Sigma, \frac{1}{d} \sigma^2 \right) \)

do not preserve ellipticity in the elastic domain

is not a flow rule for \( C_p \), but for \( E_p^{\text{log}} := \log C_p \in \mathfrak{sl}(3) \cap \text{Sym}(3) \)

associated plasticity: \( f = \partial \chi \)

algorithmic format identical to small strain plasticity

Small strain plasticity

\[
W_{\text{lin}} = W_{\text{lin}}(\varepsilon - \varepsilon_p)
\]

\[
\frac{1}{d} \partial \mathfrak{h} \left[ \varepsilon_p \right] \in \partial \chi \left( \text{dev}^3 \Sigma_{\text{lin}} \right) = f(\varepsilon, \varepsilon_p)
\]

\[
\Sigma_{\text{lin}} = -D_{\varepsilon_p}W_{\text{lin}}(\varepsilon - \varepsilon_p) \in \text{Sym}(3)
\]

thermodynamically correct, \( \text{tr}(\varepsilon_p) = 0 \)

convex elastic domain \( \mathcal{E}_c \left( \Sigma_{\text{lin}}, \frac{1}{d} \sigma^2 \right) \)

preserves ellipticity in the elastic domain

associated plasticity: \( f = \partial \chi \)

Figure 2: An additive logarithmic model, additive small strain plasticity. All these models are associative, since all flow rules are in the format \( \frac{d}{dt}[P]P^{-1} = -\partial \chi \) or \( \frac{1}{d} \mathfrak{h} \left[ \varepsilon_p \right] \in \partial \chi \).

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