A uniqueness result for the motion of micropolar solid–fluid mixtures in unbounded domain

Ionel-Dumitrel Ghiba · Cătălin Galeş

Abstract This paper deals with the Eringen’s theory for binary mixtures between elastic micropolar solids and incompressible micropolar fluids (Eringen in J Appl Phys 94:4184–4190, 2003). Using the weighted energy method, an uniqueness result in the case of unbounded domains for small displacement of the solid and for non-slow flow of fluid is presented.

Keywords Solid–fluid micropolar mixtures · Uniqueness theorem · Weighted energy method · Unbounded domains

Mathematics Subject Classification (2000) 35M10 · 74F20 · 35Q35 · 35Q72

1 Introduction

The theory of micropolar continua was introduced by Eringen as a special case of micromorphic continua [3]. In the micropolar continuum theory, the rotational degrees of freedom play a central role. Thus, we have six degrees of freedom, instead of the three

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I.-D. Ghiba Department of Mathematics, West University of Timişoara, Splaiul Independenţei no. 1, 300223 Timişoara, Romania
e-mail: ghiba Dumitrel@yahoo.com

C. Galeş Faculty of Mathematics, A.I. Cuza University of Iaşi, Blvd. Carol I, no. 11, 700506 Iasi, Romania
e-mail: cgales@uaic.ro
ones considered in classical elasticity and fluid mechanics. Moreover, in micropolar theories, to characterize the force applied on the surface element there are two tensors: a stress tensor and a couple stress tensor.

Taking into account the microstructural motions, Twiss and Eringen [1,2] introduced the mixture theory of materials with microstructure. In the last years a special attention has been paid to include some terms in the basic formulation of the theory of mixtures in order to reflect the microstructure of the constituents [4,5]. A description of the progress of continuum theories of mixtures can be found in the book [6].

In the present paper we consider the isothermal theory of the binary homogeneous mixtures consisting of a micropolar elastic solid and a micropolar viscous fluid as introduced by Eringen [4]. This theory can be successfully applied to the consolidation problems in the building industry, earthquake problems, oil exploration problems and to predict the behaviour of cellular materials.

In the framework of the theory developed by Eringen [4], uniqueness and continuous dependence results for bounded domains in the case of small displacement of the solid have been derived by Ghiba [7].

The goal of this paper is to establish a uniqueness result for exterior domains in the case of small displacement of the solid. We prove this result under positiveness assumptions upon both the internal energy density and the dissipation potential, and by using the weighted energy method [8–15].

2 Basic formulation

Let $\Omega$ be a regular region of the Euclidean three-dimensional space. We denote by $B$ the exterior of the fixed bounded region $\Omega$. We consider a point $O \in \Omega$, and let $\Gamma(O; R)$ be the open ball of center $O$ and radius $R$ and boundary $\partial \Gamma_R$. We consider a Cartesian coordinate system with the origin in the point $O$. We refer the motion of a continuum to this fixed system of rectangular Cartesian axes $Ox_k (k = 1, 2, 3)$. The Latin subscripts have the integer values 1, 2 and 3, whereas the Greek indices are understood to range over the integers $(1, 2)$. We assume that $B$ is occupied by a binary homogeneous mixture consisting of an isotropic micropolar elastic solid and an incompressible micropolar viscous fluid. We suppose that the mixture is chemical inert.

We introduce the notations: $r = |x|$, $R_0 = \inf \{ R > 0; \Omega \subset \Gamma(O; R) \}$ and for $R > R_0$ we consider the set $\Omega_R = \Gamma(O; R) \setminus \Omega$. We denote by comma followed by a subscript the partial derivative with respect to the corresponding Cartesian coordinate. The superscripts $\sigma = s, f$ denote respectively the micropolar elastic solid and the incompressible micropolar fluid and summation over repeated subscripts is implied.

Let $\rho^{\sigma}$ denote the density of the constituent $\sigma$ at time $t = 0$ and let $I = [0, T), T > 0$ be a time interval.

We suppose the motion of the solid to be small. The constitutive equations of a binary homogeneous mixture consisting of an isotropic micropolar elastic solid and an incompressible micropolar viscous fluid [4] are
\( t_{ji} = \lambda^s e^s_{kk} \delta_{ji} + (\mu^s + k^s) e^s_{ji} + \mu^s e^s_{ij}, \)

\( m_{ji} = \alpha^s \gamma^s_{kk} \delta_{ji} + \beta^s \gamma^s_{ji} + \gamma^s \gamma_{ij}, \)

\( t_{ji}^f = -\pi^f \delta_{ji} + (\mu^f + k^f) a_{ji}^f + \mu^f a_{ij}^f, \)

\( m_{ji}^f = \alpha^f b^f_{kk} \delta_{ji} + \beta^f b^f_{ji} + \gamma^f b_{ij}, \)

\( \hat{\rho}^i = -\hat{\rho}^f = -\xi (v_i^i - v_i^f), \)

\( \hat{m}_i = -\hat{m}_i^f = -\sigma (v_i^i - v_i^f), \)

where \( u_i^\sigma \) is the displacement of the constituent \( \sigma; \phi_i^s \) is the microrotation vector of the micropolar solid; \( v_i^f \) is the velocity of the constituent \( \sigma; \) \( v_i^\sigma \) is the microrotation rate of the constituent \( \sigma; \) \( \pi^f \) is the dynamic pressure in the fluid species; \( t_{ji}^\sigma \) is the stress tensor of the constituent \( \sigma; \) \( m_{ji}^\sigma \) is the couple stress tensor of the constituent \( \sigma; \) \( \hat{\rho}^\sigma \) is the force exerted on the \( \sigma \) constituent from the other constituent; \( \hat{m}^\sigma \) is the couple exerted on the \( \sigma \) constituent from the other constituent; \( \lambda^s, \mu^s, k^s, \alpha^s, \beta^s, \gamma^s \) are the micropolar elastic constants for micropolar elastic solid; \( \mu^f, k^f, \alpha^f, \beta^f, \gamma^f \) are the micropolar fluid viscosities; \( \xi \) is the momentum generation coefficient due to the velocity difference; \( \sigma \) is the momentum generation coefficient due to the difference in gyration and

\[ \epsilon_{ij}^s = u_{j,i}^s + \varepsilon_{jik} \phi_k^s, \]

\[ \gamma_{ij}^s = \phi_{i,j}^s, \]

\[ a_{ij}^f = v_{j,i}^f + \varepsilon_{jik} v_k^f, \quad b_{ij}^f = v_{i,j}^f. \]

In the case of small deformation of the solid, the equations of the theory of micropolar solid–fluid mixtures [4] are

\[ (\lambda^s + \mu^s) u_{j,ij}^s + (\mu^s + k^s) u_{i,jj}^s + k^s \varepsilon_{ijk} \phi_{k,j}^s - \xi (v_i^i - v_i^f) \]

\[ + F_i^s = \rho^s \frac{\partial}{\partial t^2} u_i^s, \]

\[ (\alpha^s + \beta^s) \phi_{j,ij}^s + \gamma^s \phi_{i,jj}^s + k^s (\varepsilon_{ijk} u_{k,j}^s - 2 \phi_{i,j}^s) - \sigma (v_i^s - v_i^f) \]

\[ + L_i^s = \rho^s J^s \frac{\partial}{\partial t^2} \phi_i^s, \]

\[ -\pi_{i,i}^f + (\mu^f + k^f) v_{i,jj}^f + k^f \varepsilon_{ijk} v_{k,j}^f + \xi (v_i^i - v_i^f) \]

\[ + F_i^f = \rho^f \left( \frac{\partial}{\partial t} v_i^f + v_{i,j}^f v_j^f \right), \]

\[ (\alpha^f + \beta^f) v_{j,ij}^f + \gamma^f v_{i,jj}^f + k^f (\varepsilon_{ijk} v_{k,j}^f - 2 v_i^f) + \sigma (v_i^s - v_i^f) \]

\[ + L_i^f = \rho^f J^f \left( \frac{\partial}{\partial t} v_i^f + v_{i,j}^f v_j^f \right), \]

\[ v_{i,i}^f = 0 \]

in \( B \times I \), where \( F_i^\sigma \) are the body forces, \( L_i^\sigma \) are the body couples and \( J^\sigma \) are the microinertia densities.
We assume that $\rho^\sigma$ and $j^\sigma$ are positive constants and $F^\sigma_i$, $L^\sigma_i$ are continuous functions on $B \times I$.

To these equations we adjoin the following boundary conditions

$$u^s_i(x, t) = u^{s*}_i(x, t), \quad \phi^s_i(x, t) = \phi^{s*}_i(x, t),$$

$$v^f_i(x, t) = v^{f*}_i(x, t), \quad v^f_i(x, t) = v^{f*}_i(x, t) \quad \text{on} \quad \partial \Omega \times I,$$  \hfill (2.5)

and the following initial conditions

$$u^s_i(x, 0) = g^s_i(x), \quad v^f_i(x, 0) = h^f_i(x),$$

$$\phi^s_i(x, 0) = a^s_i(x), \quad \nu^f_i(x, 0) = b^f_i(x) \quad \text{in} \quad B,$$  \hfill (2.6)

where $u^{s*}_i, \phi^{s*}_i, v^{f*}_i, v^{f*}_i, g^s_i, h^f_i, a^s_i$ and $b^f_i$ are prescribed continuous functions. The boundary values $u^{s*}_i, v^{f*}_i$ are such that $\frac{\partial u^{s*}_i}{\partial n} n_i = 0$, $v^{f*}_i n_i = 0$. These relations give sufficient conditions for the boundary to be a material surface.

We say that $(u^s, v^f, \phi^s, v^f, \pi^f)$ is an admissible process on $\overline{B} \times I$ provided: (a) $u^s_i$ and $\phi^s_i$ are of class $C^{2,2}$ on $B \times I$; (b) $v^f_i$ and $v^f_i$ are of class $C^{2,1}$ on $B \times I$; (c) $u^s_i, v^f_i, \phi^s_i, v^f_i$ are of class $C^0$ on $\partial B \times I$; (d) $\pi^f$ is of class $C^{1,0}$ on $B \times I$.

We introduce the following bilinear forms

$$\mathcal{W}_1^s(\xi_{ij}, \eta_{ij}) = \lambda^s \xi_{kk} \eta_{ii} + \mu^s \xi_{ji} \eta_{ij} + (\mu^s + k^s) \xi_{ij} \eta_{ij},$$

$$\mathcal{W}_2^s(\xi_{ij}, \eta_{ij}) = \alpha^s \xi_{kk} \eta_{ii} + \beta^s \xi_{ji} \eta_{ij} + \gamma^s \xi_{ij} \eta_{ij},$$

$$\mathcal{W}_1^f(\xi_{ij}, \eta_{ij}) = \mu^f \xi_{ji} \eta_{ij} + (\mu^f + k^f) \xi_{ij} \eta_{ij}. \quad (2.7)$$

Throughout this paper we suppose that the constitutive coefficients satisfy the inequalities

$$3\lambda^s + 2\mu^s + k^s > 0, \quad 2\mu^\sigma + k^\sigma > 0, \quad k^\sigma > 0, \quad \alpha^s > 0, \quad \xi > 0,$$

$$3\alpha^s + 3\beta^s + 3\gamma^s > 0, \quad \gamma^\sigma + \beta^\sigma > 0, \quad \gamma^\sigma - \beta^\sigma > 0. \quad (2.8)$$

The above inequalities are necessary and sufficient conditions for the internal energy

$$\mathcal{E}(u^s, \phi^s) = \frac{1}{2} \left[ \mathcal{W}_1^s(\varepsilon_{ij}^s, \varepsilon_{ij}^s) + \mathcal{W}_2^s(\gamma_{ij}^s, \gamma_{ij}^s) \right] \quad (2.9)$$

and for the dissipation potential

$$\Phi(v^s, v^s, v^f, v^f) = \mathcal{W}_1^f(a_{ij}^s, a_{ij}^s) + \mathcal{W}_2^f(b_{ij}^s, b_{ij}^s) + \xi (v_i^s - v_i^f)(v_i^s - v_i^f)$$

$$+ \sigma (v_i^s - v_i^f)(v_i^s - v_i^f) \quad (2.10)$$

to be positive defined for every admissible process.
Moreover, we have the estimates
\[
\begin{align*}
\mathcal{W}_1^s(\xi_{ij}, \xi_{ij}) & \leq \sigma^s_M \xi_{ij} \xi_{ij}, \\
\mathcal{W}_2^s(\xi_{ij}, \xi_{ij}) & \leq \delta^s_M \xi_{ij} \xi_{ij}, \\
\sigma^f_M \xi_{ij} \xi_{ij} & \leq \mathcal{W}_1^f(\xi_{ij}, \xi_{ij}), \\
\delta^f_M \xi_{ij} \xi_{ij} & \leq \mathcal{W}_2^f(\xi_{ij}, \xi_{ij}),
\end{align*}
\]
where
\[
\begin{align*}
\sigma^s_M &= \max\{2\mu^s + k^s, k^s, 3\lambda^s + 2\mu^s + k^s\}, \\
\sigma^f_M &= \max\{2\mu^f + k^f, k^f\}, \\
\sigma^f_m &= \min\{2\mu^f + k^f, k^f\}, \\
\delta^s_M &= \max\{3\alpha^s + \beta^s + \gamma^s, \gamma^s + \beta^s, \gamma^s - \beta^s\}, \\
\delta^f_m &= \min\{3\alpha^f + \beta^f + \gamma^f, \gamma^f + \beta^f, \gamma^f - \beta^f\}.
\end{align*}
\]

We denote by $\mathcal{P}$ the initial-boundary values problem defined by the equations (2.4), the initial conditions (2.5) and the boundary conditions (2.6).

### 3 Uniqueness result

In this section we study the uniqueness problem in connection with the classical solutions of the boundary-initial value problem $\mathcal{P}$. By a solution of the boundary-initial value problem $\mathcal{P}$, corresponding to the same external data system \( \{F_i^s, L_i^s, u_i^{s*}, \phi_i^{s*}, v_i^{s*}, v_i^{s*}, g_i^s, h_i^s, a_i^s, b_i^s\} \), we mean an admissible process that satisfies the equations (2.4), the initial conditions (2.6) and the boundary conditions (2.5).

Let us introduce the quantities
\[
\mathcal{K}^\sigma = \frac{1}{2} \left( \rho^\sigma v_i^\sigma v_i^\sigma + \rho^\sigma j^\sigma v_i^\sigma v_i^\sigma \right),
\quad \mathcal{K} = \sum_{\sigma=s,f} \mathcal{K}^\sigma.
\]

**Theorem 1** Let \((u^{(1)}(t), \phi^{s(1)}(t), v^{f(1)}(t), \pi^{f(1)}(t)) and (u^{(2)}(t), \phi^{s(2)}(t), v^{f(2)}(t), \pi^{f(2)}(t))\) be two solutions of the problem $\mathcal{P}$, corresponding to the same external data system \( \{F_i^s, L_i^s, u_i^{s*}, \phi_i^{s*}, v_i^{s*}, v_i^{s*}, g_i^s, h_i^s, a_i^s, b_i^s\} \), and let us assume that

(i) $\rho^\sigma$ and $j^\sigma$ are positive constants;

(ii) The internal energy density and the dissipation potential are positive definite;

(iii) $|v_i^{f(\alpha)}| \leq A_1$, $|v_i^{f(\alpha)}| \leq A_2$ for each \((x, t) \in B \times I\), where $A_1$ and $A_2$ are positive constants;

(iv) $v_i^{s(\alpha)}$, $v_i^{s(\alpha)}$, $u_i^{s(\alpha)}$, $v_i^{s(\alpha)}$, $\phi_i^{s(\alpha)}$ and $v_i^{f(\alpha)}$ may be unbounded but there exist the real numbers $k_q$ and the positive real numbers, $M_q$, $q = 1, \ldots, 6$ and $r$ such that

\[
\begin{align*}
|v_i^{s(\alpha)}| & \leq M_1 r^k_1, \\
|v_i^{s(\alpha)}| & \leq M_2 r^k_2, \\
u_i^{s(\alpha)} & \leq M_3 r^k_3, \\
v_i^{f(\alpha)} & \leq M_4 r^k_4, \\
|\phi_i^{s(\alpha)}| & \leq M_5 r^k_5, \\
|v_i^{f(\alpha)}| & \leq M_6 r^k_6 \text{ for all } r > \bar{r}.
\end{align*}
\]
(v) the pressure difference satisfies the Serrin’s condition

\[ |\pi^{f(1)} - \pi^{f(2)}| \leq A_3 r^{-\varepsilon - \frac{1}{2}}, \]

as \( r \to \infty \), for a positive constant \( A_3 \) and any preassigned \( \varepsilon \in (0, 1) \),

then

\[
\begin{align*}
\mathbf{u}^{s(1)} &= \mathbf{u}^{s(2)}, \quad \phi^{s(1)} = \phi^{s(2)}, \\
\mathbf{v}^{f(1)} &= \mathbf{v}^{f(2)}, \quad \mathbf{v}^{f(1)} = \mathbf{v}^{f(2)}, \quad \pi^{f(1)} = \pi^{f(2)} + \pi^{f},
\end{align*}
\]

where

\[ \pi^{f}_{,i} = 0. \]  

\[ (3.5) \]

**Proof** We denote by \((\mathbf{u}^{s}, \phi^{s}, \mathbf{v}^{f}, \mathbf{v}^{f}, \pi^{f})\) the difference between the two solutions. This difference is solution of the problem defined by the following equations

\[
\begin{align*}
(\lambda^{s} + \mu^{s})u_{,ij}^{s} &+ (\mu^{s} + k^{s})u_{,ij}^{s} + k^{s} \epsilon_{ijk} \phi^{s}_{,j} - \xi(v_{i}^{s} - v_{i}^{f}) = \rho^{s} \frac{\partial^{2}}{\partial t^{2}} u_{i}^{s}, \\
(\alpha^{s} + \beta^{s})\phi_{,i,j}^{s} &+ \gamma^{s} \phi_{,i,j}^{s} + k^{s} (\epsilon_{ijk} u_{k,j}^{s} - 2 \phi_{i}^{s}) - \sigma (v_{i}^{s} - v_{i}^{f}) = \rho^{s} j^{s} \frac{\partial^{2}}{\partial t^{2}} \phi_{i}^{s}, \\
-\pi^{f}_{,i} &+ (\mu^{f} + k^{f})v_{i,jj}^{f} + k^{f} \epsilon_{ijk} v_{k,j}^{f} + \xi (v_{i}^{s} - v_{i}^{f}) \\
&= \rho^{f} \left( \frac{\partial}{\partial t} v_{i,j}^{f} + v_{i,j}^{f(1)} v_{j}^{f} + v_{i,j}^{f(2)} \right), \\
(\alpha^{f} + \beta^{f})v_{,i,j}^{f} &+ \gamma^{f} v_{i,jj}^{f} + k^{f} (\epsilon_{ijk} v_{k,j}^{f} - 2 v_{i,j}^{f}) + \sigma (v_{i}^{s} - v_{i}^{f}) \\
&= \rho^{f} j^{f} \left( \frac{\partial}{\partial t} v_{i,j}^{f} + v_{i,j}^{f(1)} v_{j}^{f} + v_{i,j}^{f(2)} \right), \\
v_{i,i}^{f} &= 0,
\end{align*}
\]

with homogeneous boundary conditions on \( \partial \Omega \) and null initial data.

We introduce the function

\[ g(r) = \exp(-dr), \quad d > 0 \]  

\[ (3.7) \]

and with the help of this function we define the quantity

\[
G = \left[ t_{ji, j}^{s} - \xi(v_{i}^{s} - v_{i}^{f}) \right] g v_{i}^{s} + \left[ m_{ji, j}^{s} + \epsilon_{ijk} t_{jk}^{s} - \sigma (v_{i}^{s} - v_{i}^{f}) \right] g v_{i}^{s}
\]

\[ + \left[ t_{ji, j}^{f} + \xi(v_{i}^{s} - v_{i}^{f}) \right] g v_{i}^{f} + \left[ m_{ji, j}^{f} + \epsilon_{ijk} t_{jk}^{f} + \sigma (v_{i}^{s} - v_{i}^{f}) \right] g v_{i}^{f}, \]  

\[ (3.8) \]

\[ 1 \] This condition was used by [8] at the Serrin’s suggestion.
where \( f_{ji}^g \) and \( m_{ji}^g \) correspond to the difference \((u^s, \phi^s, v^f, v^f, \pi^f)\).

Using the equations of motion (3.6), we obtain

\[
G = g \frac{\partial}{\partial t} \mathcal{K} + \left( \mathcal{K} f g v_j^f(2) \right)_{,j} - g_{,j} \mathcal{K} f v_j^f(2) + \rho f \left[ g \left( v_i^f(1) v_i^f + j f v_i^f(1) v_i^f \right) v_j^f \right]_{,j} \\
- \rho f g_{,j} \left( v_i^f(1) v_i^f + j f v_i^f(1) v_i^f \right) v_j^f - \rho f g \left( v_i^f(1) v_i^f + j f v_i^f(1) v_i^f \right) v_j^f.
\]

(3.9)

On the other hand, using the constitutive equations (2.1), we have

\[
G = Q_{j,j} - g \frac{\partial}{\partial t} \mathcal{E}(u^s, \phi^s) - g \Phi(v^s, v^f, v^f) - \mathcal{W}_1^f(e_{ij}^s, g_i v_j^f) - \mathcal{W}_2^f(\gamma_{ij}^s, v_i^s g_j, j) - \mathcal{W}_1^f(a_{ij}^f, g_i v_j^f) - \mathcal{W}_2^f(b_{ij}^f, v_i^f g_j, j) + \pi^f g_{,j} v_j^f,
\]

(3.10)

where

\[
Q_j = \lambda^s e_{ii} g v_i^s + (\mu^s + k^s) e_{ij}^s g v_i^s + \mu^s e_{ij}^s g v_i^s + \alpha^s \gamma_{ii}^s g v_j^s \\
+ \beta^s \gamma_{ij}^s g v_i^s + \gamma^s \gamma_{ij}^s g v_i^s - \pi^f g v_j^f + (\mu^f + k^f) a_{ij}^f g v_i^f \\
+ \mu^f a_{ij}^f g v_i^f + \alpha^f b_{ii}^f g v_j^f + \beta^f b_{ij}^f g v_j^f + \gamma^f b_{ij}^f g v_j^f.
\]

(3.11)

Let remark that, in view of the incompressibility condition (2.4), we have

\[
(v_{j,i}^f g v_j^f)_{,j} = v_{j,i}^f g_{,j} v_j^f + v_{j,i}^f g v_j^f
\]

(3.12)

and thus, using the relations (3.9) and (3.10), we deduce

\[
g \frac{\partial}{\partial t} \left[ \mathcal{K}(t) + \mathcal{E}(u^s, \phi^s) \right] + g \Phi(v^s, v^f, v^f) \\
= \mathcal{K}_{j,j} - \mathcal{W}_1^f(e_{ij}^s, g_i v_j^f) - \mathcal{W}_2^f(\gamma_{ij}^s, v_i^s g_j, j) \\
- \mathcal{W}_1^f(a_{ij}^f, g_i v_j^f) - \mathcal{W}_2^f(b_{ij}^f, v_i^f g_j, j) + \pi^f g_{,j} v_j^f + g_{,j} \mathcal{K} f v_j^f(2) \\
+ \rho^f g_{,j} \left( v_i^f(1) v_i^f + j f v_i^f(1) v_i^f \right) v_j^f + \rho^f g \left( v_i^f(1) v_i^f + j f v_i^f(1) v_i^f \right) v_j^f \\
+ \epsilon_1 v_{j,i}^f g v_j^f + \epsilon_1 v_{j,i}^f g v_j^f,
\]

(3.13)

where

\[
\mathcal{K}_j = Q_j - \mathcal{K} f g v_j^f(2) - \rho^f g \left( v_i^f(1) v_i^f + j f v_i^f(1) v_i^f \right) v_j^f - \epsilon_1 v_{j,i}^f g v_j^f
\]

(3.14)

and \( \epsilon_1 \) is a positive constant.

We remark that

\[
|g_{,i}| \leq d g, \quad g_{,ii} \leq d^2 g, \quad g_{,i} g_{,i} = d^2 g^2.
\]

(3.15)
In the following we use the Schwarz’s inequality, the arithmetic geometric mean inequality and the relations (2.11) and (3.15) to obtain

$$\mathcal{W}^s_i(e^s_{ij}, g, i v^s_j) \leq \frac{\varepsilon_2}{2} g \mathcal{W}^s_i(e^s_{ij}, e^s_{ij}) + \frac{1}{2 \varepsilon_2 g} \mathcal{W}^s_i(g, i v^s_j, g, i v^s_j)$$

$$\leq \frac{\varepsilon_2}{2} g \mathcal{W}^s_i(e^s_{ij}, e^s_{ij}) + \frac{1}{2 \varepsilon_2} \sigma^s_M d^2 g v^s_j v^s_j,$$

(3.16)

for every $\varepsilon_2 > 0$.

Moreover, in view of the hypotheses of the theorem, the Schwarz’s inequality and the arithmetic geometric mean inequality, we get

$$\mathcal{W}^s_i(y^s_{ij}, v^s_i g, j) \leq \frac{\varepsilon_2}{2} g \mathcal{W}^s_i(y^s_{ij}, y^s_{ij}) + \frac{1}{2 \varepsilon_2} \delta^s_M d^2 g v^s_i v^s_j,$$

$$\mathcal{W}^s_i(a^s_{ij}, g, i v^s_j) \leq \frac{\varepsilon_3}{2} g \mathcal{W}^s_i(a^s_{ij}, a^s_{ij}) + \frac{1}{2 \varepsilon_3} \sigma^s_M d^2 g v^s_f v^s_j,$$

$$\mathcal{W}^s_i(b^s_{ij}, v^s_j g, j) \leq \frac{\varepsilon_3}{2} g \mathcal{W}^s_i(b^s_{ij}, b^s_{ij}) + \frac{1}{2 \varepsilon_3} \delta^s_M d^2 g v^s_f v^s_j,$$

$$g, j v^s_i (1) v^s_i v^s_j \leq \frac{1}{2} \left( \frac{\sqrt{3} A_1}{g d} g, j g, j v^s_i v^s_i + \frac{g d}{\sqrt{3} A_1} v^s_i (1) v^s_i v^s_j \right) \leq \sqrt{3} A_1 g d v^s_i v^s_i,$$

$$g, j v^s_j (2) \leq 3 d g A_1,$$

$$g, j v^s_i (1) v^s_i v^s_j \leq \frac{d g}{2} \left( 3 e_4 A_2^2 v^s_i v^s_i + \frac{1}{e_4} v^s_i v^s_i \right),$$

$$v^s_i (1) v^s_i v^s_j \leq \frac{1}{2} \left( \frac{3 A_1^2}{e_5} v^s_i v^s_i + e_5 v^s_i v^s_i \right),$$

$$v^s_i (1) v^s_i v^s_j \leq \frac{1}{2} \left( \frac{3 A_5^2}{e_6} v^s_i v^s_i + e_6 v^s_i v^s_i \right),$$

$$g, j v^s_i v^s_i \leq \frac{d g}{2} \left( \frac{e_7}{\rho f} v^s_i v^s_i v^s_i + \frac{\rho f}{e_7} v^s_i v^s_i \right),$$

$$g, i v^s_i \pi^f \leq \frac{1}{2} \left[ \frac{e_8 d^2 g (\pi^f)^2}{\sigma^s_M v^s_j v^s_j} + \frac{1}{e_8} g v^s_i v^s_i \right].$$

(3.17)

Thus, from (3.13), (3.16) and (3.17) we deduce

$$g \frac{\partial}{\partial t} \left[ K(t) + \mathcal{E}(u^s, \phi^s) + g \Phi(v^s, v^s, v^f, v^f) \right]$$

$$\leq K_{ij} + \frac{\varepsilon_2 g}{2} \left( \mathcal{W}^s_i(e^s_{ij}, e^s_{ij}) + \mathcal{W}^s_i(v^s_i, v^s_i) \right) + \frac{\varepsilon_3 g}{2} \left( \mathcal{W}^s_i(a^s_{ij}, a^s_{ij}) + \mathcal{W}^s_i(b^s_{ij}, b^s_{ij}) \right)$$

$$+ \frac{g d^2}{2 \varepsilon_2} \left( \sigma^s_M v^s_j v^s_j + \delta^s_M v^s_j v^s_j \right) + \frac{g d^2}{2 \varepsilon_3} \left( \sigma^s_M v^s_j v^s_j + \delta^s_M v^s_j v^s_j \right)$$

$$+ \frac{1}{2} \left[ e_8 d^2 g (\pi^f)^2 + \frac{1}{e_8} g v^s_i v^s_i \right] + 3 g d A_1 \mathcal{K}^f + \sqrt{3} \rho f g d A_1 v^s_i v^s_i.$$
Using the following inequality (see [16]):
\[
in \frac{\varepsilon_2}{\rho_f} \geq 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8 + \varepsilon_9 + \varepsilon_{10} + \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{13},
\]
and introducing the notation
\[
\tilde{a}(\varepsilon_1) = \frac{1}{\varepsilon_8 \rho_f^2} + \sqrt{5} d A_1 + \sqrt{3} j f \sqrt{d} A_2 + \frac{3 A_1^2}{\varepsilon_5} + \frac{3 j f A_2^2}{\varepsilon_6} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8 + \varepsilon_9 + \varepsilon_{10} + \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{13},
\]
then, from (2.9), (2.10) and (3.18) we obtain
\[
g \frac{\partial}{\partial t} \left[ \mathcal{K} + \mathcal{E}(\mathbf{u}^s, \phi^s) \right] + g \Phi(\mathbf{v}^s, \mathbf{v}^f, \mathbf{v}^f, \mathbf{v}^f) \leq K_{i,j} + \frac{g \sqrt{c_M}}{R_0} \mathcal{E}(\mathbf{u}^s, \phi^s) + g d^2 R_0 \sqrt{c_M} \mathcal{K}^s + \frac{\varepsilon_1}{\varepsilon_3} + \frac{d^2 c_M}{\varepsilon_3} + 3 d A_1 \mathcal{K}^f + \varepsilon_1 g(v^f_{i,j}, v^f_{i,j}) + \varepsilon_1 g(v^f_{i,j}, v^f_{i,j}) + \frac{2 \delta_m g \varepsilon_1}{\sigma_m} \left( v^f_{i,j}, v^f_{i,j} \right) + \frac{\varepsilon_1 g^2}{\sigma_m^2} \left( v^f_{i,j}, v^f_{i,j} \right) \left( \pi^f \right)^2,
\]
Using the following inequality (see [16]):
\[
a_{i,j}^f a_{i,j}^f \geq \frac{1}{2} (v^f_{i,j}, v^f_{i,j} + v^f_{i,j}, v^f_{i,j}),
\]
and the relations (2.10), (2.11) we deduce
\[
g \left\{ \frac{\partial}{\partial t} \left[ \mathcal{K} + \mathcal{E}(\mathbf{u}^s, \phi^s) \right] + \left[ 1 - \left( \frac{\varepsilon_3}{2} + \frac{\varepsilon_1}{\sigma_m} \right) \right] \Phi(\mathbf{v}^s, \mathbf{v}^f, \mathbf{v}^f) \right\} \leq K_{i,j} + \chi g \mathcal{K} + \mathcal{E}(\mathbf{u}^s, \phi^s) + \frac{\varepsilon_8 d^2 g}{2} (\pi^f)^2,
\]
where
\[ \kappa = \kappa(\varepsilon_1, \varepsilon_3) = \max \left\{ \frac{\sqrt{c_M^f}}{R_0}, d^2 R_0 \sqrt{c_M^s}, \bar{a}(\varepsilon_1) + \frac{d^2 c_M^f}{\varepsilon_3} + 3dA_1 \right\}. \] (3.24)

Now we choose the arbitrary constants \( \varepsilon_1 \) and \( \varepsilon_3 \) such that
\[ 0 < \varepsilon_1 < \frac{\sigma_m^f}{2} \quad \text{and} \quad 0 < \varepsilon_3 < 2\left(1 - 2\frac{\varepsilon_1}{\sigma_m^f}\right) \] (3.25)
and thus, in view of the positivity of the dissipation potential, we deduce
\[ \int_{\Omega_1} \rho \frac{\partial}{\partial t} \left[ K + \mathcal{E}(\mathbf{u}^s, \phi^s) \right] dv \leq \kappa \int_{\Omega_1} \rho \left[ K + \mathcal{E}(\mathbf{u}^s, \phi^s) \right] dv + \varepsilon_8 d^2 \int_{\Omega_1} (\pi f)^2 dv + \int_{\partial\Omega_1} K j n_j da, \] (3.26)
where \( \partial\Omega_1 = S_R \cup \partial B \).

In view of the boundary conditions on \( \partial B \) and the hypotheses (iii) and (iv) of the theorem, it follows that for \( R \to \infty \) the boundary integral vanishes, and thus one obtains
\[ \int_{B} \rho \frac{\partial}{\partial t} \left[ K + \mathcal{E}(\mathbf{u}^s, \phi^s) \right] dv \leq \kappa \int_{B} \rho \left[ K + \mathcal{E}(\mathbf{u}^s, \phi^s) \right] dv + \frac{\varepsilon_8 d^2}{2} \int_{B} (\pi f)^2 dv. \] (3.27)

Now, we use the assumption (v) to obtain
\[ \frac{dE}{dt} \leq \kappa E + \epsilon d^2 \epsilon, \quad t \in I \] (3.28)
where
\[ E(t) = \int_{B} g[K + \mathcal{E}(\mathbf{u}^s, \phi^s)]dv, \]
\[ \epsilon = 2\pi A_3^2 \varepsilon_8 \int_{0}^{\infty} e^{-t} t^{1-2\epsilon} dt. \] (3.29)
By a direct integration of the relation (3.28), we deduce

\[ E(t) \leq \frac{c}{\kappa} d^{2e} \exp(\kappa t), \quad t \in I. \]  

(3.30)

For \( R \geq R_0 \), we have

\[ E(t) \geq \left\{ \int_{\Omega_R} \left[ K(t) + \mathcal{E}(u^s, \phi^s) \right] \,dv \right\} \exp(-dR), \quad t \in I. \]  

(3.31)

Thus, we have the following estimate

\[ \int_{\Omega_R} \left[ K(t) + \mathcal{E}(u^s, \phi^s) \right] \,dv \leq \frac{c}{\kappa} d^{2e} \exp(\kappa t + dR), \quad t \in I. \]  

(3.32)

In view of the relation (3.24), it is clear that \( \kappa \) do not tend to zero (or infinity) when \( d \to 0 \). Thus, allowing \( d \to 0 \), and taking into account the positivity of the internal energy density, we deduce that \( K(t) = 0 \), for all \( t \in I \), and in consequence we obtain the relations (3.4)\(_{1-4}\). The relation (3.4)\(_5\) results from (3.6)\(_3\) and the proof is complete.

\[ \square \]

**Remark** Setting formally the coupling coefficients \( \xi \) and \( \varpi \) to tend to zero in the field equations and following the same strategy as in the above theorem, then one can obtain uniqueness results for exterior domains in the theory of micropolar elastic solids and respectively, in the theory of micropolar viscous fluids. Thus, the above result can also be interpreted as an alternative to the method used by Padula and Russo [10] and by Carbonaro [17] in the proof of uniqueness for the micropolar motions in unbounded regions.

**References**