Abstract

This paper is devoted to the study of mixtures which have as constituents a micropolar thermoelastic solid and a micropolar incompressible fluid. This unified paper contains extended proofs and corrected forms of the results previously established by the author.

In the first part we present the basic equations of the theory. We study the following mathematical problems: existence, uniqueness and continuous data dependence of the solution. The results shows that the considered model is in concordance with the physical reality. Moreover, we present a result which shows that the difference between the Cesáro means of the kinetic energy and the internal energy and also the Cesáro mean of the thermal energy go to zero when the time goes to infinity. Another problem is the problem of the asymptotic stability, as defined by Galdi and Rionero [118], of the thermodynamical process.

Then, we consider that the constituents of the mixtures are not heat conducting. Using the method introduced by [210], we establish a representation of Galerkin type for the dynamical problem and with the help of this Galerkin representations, we determine the fundamental solutions for the three–dimensional problem governing the motion of a micropolar solid-fluid mixture in the case of steady state vibrations. There representations are useful to establish representations of the Somigliana type. Inspired by these representations we define the potentials of single layer and double layer and then we use them to reduce the boundary value problems to singular integral equations [206]. We prove that Fredholm’s theorems are valid for these singular integral equations. Existence and uniqueness theorems are presented for interior and exterior problems. The method [189, 192] is a constructive one and can be used in order to obtain numerical solutions of the problems.

In the last part of the chapter, we consider a mixture consisting of an isotropic micropolar solid and a compressible fluid. First, we establish estimates of Saint–Venant type for bounded bodies and alternative estimates of Phragmén–Lindelöf type for unbounded bodies in terms of two appropriate time–weighted surface power functions. Further, for unbounded bodies we are able to establish an estimate which proves that the measure associated to the solution decays faster than an exponential of a second degree polynomial, provided an appropriate class of mixtures is considered. This estimate shows that at
large distance from the support of the external given data, the spatial decay of processes is influenced only by the thermal effect and by the viscosities of the fluid.
1 Micropolar solid–fluid mixtures

1.1 Kinematics. The basic equations

We consider a binary mixture of a thermoelastic micropolar solid \( B^s \) and a micropolar fluid \( B^f \). The both constituents \( B^\sigma (\sigma = s, f) \) occupy the region \( B \) of the tridimensional Euclian space. In this Section we present the equations which describe the behavior of such a continuum using the work of Eringen [104].
According with the theory of continuum with microstructure, the position of a material point $M$ of the $\sigma$ constituent, at the moment $t$, is given by

$$x^{\sigma}_{(M)} = x^{\sigma} + \xi^{\sigma}, \quad (1.1)$$

where $x^{\sigma}$ is the center of mass of the particle which include the material point and gives the position of the particle at the macrostructure level and $\xi^{\sigma}$ is the position of the material point with respect to the center of mass of the particle.

We refer the mixture, both in deformable and in natural state, to the same system of Cartesian coordonates. Obviously, after the deformation process the position of the material point $M_0$ from the natural state become $M$ in the deformation state, and will be given by

$$X^{\sigma}_{(M_0)} = X^{\sigma} + \Xi^{\sigma}, \quad (1.2)$$

where $X^{\sigma}$ and $\Xi^{\sigma}$ have similar significations with $x^{\sigma}$ and $\xi^{\sigma}$.

We assume that the each constituent has the own movement and that at all moment of time, all location of the mixture is occupies by particles belonging to each constituent.

The material particles have small diameter compared with the macroscopic scale and thus, it is natural to suppose that $\xi^{\sigma}$ depends linearly by $\Xi^{\sigma}$, i.e.

$$\xi^{\sigma}_{i} = \chi_{ik}^{\sigma}(X^{\sigma}, t)\Xi^{\sigma}_{k}. \quad (1.3)$$

In the theory of micropolar continuum, the matrix $\chi$ is considered to be such that the following relations are satisfied

$$\chi_{ik}^{\sigma}\chi_{il}^{\sigma} = \delta_{kl}, \chi_{ik}^{\sigma}\chi_{jk}^{\sigma} = \delta_{ij}. \quad (1.4)$$

The deformation of the mixture is describes by the maps [104]

$$x = \Lambda^{\sigma}(X^{\sigma}, t), \quad X^{\sigma}_{(M_0)} = \chi_{ik}^{\sigma}(X^{\sigma}, t), \quad (X^{\sigma}, t) \in B^{\sigma} \times [0, \infty), \quad (1.5)$$

where $B^{\sigma}$ is the domain occupied by the $\sigma$ constituent in the undistorted state.

We will suppose that all the functions which arise in our equations are sufficiently regular. Using the arguments presented in the first part of the previous chapter, we assume that the maps (1.5) are one to one for all times $t$. Thus, we have

$$X^{\sigma} = (\Lambda^{\sigma})^{-1}(x, t), \quad \Xi^{\sigma}_{k} = \chi_{ik}^{\sigma}(x, t)\xi^{\sigma}_{i}, \quad (x, t) \in B \times [0, \infty). \quad (1.6)$$

The above relations ensure that the impenetrability condition is satisfied for each constituent both to the macrostructure and to the microstructure level.

The velocity and the acceleration at the time $t$, of the center of mass of the $\sigma$ constituent which in has occupied the $X^{\sigma}$ position in the natural state, are defined by

$$v^{\sigma} = \frac{\partial \Lambda^{\sigma}}{\partial t}(X^{\sigma}, t), \quad a^{\sigma} = \frac{\partial^2 \Lambda^{\sigma}}{\partial t^2}(X^{\sigma}, t). \quad (1.7)$$

From (1.1) it follows that the velocity of a material point of the $\sigma$ constituent will be

$$\dot{x}^{\sigma}_{i(M)} = v^{\sigma}_{i} + \nu_{ik}^{\sigma}\xi^{\sigma}_{i}, \quad (1.8)$$
where

\[ \nu_{ij}^\sigma = \dot{\chi}_{ik}^\sigma \chi_{jk}^\sigma, \] (1.9)

is the *gyration tensor*.

In view of the relations (1.4) we deduce that \( \nu_{ij}^\sigma \) is an antisymmetric tensor, and thus, as in classical mechanics we introduce the vector

\[ \nu_i^\sigma = -\frac{1}{2} \varepsilon_{ikl} \nu_k^\sigma, \] (1.10)

which will be called *microrotation velocity*.

Let us introduce the density \( \rho^\sigma(x, t) \) of the \( \sigma \) constituent at the moment \( t \). We define the density \( \rho(x, t) \) and the velocity \( \mathbf{v}(x, t) \), of the mixtures by the following relations

\[ \rho(x, t) = \sum_{\sigma=s,f} \rho^\sigma(x, t), \quad \rho \mathbf{v} = \sum_{\sigma=s,f} \rho^\sigma \mathbf{v}^\sigma. \] (1.11)

Because the maps defined by (1.5) are one to one, it follows that

\[ J^\sigma = \det \left( \frac{\partial x_k}{\partial X^\sigma_j} \right) > 0. \] (1.12)

Moreover, we have the Euler’s relations

\[ \dot{j}^\sigma = j^\sigma \dot{\nu}_{i,j}. \] (1.13)

In mathematical formulation of the fundamental principles of the mechanics and thermodynamics of the considered mixtures we will take into account the following:

(i) each constituent is considered to be an independent micropolar continuum whose behavior is influenced by the presence of the other constituent;

(ii) the mixture itself is a micropolar continuum.

We also assume that the mixture is chemical inert and that the both constituents have the same temperature. Let us consider the domains \( P^\sigma \subset B^\sigma \) which after the deformation become \( P \subset B \).

In the following we suppose that each constituent are microisotrop continua, i.e. the microinertial tensors are

\[ i_{ij}^\sigma = j^\sigma \delta_{ij}. \] (1.14)

We postulate the balance law of the energy for the \( \sigma \) constituent, at all time \( t \) and for each domain \( P \subset B^\sigma \), given by

\[ \frac{d}{dt} \int_P \rho^\sigma (e^\sigma + \frac{1}{2} v_i^\sigma v_i^\sigma + \frac{1}{2} j^\sigma \nu_i^\sigma \nu_i^\sigma) dv = \int_P \rho^\sigma (f_i^\sigma v_i^\sigma + l_i^\sigma \nu_i^\sigma) dv \\
+ \int_{\partial P} (t_i^\sigma v_i^\sigma + m_i^\sigma \nu_i^\sigma) da + \int_P (\hat{p}_i^\sigma v_i^\sigma + \hat{m}_i^\sigma \nu_i^\sigma + \hat{e}^\sigma) dv \\
+ \int_{\partial P} q^\sigma da + \int_P \rho^\sigma h^\sigma dv, \] (1.15)
where \(e^\sigma, f^\sigma, \ell^\sigma_i\) and \(h^\sigma\) are the internal energy density, the body force, the body couple and the energy source, respectively, associated with the unit volume of the \(\sigma\) constituent, \(t^\sigma_i = t^\sigma_{ji} n_j\) is the stress vector on the unit area, \(m^\sigma_i = m^\sigma_{ji} n_j\) is the couple stress on the unit area, \(\tilde{p}^\sigma_i\) is the force exerted on the \(\sigma\) constituent by the other constituent, \(\tilde{m}^\sigma_i\) is the couple exerted on the \(\sigma\) constituent by the other constituent, and \(\tilde{\varepsilon}^\sigma\) is the energy received by the \(\sigma\) constituent from the other constituent.

Green and Rivlin [142] showed that, in the theory of classical termoelastic materials, the principle of conservation of mass, the momentum conservation principle and the moment of momentum conservation principle can be obtained from the balance of the total energy, imposing certain conditions of invariance to superposed rigid motions.

We know that the considered continuum are now in a state after certain movement of the particles. Let us consider another movement of the particles obtained by superposing to this movement a rigid motion to different moments, i.e.

\[
x_i(X, t) = c_i(t) + Q_{ik}(t) X^\sigma_k,
\]

\[
\chi^\sigma_{ik}(X, t) = Q_{ik}(t),
\]

where \(c_i\) is a function of time and \(Q_{ik}\) is an orthogonal transformation.

Corresponding to these motion we have

\[
v^\sigma_{i,k} = \varepsilon_{ijk} b_j x_k + \tilde{c}_i, \quad \nu^\sigma_i = b_i
\]

where \(b_i\) is the axial vector of the antisymmetric tensor \(\Omega_{ij} = \dot{Q}_{ik} Q_{jk}\) and \(\tilde{c}_i = c_i - Q_{ik} Q_{j} c_j\).

Imposing the invariance condition to a superposed rigid motion for the quantities \(\rho^\sigma, j^\sigma, f^\sigma_i, l^\sigma_i, m^\sigma_i, \tilde{p}^\sigma_i, \tilde{m}^\sigma_i, \tilde{\varepsilon}^\sigma, e^\sigma, q^\sigma\) and \(h^\sigma\) (see for example [142]) we obtain that \(j^\sigma\) are constants in time and we have the following equations

\[
\dot{\rho}^\sigma + \rho^\sigma \varepsilon_{i,ij} = 0,
\]

\[
t^\sigma_{ji,j} + \rho^\sigma f^\sigma_i + \tilde{p}^\sigma_i = \rho^\sigma \dot{\nu}_i^\sigma,
\]

\[
m^\sigma_{ji} + \varepsilon_{ijk} l^\sigma_j + \tilde{m}^\sigma_i + \rho^\sigma l^\sigma_j = \rho^\sigma j^\sigma \dot{\nu}_i^\sigma, \quad \sigma = s, f.
\]

Let now introduce these equations in (1.15) to prove that

\[
\int_P \rho^\sigma \dot{e}^\sigma dv = \int_P (t^\sigma_{ji} \nu_{i,j} - \varepsilon_{ijk} l^\sigma_j + m^\sigma_{ji} \nu_{i,j} + \tilde{\varepsilon}^\sigma + \rho^\sigma h^\sigma) dv + \int_{\partial P} q^\sigma da.
\]

As in classical theory, we deduce that there is a vector \(q^\sigma_i\) such that

\[
q^\sigma_i = q^\sigma_{i,n_i}.
\]

Hence, the local form of the energy equation is

\[
\rho^\sigma \dot{e}^\sigma = t^\sigma_{ji} \nu_{i,j} + m^\sigma_{ji} b^\sigma_{ij} + q^\sigma_{i,n_i} + \rho^\sigma h^\sigma + \tilde{\varepsilon}^\sigma,
\]

where

\[
a^\sigma_{ji} = \nu_{i,j} + \varepsilon_{ijk} \nu_{i,k}, \quad b^\sigma_{ij} = \nu^\sigma_{i,j}.
\]
are the microdeformation rate tensors.

For the mixture we have a similar form of the continuity equation with (1.18). So, we have

$$\frac{\partial \rho}{\partial t} + (\rho \mathbf{v})_i = 0.$$  \hspace{1cm} (1.25)

If \( f^\sigma \) is a function defined along the path of the \( \sigma \) constituent, then, as a consequence of the equation (1.18), we have

$$\frac{d}{dt} \int_{P^\sigma} \rho^\sigma f^\sigma dv = \int_{P^\sigma} \rho_0^\sigma \dot{f}^\sigma dV = \int_{P^\sigma} \rho^\sigma \dot{f}^\sigma dv,$$  \hspace{1cm} (1.26)

where by \( \dot{f}^\sigma \) we denote the material derivative of \( f^\sigma \) while \( P^\sigma \) is the domain occupied by the \( \sigma \) constituent which becomes \( P \) after the deformation process.

We define

$$\rho f = \sum_{\sigma=s,f} \rho^\sigma f^\sigma.$$  \hspace{1cm} (1.27)

By direct summation upon \( \sigma \) in (1.26) we obtain

$$\frac{d}{dt} \int_P \rho f dv = \int_P \sum_{\sigma=s,f} \rho^\sigma \dot{f}^\sigma dv,$$  \hspace{1cm} (1.28)

for all \( P \subset B \).

On the other hand, because \( f \) can be viewed as a quantity defined on the path of each constituent, we can write

$$\frac{d}{dt} \int_P \rho f dv = \frac{d}{dt} \int_P \sum_{\sigma=s,f} \rho^\sigma f dv = \frac{d}{dt} \int_{P^\sigma} \sum_{\sigma=s,f} \rho_0^\sigma f dV = \frac{d}{dt} \int_{P^\sigma} \rho_0^\sigma f dV = \frac{d}{dt} \int_P \rho \dot{f} dv.$$  \hspace{1cm}

We combine the above relations to deduce that

$$\rho \dot{f} = \sum_{\sigma=s,f} \rho^\sigma \dot{f}^\sigma.$$  \hspace{1cm} (1.29)

A similar formula with the above relation was used by Green and Naghdi [144] in describe a theory of mixtures. This relation will be used in the calculus of some derivatives, when we formulate the energy balance of the total energy of the mixture and moreover when we compare the equations obtained for the mixture with those obtained for the each constituent of the mixture.

In the following we consider the mixture as a micropolar continuum and we postulate the
following law of conservation of the total energy
\[
\frac{d}{dt} \int_{\mathcal{P}} \sum_{\sigma = s,f} \rho^\sigma (e^\sigma + \frac{1}{2} v_i^\sigma v_i^\sigma + \frac{1}{2} j^\sigma v_i^\sigma v_i^\sigma) dv
\]
\[
= \int_{\mathcal{P}} \sum_{\sigma = s,f} \rho^\sigma (f_i^\sigma v_i^\sigma + l_i^\sigma v_i^\sigma) dv
\]
\[
+ \int_{\partial \mathcal{P}} \sum_{\sigma = s,f} (t_i^\sigma v_i^\sigma + m_i^\sigma v_i^\sigma) da
\]
\[
+ \int_{\partial \mathcal{P}} \sum_{\sigma = s,f} q_i^\sigma da + \int_{\mathcal{P}} \sum_{\sigma = s,f} \rho^\sigma h^\sigma dv,
\]
(1.30)
for all $\mathcal{P} \subset \mathcal{B}$ and all time.

We define the total internal density energy $e$, the heat source $h$ and the heat flux for the mixture by the relations
\[
\rho e = \sum_{\sigma = s,f} \rho^\sigma e^\sigma, \quad \rho h = \sum_{\sigma = s,f} \rho^\sigma h^\sigma, \quad q = \sum_{\sigma = s,f} q^\sigma.
\]
(1.31)

Using the local forms of the basic equations (1.19) and (1.20) and moreover the equation (1.28), we obtain
\[
\rho \dot{e} = \sum_{\sigma = s,f} (t_i^\sigma a_i^\sigma + m_i^\sigma b_i^\sigma) + q_{i,i} + \rho \dot{h} - \sum_{\sigma = s,f} (\dot{p}_i^\sigma v_i^\sigma + \dot{m}_i^\sigma v_i^\sigma).
\]
(1.32)

We compare the equation which derive from (1.30) with those obtained for each constituent and we deduce that
\[
\sum_{\sigma = s,f} \dot{p}_i^\sigma = 0, \quad \sum_{\sigma = s,f} \dot{m}_i^\sigma = 0, \quad \sum_{\sigma = s,f} (\dot{p}_i^\sigma v_i^\sigma + \dot{m}_i^\sigma v_i^\sigma + \dot{\varepsilon}^\sigma) = 0.
\]
(1.33)

The mixtures theories can be divided into two categories: theories that occurs only one temperature, depending on $x$ and $t$, and theories that occurs different temperatures for the constituents. The theory considered in this book is included in the first category. We mention that many papers consider mixtures which have constituents with different temperature (see the paper of Dunwoody and Müller [88], Bowen and García [16], Craine, Green and Naghdi [71], Bowen and Chen [17, 18] etc.). As a consequence of the presence of one temperature, we will consider the energy equations only for entire mixture and we will formulate the second principle of the thermodynamics only for entire mixture.

We introduce the Helmholtz’s free energy by
\[
\Psi = e - \theta \eta,
\]
(1.34)
where $\theta > 0$ is the absolute temperature and $\eta$ is the entropy.

Taking into account the definition of the quantities associated to the mixture (1.31) as means of the quantities associated to the constituents, it follows that there is a function $\Psi^\sigma$ given by
\[
\Psi^\sigma = e^\sigma - \theta \eta,
\]
(1.35)
so that
\[ \rho \Psi = \sum_{\sigma=s,f} \rho^\sigma \Psi^\sigma. \] (1.36)

In view of the relations (1.33), the equation (1.32) can be written in the form
\[ -\left( \sum_{\sigma=s,f} \rho^\sigma \dot{\Psi}^\sigma + \rho \dot{\theta} \eta + \rho \dot{\theta} \eta \right) + q_{i,i} + \rho h + \sum_{\sigma=s,f} (t_{ji}^\sigma a_{ji}^\sigma + m_{ji}^\sigma b_{ij}^\sigma) \]
\[ -\hat{\rho}^s_i (v^s_i - v^f_i) - \hat{m}^s_i (\nu^s_i - \nu^f_i) = 0. \] (1.37)

We consider the following Clausius-Duhem’s type inequality to express the entropy principle [104]
\[ \int_\mathcal{P} \rho \dot{\eta} dv - \int_\mathcal{P} \rho h \theta dv - \int_{\partial \mathcal{P}} q \theta da \geq 0, \]
for all \( \mathcal{P} \subset \mathcal{B} \) and all time.

The local form of this second principle of thermodynamics of the considered type of mixtures is
\[ \rho \dot{\theta} \eta - q_{i,i} + \frac{q_i}{\theta} \dot{\theta}_i - \rho h \geq 0, \] (1.39)
which together with the relation (1.37) gives us the relation
\[ -\left( \sum_{\sigma=s,f} \rho^\sigma \dot{\Psi}^\sigma + \rho \dot{\theta} \eta \right) + \sum_{\sigma=s,f} (t_{ji}^\sigma a_{ji}^\sigma + m_{ji}^\sigma b_{ij}^\sigma) \]
\[ -\hat{\rho}^s_i (v^s_i - v^f_i) - \hat{m}^s_i (\nu^s_i - \nu^f_i) + \frac{q_i}{\theta} \dot{\theta}_i \geq 0. \] (1.40)

In conclusion, the equations which describe the mechanical behavior of micropolar solid-fluid mixture are: the continuity equation (1.18); the equation of motion (1.19) and (1.20); the energy equation (1.37) and the inequality of entropy (1.40). For the first time, these equations was established by Twiss and Eringen [262] and recently was reconsidered by Eringen in the paper [104].

When the displacement of the solid, the velocity of the fluid, the microrotation of the solid particles, the microrotation rate of the fluid particles and the variation of the temperature are small, it is possible to give a linear approximation of the theory previously considered in this chapter. To this aim, we introduce the displacement vector \( u^s_i \), the microrotation tensor \( \phi^s_{ij} \) and the variation of temperature \( T \):
\[ u^s_i = x_i - X^s_i, \quad \phi^s_{ij} = \delta_{ij} - x^s_{ij}, \quad T = \theta - T_0, \quad |T| \leq T_0, \quad T_0 > 0, \] (1.41)
where \( T_0 \) is the ambient temperature.

In linear theory, we assume that
\[ u^s_i = \hat{u}^s_i \varepsilon, \quad \phi^s_{ij} = \hat{\phi}^s_{ij} \varepsilon, \quad v^f_i = \hat{v}^f_i \varepsilon, \quad \nu^f_i = \hat{\nu}^f_i \varepsilon, \quad T = \hat{T} \varepsilon. \] (1.42)
where \( \varepsilon \) is a very small parameter without dimension for which \( \varepsilon^n \approx 0 \) for \( n > 1 \) and \( \hat{u}^s_i, \hat{\phi}^s_{ij}, \hat{v}^f_i, \hat{\nu}^f_i, \hat{T} \) do not depend by \( \varepsilon \).
Taking into account the relation (1.4) we deduce that, in the linear theory, $\phi^s_{ij}$ is an antisymmetric tensor. We introduce the axial vector $\phi^s_i$, called microrotation vector, by

$$\phi^s_i = -\frac{1}{2}\varepsilon_{ikt}\phi^s_{kl}. \quad (1.43)$$

In view of relations (1.10), (1.41) and (1.43) we can say that

$$\nu^s_i \simeq \dot{\phi}^s_i \quad (1.44)$$

only for infinitesimal microdisplacement.

In the following we use the following notation $f' = \frac{\partial f}{\partial t}$.

The fact that the mixture has as constituents a solid and a fluid was not considered until now. This fact will be essential when we consider the constitutive equations.

According with the definition of the micropolar elastic materials and of the micropolar fluids (see [102] and [103]) we consider the following list of independent variables

$$Y = (\rho^s, \theta, e^s_{ij}, \gamma^s_{ij}, a^f_{kl}, b^f_{kl}, v^s_i - v^f_i, \nu^s_i - \nu^f_i), \quad (1.45)$$

and the following dependent variables

$$Z = (\Psi^s, \Psi^f, \eta, t^s_{ij}, m^s_{ij}, t^f_{ij}, m^f_{ij}, -\dot{p}^s_i, -\dot{m}^s_i), \quad (1.46)$$

where

$$e^s_{kl} = u^s_{i,k} + \varepsilon_{ikm}\phi^s_{mk}, \quad \gamma^s_{kl} = \phi^s_{k,l}, \quad (1.47)$$

are the strain measure for the solid.

We remark the presence of the relative velocities $v^s_i - v^f_i$ and $\nu^s_i - \nu^f_i$ in the list of constitutive independent variables. These quantities are not objective and the theirs presence is a linear approximation of the model. This fact is considered by Eringen [104] to be in concordance with the physical reality because the theories involving the fluids must take into account the Darcy’s law.

We assume that the constitutive dependent functionals are of class $C^2$ in theirs domains.

The Helmholtz’s free energies are functions of the constitutive independent variables $\theta, e^s_{ji}, \gamma^s_{ji}$ and $\rho^f$, as in the following relations

$$\Psi^s = \Psi^s(\theta, e^s_{ij}, \gamma^s_{ij}), \quad \Psi^f = \Psi^f(\theta, \rho^f). \quad (1.48)$$

By the well known procedure, from (1.40) we obtain

$$\eta = -\sum_{\sigma=s,f} \frac{\partial \Psi^\sigma}{\partial \theta}, \quad t^s_{ji} = \rho^s \frac{\partial \Psi^s}{\partial e^s_{ji}}, \quad m^s_{ji} = \rho^s \frac{\partial \Psi^s}{\partial \gamma^s_{ji}},$$

$$t^f_{ji} = -\pi^f \delta_{ji} + D t^f_{ij}, \quad \pi^f = -\rho^f \frac{\partial \Psi^f}{\partial \rho^f},$$

$$\Psi^s = \Psi^s(\theta, e^s_{ji}, \gamma^s_{ji}), \quad \Psi^f = \Psi^f(\theta, \rho^f),$$

$$D t^f_{ij} a^f_{ji} + m^f_{ji} B^f_{ij} - \ddot{p}^f_i (v^s_i - v^f_i) - \ddot{m}^s_i (\nu^s_i - \nu^f_i) + \frac{\theta^i}{\theta} \geq 0 \quad (1.49)$$
This last inequality show us that to describe the dissipative part of the constitutive equations, we have to use the following list of constitutive variables

\[
\vec{Y} = (\alpha_{fi}, B_{ij}, v^s_i - v^f_i, \nu^s_i - \nu^f_i, \theta^i, \theta^i).
\] (1.50)

In the following we call \(\vec{Y}\) *thermodynamic forces* and

\[
J(\vec{Y}) = (D_{tji}, m_{ji}, -\hat{p}^s_i, -\hat{m}^s_i, q_i),
\] (1.51)

*thermodynamic fluxes.* The inequality (1.49) can be written in the form

\[
\vec{Y} \cdot J(\vec{Y}) \geq 0.
\] (1.52)

Because \(J\) has as components \(C^2\) functions of theirs domains, we can use the decomposition given by Edelen [89]

\[
J(\vec{Y}) = \text{grad}_\vec{Y} \Phi(\vec{Y}) + W(\vec{Y}),
\] (1.53)

where \(W(\vec{Y})\) does not influence the dissipation effect, i.e.

\[
W(\vec{Y}) \cdot \vec{Y} = 0,
\] (1.54)

and the dissipation potential is given by

\[
\Phi(\vec{Y}) = \int_0^1 \vec{Y} \cdot J(\vec{Y}) \frac{d\tau}{\tau}.
\] (1.55)

From the inequality (1.52) it follows that the dissipation potential must be positive for all thermodynamical forces.

We remark from (1.54) that \(W\) does not has a contribution to the dissipation of energy and will be not considered in what follows. Thus, we have

\[
D_{tji} = \frac{\partial \Phi}{\partial \alpha_{fi}}, \quad m_{ji} = \frac{\partial \Phi}{\partial B_{ij}}, \quad q_i = \frac{\partial \Phi}{\partial (\theta^i/\theta)},
\]

\[
\hat{p}^s_i = -\frac{\partial \Phi}{\partial (v^s_i - v^f_i)}, \quad \hat{m}^s_i = -\frac{\partial \Phi}{\partial (\nu^s_i - \nu^f_i)}.
\] (1.56)

The dissipation potential has to satisfy the objectivity principle. Thus, \(\Phi\) depends by \(\vec{Y}\) only by means the corresponding invariants [101].

Hence, Eringen [104] proposed the following form of the *dissipation potential*

\[
\Phi = \Phi_1 + \Phi_2,
\] (1.57)

with

\[
2\Phi_1 = \lambda^f (\text{tr } a^f)^2 + \mu^f \text{tr } (a^f)^2 + (\mu^f + k^f) \text{tr } (a^f a^f^T) + \xi (v^s - v^f)(v^s - v^f) + 2\zeta (v^s - v^f) \frac{\nabla \theta}{\theta} + K \frac{\nabla \theta \nabla \theta}{\theta},
\] (1.58)

\[
2\Phi_2 = \alpha^f (\text{tr } b^f)^2 + \beta^f \text{tr } (b^f)^2 + \gamma^f \text{tr } (b^f b^f^T) + \omega (v^s - v^f)(v^s - v^f).
\]
In our analysis we consider that the fluid is incompressible. So, \( \rho^f(x, t) = \rho^f_0(X^f) \), and the pressure becomes a unknown function.

The functional \( \Psi \) will be considered a second degree polynomial form in terms of the actual constitutive variables

\[
\rho^s_0 \Psi^s = S^s_0 - \eta^s_0 T - \frac{C^s_0}{2T_0} T^2 - A^s_{ij} T e^s_{ij} - B^s_{ijkl} \gamma^s_{ijkl} + \frac{1}{2} (A^s_{ijkl} \epsilon^s_{ij} \epsilon^s_{kl} + B^s_{ijkl} \gamma^s_{ij} \gamma^s_{kl} + 2C^s_{ijkl} \epsilon^s_{ij} \epsilon^s_{kl}),
\]

(1.59)

where

\[
A^s_{ijkl} = A^s_{klij}, \quad B^s_{ijkl} = B^s_{klij}.
\]

(1.60)

We consider that in the natural state we do not have tensions and the heat flux and the entropy are null. Hence,

\[
S^\sigma_0 = 0, \quad \eta^\sigma_0 = 0.
\]

(1.61)

In view of the relations (1.49), (1.56), (1.58) and (1.59), we deduce

\[
\begin{align*}
\rho_0 \eta^s &= \frac{C^s_0}{T_0} T + A^s_{ij} \epsilon^s_{ij} + B^s_{ij} \gamma^s_{ij}, \\
t^s_{ij} &= -A^s_{ij} T + A^s_{ijkl} \epsilon^s_{kl} + C^s_{ijkl} \gamma^s_{kl}, \\
m^s_{ij} &= -B^s_{ij} T + B^s_{ijkl} \gamma^s_{kl} + C^s_{ijkl} \epsilon^s_{kl}, \\
t^f_{ji} &= -\pi^f \delta_{ji} + (\mu^f + k^f) a^f_{ij} + \mu^f a^f_{ji}, \\
m^f_{ji} &= \alpha^f b^f_{kk} \delta_{ij} + \beta^f b^f_{ij} + \gamma^f b^f_{ji}, \\
\hat{p}^s_i &= -\hat{p}^f_i = -\xi (v^s_i - v^f_i) - \zeta \frac{\theta_i}{\theta}, \\
\hat{m}^s_i &= -\hat{m}^f_i = -\omega (v^s_i - v^f_i), \\
q_i &= \zeta (v^s_i - v^f_i) + K \frac{\theta_i}{\theta},
\end{align*}
\]

where

\[
C_0 = \sum_{\sigma=s,f} C^\sigma_0.
\]

(1.63)

In the case of isotropic elastic solids we have

\[
\begin{align*}
A^s_{ij} &= \beta_0 \delta_{ij}, \quad B^s_{ij} = 0, \quad C^s_{ijkl} = 0, \\
A^s_{ijkl} &= \lambda^s \delta_{ij} \delta_{kl} + (\mu^s + k^s) \delta_{ik} \delta_{jk}, \\
B^s_{ijkl} &= \alpha^s \delta_{ij} \delta_{kl} + \beta^s \delta_{il} \delta_{jk} + \gamma^s \delta_{ik} \delta_{jl},
\end{align*}
\]

(1.64)

and the expressions of \( \rho_0 \eta, t^s_{ij} \) and \( m^s_{ij} \) are reduced to

\[
\begin{align*}
\rho_0 \eta &= \frac{C^s_0}{T_0} T + \beta_0 \epsilon^s_{kk}, \\
t^s_{ij} &= -\beta_0 T + \lambda^s \epsilon^s_{kk} \delta_{ij} + (\mu^s + k^s) \epsilon^s_{ij} + \mu^s \epsilon^s_{ji}, \\
m^s_{ij} &= \alpha^s \gamma^s_{kk} \delta_{ij} + \beta^s \gamma^s_{ij} + \gamma^s \gamma^s_{ji}.
\end{align*}
\]

(1.65)

\[\text{12}\]
### 1.2 The initial boundary values problem

The fundamental system of partial differential equations of the linear theory of micropolar mixtures which have as constituents a micropolar elastic solid and a incompressible fluid are:

- the geometrical equations (1.24) and (1.47),
- the constitutive equations (1.62),
- the motion equations (1.19), (1.20),
- the energy equation (1.37).

If the rotation of the fluid’s particle are not small (see the relations (1.10), (1.41) and (1.43)), then the microrotation rate is not the partial derivative of the microrotation vector with respect to time.

In this case, we introduce the functions

\[ u^f_i = \int_0^t v^f_i \, ds, \quad \phi^f_i = \int_0^t \nu^f_i \, ds. \]  

We will see that in the definition of the principal energies associated to the solution of the problem which will be studied and in the expressions of the main results, the important roles will be played by \( v^f_i \) and \( \nu^f_i \), the quantities \( u^f_i \) and \( \phi^f_i \) are used only in the auxiliary results.

In terms of the new quantities, the basic equations are:

- **the equations of motion**

\[
\begin{align*}
\rho_0^\sigma u^{\sigma\sigma}_i &= \sigma_{ji,j}^\sigma + \rho_0^\sigma f_i^\sigma + \rho_0^f g_i^\sigma, \\
\sum_{\sigma=s,f} \rho_0^\sigma f_i^\sigma &= 0, \\
\rho_0^f \phi^{\sigma\sigma}_i &= m_{ji,j}^\sigma + \varepsilon_{ijk} t_{jk}^\sigma + \rho_0^f m_i^\sigma, \\
\sum_{\sigma=s,f} \rho_0^f m_i^\sigma &= 0.
\end{align*}
\]

- **the energy equations**

\[ \rho_0 T_0 q^f_i = q_{i,i} + \rho_0 h, \]  

- **the incompressibility condition**

\[ u'^f_{i,i} = 0. \]  

The constitutive equations for a homogeneous mixture of a micropolar isotropic elastic solid and a viscous incompressible fluid are

\[
\begin{align*}
t_{ji}^\sigma &= -\beta_0 T \delta_{ji} + \lambda^\sigma u^\sigma_{k,k} \delta_{ji} + \mu^\sigma (u^\sigma_{j,i} + u^\sigma_{i,j}) + k^\sigma (u^\sigma_{i,j} + \varepsilon_{ijk} \phi^\sigma_k), \\
m_{ji}^\sigma &= \alpha^\sigma \phi^\sigma_{k,k} \delta_{ji} + \beta^\sigma \phi^\sigma_{j,i} + \gamma^\sigma \phi^\sigma_{i,j}, \\
t_{ji}^f &= -\pi^f \delta_{ji} + \mu^f (u'^f_{j,i} + u'^f_{i,j}) + k^f (u'^f_{i,j} + \varepsilon_{ijk} \phi'^f_k), \\
m_{ji}^f &= \alpha^f \phi'^f_{k,k} \delta_{ji} + \beta^f \phi'^f_{j,i} + \gamma^f \phi'^f_{i,j}.
\end{align*}
\]
We can easily see that this the quadratic is semipositive defined if and only if

\[ a_i \text{ must be a semipositive defined quadratic form in terms of the independent constitutive variables} \]

This is true if and only if

\[ e_{ij} \text{ are dimensionless. For this, we define} \]

\[ \Phi(\bm{v}^s, \bm{v}^f) = \mu^f a_{ij}^f a_{ij} + (\mu^f + k^f) a_{ij}^f a_{ij} \]

\[ + \alpha f b_{ij}^f b_{ij}^f + \beta f b_{ij}^f b_{ij}^f + \gamma f b_{ij}^f b_{ij}^f + \xi (u_i^s - u_i^f)(u_i^s - u_i^f) \]

\[ + 2 \zeta (u_i^s - u_i^f) \frac{T^\alpha}{T_0} + \varpi (\phi_i^s - \phi_i^f)(\phi_i^s - \phi_i^f) + \frac{K}{T_0} T_i \]

Because we have considered the case of homogeneous constituents, the constitutive coefficients are considered constants.

The local form of the Clausius–Duhem inequality [104] implies that the dissipation potential

\[ \Phi(\bm{v}^s, \bm{v}^f) = \mu^f a_{ij}^f a_{ij} + (\mu^f + k^f) a_{ij}^f a_{ij} \]

\[ + \alpha f b_{ij}^f b_{ij}^f + \beta f b_{ij}^f b_{ij}^f + \gamma f b_{ij}^f b_{ij}^f + \xi (u_i^s - u_i^f)(u_i^s - u_i^f) \]

\[ + 2 \zeta (u_i^s - u_i^f) \frac{T^\alpha}{T_0} + \varpi (\phi_i^s - \phi_i^f)(\phi_i^s - \phi_i^f) + \frac{K}{T_0} T_i \]

must be a semipositive defined quadratic form in terms of the independent constitutive variables

\[ a_{ij}^f, b_{ij}^f, v_i^s - v_i^f, \nu_i^s - \nu_i^f \text{ and} \frac{T^\alpha}{T_0}. \]

This is true if and only if

\[ K \geq 0, \quad \xi K - \zeta^2 \geq 0, \quad \varpi \geq 0, \]

\[ 2 \mu^f + k^f \geq 0, \quad k^f \geq 0, \]

\[ 3 \alpha^f + \beta^f + \gamma^f \geq 0, \quad \gamma^f + \beta^f \geq 0, \quad \gamma^f - \beta^f \geq 0. \]

We define the internal energy density to be the following quadratic form in terms of \( e_{ij}^s, \gamma_{ij}^s \)

\[ E(\bm{u}, \Phi) = \frac{1}{2} [\lambda^s e_{kk}^s e_{ii}^s + \mu^s e_{jj}^s e_{ij}^s + (\mu^s + k^s) e_{ij}^s e_{ij}^s \]

\[ + \alpha^s \gamma_{kk}^s \gamma_{ii}^s + \beta^s \gamma_{jj}^s \gamma_{ij}^s + \gamma^s \gamma_{ij}^s \gamma_{ij}^s]. \]

We can easily see that this the quadratic is semipositive defined if and only if

\[ 3 \lambda^s + 2 \mu^s + k^s \geq 0, \quad 2 \mu^s + k^s \geq 0, \quad k^s \geq 0, \]

\[ 3 \alpha^s + \beta^s + \gamma^s \geq 0, \quad \gamma^s + \beta^s \geq 0, \quad \gamma^s - \beta^s \geq 0. \]

In the following we rewrite the basic equations in a equivalent form in which all the quantities are dimensionless. For this, we define

\[ \bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{\tau}, \quad \bar{u} = \frac{u}{L}, \quad \bar{\phi} = \phi, \]

\[ \bar{T} = \frac{T}{T_0}, \quad \bar{T}_0 = \frac{T_0}{T}, \quad \bar{\rho} = \frac{L^3}{m} \rho_0, \quad (\sigma = s, f), \]

\[ \begin{align*}
\dot{p}_i^s &= -\xi (u_i^s - u_i^f) - \frac{\zeta}{T_0} T_i, \\
\dot{m}_i^s &= -\varpi (\phi_i^s - \phi_i^f), \\
\rho_0 \eta &= \frac{C_0}{T_0} T + \beta_0 u_{i,i}, \\
q_i &= \zeta (u_{i}^s - u_{i}^f) + \frac{K}{T_0} T_i.
\end{align*} \]
where $L, \tau, \hat{T}$ and $m$ are constants which characterize the length, the time, the temperature and the mass, respectively. With the help of these quantities, we can rewrite the basic system of equations (1.67)–(1.71) in the form

\begin{equation}
\begin{align*}
\bar{\rho}'\bar{u}''_i &= (\bar{\lambda} + \bar{\mu}\bar{\kappa})\bar{u}'_i - (\bar{\mu}' + \bar{\kappa})\bar{u}'_i + \bar{\kappa}\varepsilon_{ijk}\bar{\phi}_{k,j} - \bar{\xi}(\bar{u}'_i - \bar{u}'_i) \\
&\quad - \left(\bar{\xi}_0 + \bar{\beta}_0\right)T_i + \bar{\rho}' \bar{f}'_i, \\
\bar{\rho}_0\bar{\phi}''_i &= (\bar{\alpha} + \bar{\beta}\bar{\kappa})\bar{\phi}'_i + \bar{\gamma}\bar{\phi}'_i + \bar{\kappa}(\varepsilon_{ijk}\bar{u}'_k - 2\bar{\phi}'_i) \\
&\quad - \bar{\omega}(\bar{\phi}'_i - \bar{\phi}'_i) + \bar{\rho}_0\bar{f}_i, \\
\bar{\rho}_0\bar{u}'_i + \bar{\rho}_0\bar{f}'_i &= (\bar{\alpha} + \bar{\beta}\bar{\kappa})\bar{\phi}'_i + \bar{\gamma}\bar{\phi}'_i + \bar{\kappa}(\varepsilon_{ijk}\bar{u}'_k - 2\bar{\phi}'_i) \\
&\quad + \bar{\omega}(\bar{\phi}'_i - \bar{\phi}'_i) + \bar{\rho}_0\bar{f}_i, \\
0 &= -\bar{C}_0\bar{T}' + (\bar{\beta}_0 T_0 + \bar{\xi})\bar{u}'_i + \bar{\bar{K}}T_i + \bar{\rho}_0\bar{h}, \\
\bar{\bar{u}}'_i &= 0,
\end{align*}
\end{equation}

in $B \times I$, where the partial derivatives are now with respect to $\bar{x}$ and $\bar{t}$, $I$ is the time interval, and

\begin{equation}
\begin{align*}
\bar{\lambda}' &= \frac{L\tau^2}{m}\lambda', \quad \bar{\mu}' = \frac{L\tau^2}{m}\mu', \quad \bar{\kappa}' = \frac{L\tau^2}{m}\kappa', \\
\bar{\alpha}' &= \frac{\tau^2}{\bar{L}m}\alpha', \quad \bar{\beta}' = \frac{\tau^2}{\bar{L}m}\beta', \quad \bar{\gamma}' = \frac{\tau^2}{\bar{L}m}\gamma', \\
\bar{\xi}' &= \frac{L^2}{m}\xi', \quad \bar{\zeta}' = \frac{L\tau^2}{m}\zeta', \quad \bar{\omega}' = \frac{L\tau}{m}\omega', \\
\bar{\beta}_0 &= \frac{L\tau^2\bar{T}}{m}\beta_0, \quad \bar{C}_0 = \frac{L\tau^2\bar{T}}{m}C_0, \quad \bar{\bar{K}} = \frac{K\tau^3}{\bar{L}m}, \\
\bar{\bar{j}} = \frac{1}{\bar{L}^2}\bar{j}', \quad \bar{\bar{f}} = \frac{\tau^2}{\bar{L}f_0}, \quad \bar{\bar{i}} = \frac{\tau^2}{\bar{L}^2}\bar{i}, \quad (\sigma = s, f), \\
\bar{\bar{h}} &= \frac{\tau^3}{\bar{L}^2}\bar{h}', \quad \bar{f}_i = \frac{\tau^2}{\bar{L}f_i}, \quad \bar{\bar{i}}_i = \frac{\tau^2}{\bar{L}^2}\bar{i}_i, \quad (\bar{\dot{f}}_i = \bar{\dot{f}}_i), \quad (\bar{\dot{f}}_i = \bar{\dot{f}}_i).
\end{align*}
\end{equation}

To these equations we adjoin the boundary conditions

\begin{equation}
\begin{align*}
\bar{u}'_i(\bar{x}, t) &= \bar{u}'_i(\bar{x}, t), \quad \bar{\phi}'_i(\bar{x}, t) = \bar{\phi}'_i(\bar{x}, t), \\
\bar{T}(\bar{x}, t) &= \bar{T}(\bar{x}, t), \quad (\bar{x}, t) \in \partial B \times I,
\end{align*}
\end{equation}

(1.79)
and the initial conditions
\[\begin{align*}
\bar{u}_s^i (\bar{x}, 0) &= \bar{g}_s^i (\bar{x}), &\bar{u}_f^i (\bar{x}, 0) &= 0, &\bar{u}_r^i (\bar{x}, 0) &= \bar{h}_r^i (\bar{x}), \\
\bar{\phi}_s^i (\bar{x}, 0) &= \bar{a}_s^i (\bar{x}), &\bar{\phi}_f^i (\bar{x}, 0) &= 0, &\bar{\phi}_r^i (\bar{x}, 0) &= \bar{b}_r^i (\bar{x}), \\
\bar{T}(\bar{x}, 0) &= \bar{Q}(\bar{x}), &\bar{x} \in \bar{B},
\end{align*}\] (1.80)

where \(\bar{u}_s^\alpha, \bar{\phi}_s^\alpha, \bar{T}^\alpha, \bar{g}_s^\alpha, \bar{h}_s^\alpha, \bar{a}_s^\alpha, \bar{b}_r^\alpha\) and \(\bar{Q}\) are prescribed functions.

To have good boundary conditions, we will suppose that the velocities of the particles are tangents to the boundary, i.e. we impose that \(u_i^{\prime\alpha}_n = 0\). These conditions are sufficient to say that the domain occupied by the mixture is the same for each time. The conditions (1.80)_{2,4} are imposed in view of the substitutions (1.66).

We can have expressions of the constitutive equations in a dimensionless form, such that the equations (1.77) to be written in a similar way as the equations (1.67)-(1.71). Taking into account this remark, we can remove the superposed bar attached to the dimensionless quantities defined above and we can suppose in the following that all the quantities are dimensionless.

We also suppose that the functions \(f_\sigma^\alpha, \ell_\sigma^\alpha, h\) are continuous functions on \(\bar{B} \times I\), and \(\rho_\sigma^0, j_\sigma, C_0\) are positive constants.

We denote by \((P)\) the initial boundary value problem defined by the equations (1.77), the initial conditions (1.80) and the boundary conditions (1.79). We say that \((\bar{u}_s^i, u_f^i, \bar{\phi}_s^i, \bar{\phi}_f^i, \pi_f, \bar{T})\) is a admissible process on \(\bar{B} \times I\) if:

(a) \(u_i^\alpha\) and \(\phi_i^\alpha\) are of class \(C^2\) on \(\bar{B} \times I\);

(b) \(T\) is of class \(C^{2,1}\) on \(\bar{B} \times I\);

(c) \(\pi_f\) is of class \(C^{1,0}\) on \(\bar{B} \times I\).

By a solution of the problem \((P)\) we mean a admissible process which verify the equations (1.77), the initial conditions (1.80) and the boundary conditions (1.79).

## 2 Auxiliary results

In this section we establish a fundamental identity which characterize the conservation of the total energy associated to the solution of the problem \((P)\).

Let us consider a solution \(\{u_i^s, u_i^f, \phi_i^s, \phi_i^f, \pi_f, T\}\) of the problem \((P)\) corresponding to the given data \(I = \{f^\sigma, \ell^\sigma, h; u_i^{\prime\alpha}, \phi_i^{\prime\alpha}, T^\alpha, g_s^\alpha, h_s^\alpha, a_s^\alpha, b_r^\alpha, Q\}\).

We associate with this solution the kinetic energy
\[K(t) = \frac{1}{2} \int_B \sum_{\alpha=s,f} (\rho_0^\alpha u_i^{\alpha}(t) u_i^{\alpha}(t) + \rho_0^\alpha j_i^{\alpha}(t) \phi_i^{\alpha}(t) \phi_i^{\alpha}(t)) dv,\] (2.1)

the internal energy
\[\dot{U}(t) = \int_B \mathcal{E}(t) dv,\] (2.2)

the dissipation energy
\[D(t) = \int_0^t \int_B \Phi(\tau) dv d\tau,\] (2.3)

16
Then, we get
\[ S(t) = \frac{1}{2} \int_B C_0 \frac{T^2(t)}{T_0} dv, \quad (2.4) \]
the total energy
\[ E(t) = K(t) + U(t) + D(t) + S(t), \quad (2.5) \]
and the power function
\[ P(t) = \int_B \left[ \sum_{\sigma=s,f} \left( \rho_0 f^\sigma_i u^\sigma_i + \rho_0 f^\sigma_i \phi^\sigma_i \right) + \rho_0 h \frac{T}{T_0} \right] dv \]
\[ + \int_{\partial B} \left[ \sum_{\sigma=s,f} \left( t^\sigma_i u^\sigma_i + m^\sigma_i \phi^\sigma_i \right) + \frac{T}{T_0} q \right] da. \quad (2.6) \]

**Lemma 2.1 (Energy conservation)** Let \( \{u^s_i, u^f_i, \phi^s_i, \phi^f_i, \pi^f, T\} \) be a solution of the problem \((P)\) corresponding to the loads \( \mathcal{I} = \{f^\sigma_i, \ell^\sigma_i, h; u^s_i, \phi^s_i, T^s, g^s_i, h^s_i, a^s_i, b^s_i, Q\} \). Then, for every time \( t \in I \), we have
\[ E(t) = E(0) + \int_0^t P(s) ds. \quad (2.7) \]

**Proof.** From the relations \((1.77)_1, (1.71)_1\) and by using the divergence theorem, it follows that
\[ \int_B \left[ \rho_0^s u^s_i u^{s^2}_{i,k} u^s_{i,j} + \lambda^s u^s_i u^s_{i,j} + \mu^s u^s_i u^{s^2}_{i,j} + (\mu^s + k^s) u^s_{i,j} u^s_{i,j} \right. \]
\[ + k^s \varepsilon_{ijk} u^s_{i,j} \phi^k + \xi (u^s_i - u^f_i) u^s_i + \frac{\zeta}{T_0} T_i u^s_i - \beta_0 T u^s_{i=i} \left. \right] dv \]
\[ = \int_{\partial B} t^s_j n_j u^s_i dv + \int_B \rho_0^s f^s_i u^s_i dv. \]

On the other hand, in view of \((1.77)_{3,6}\) and \((1.71)_3\) we have
\[ \int_B \left[ \rho_0^f u^f_i u^{f^2}_{i,j} + \mu^f u^{f^2}_{i,j} + (\mu^f + k^f) u^f_{i,j} u^f_{i,j} + k^f \varepsilon_{ijk} u^f_{i,j} \phi^f_k \right. \]
\[ - \xi (u^s_i - u^f_i) u^f_i - \frac{\zeta}{T_0} T_i u^f_i \right] dv = \int_{\partial B} t^f_j n_j u^f_i dv + \int_B \rho_0^f f^f_i u^f_i dv. \quad (2.9) \]

Then, we get
\[ \int_B \left[ \rho_0^s \alpha^s_k \phi^s_{i,j} + \beta^s k^s \phi^s_{i,j} + \gamma^s \phi^s_{i,j} \phi^s_{i,j} \right. \]
\[ - k^s \varepsilon_{ijk} u^s_{i,j} \phi^s_k + 2 k^s \phi^s_k \phi^s_i + \varepsilon (\phi^s_i - \phi^f_i) \phi^s_i \left. \right] dv \quad (2.10) \]
and
\[ \int_{\partial B} m^s_i n_j \phi^s_i dv \]
\[ = \int_{\partial B} m^f_i n_j \phi^f_i dv \]
\[ \left. \right] dv \quad (2.11) \]
\[ = \int_{\partial B} \rho_0^f f^f_i \phi^f_i dv. \]
Finally, we use the relation (1.77) to obtain
\[\int_B \left[ \frac{C_0}{T_0} TT' + \beta_0 T u_i' + \zeta_0 \frac{T}{T_0} (u_i'' - u_i') + \frac{K}{T_0} T_i T_i \right] dv \]
\[= \int_{\partial B} \frac{T}{T_0} \left[ \zeta (u_i'' - u_i') + \frac{K}{T_0} T_i \right] n_i da + \int_B \rho_0 h_0 \frac{T}{T_0} dv. \tag{2.12}\]

Therefore, by summing the relations (2.8)-(2.12) we deduce
\[\frac{d}{dt} E(t) = P(t) \tag{2.13}\]
so that, by integration over \([0, t]\), we obtain the identity (2.7) and the proof is complete.

Let us now introduce the following bilinear forms
\[W_1^s(\xi_{ij}, \eta_{ij}) = \lambda^s \xi_{kk} \eta_{ii} + \mu^s \xi_{ji} \eta_{ij} + (\mu^s + k^s) \xi_{ij} \eta_{ij},\]
\[W_2^s(\xi_{ij}, \eta_{ij}) = \alpha^s \xi_{kk} \eta_{ii} + \beta^s \xi_{ji} \eta_{ij} + \gamma^s \xi_{ij} \eta_{ij},\]
\[W_1^f(\xi_{ij}, \eta_{ij}) = \mu^f \xi_{ji} \eta_{ij} + (\mu^f + k^f) \xi_{ij} \eta_{ij},\]
\[W((\varphi_i, \gamma_i), (\psi_i, \theta_i)) = \xi \varphi_i \psi_i + \zeta (\theta_i \varphi_i + \gamma_i \psi_i) + K \gamma_i \theta_i\]
and let us consider the corresponding quadratic forms. These quadratic forms are positive defined if the inequalities (1.73) and (1.75) are strictly satisfied.

If the above quadratic forms are positive defined, then
\[\sigma^s_m \xi_{ij} \xi_{ij} \leq W_1^s(\xi_{ij}, \xi_{ij}) \leq \sigma^s_M \xi_{ij} \xi_{ij},\]
\[\sigma^f_m \xi_{ij} \xi_{ij} \leq W_1^f(\xi_{ij}, \xi_{ij}) \leq \sigma^f_M \xi_{ij} \xi_{ij},\]
\[W((0, \gamma_i), (0, \gamma_i)) = K \gamma_i \gamma_i,\]
\[c_m(\varphi_i \varphi_i + \gamma_i \gamma_i) \leq W((\varphi_i, \gamma_i), (\varphi_i, \gamma_i)) \leq c_M(\varphi_i \varphi_i + \gamma_i \gamma_i).\]

where
\[\sigma^s_M = \max\{2\mu^s + k^s, k^s, 3\lambda^s + 2\mu^s + k^s\},\]
\[\sigma^s_m = \min\{2\mu^s + k^s, k^s, 3\lambda^s + 2\mu^s + k^s\},\]
\[\delta^s_M = \max\{3\alpha^s + \beta^s + \gamma^s, \gamma^s + \beta^s, \gamma^s - \beta^s\},\]
\[\delta^s_m = \min\{3\alpha^s + \beta^s + \gamma^s, \gamma^s + \beta^s, \gamma^s - \beta^s\},\]
\[\sigma^f_M = \max\{2\mu^f + k^f, k^f\},\]
\[\sigma^f_m = \min\{2\mu^f + k^f, k^f\},\]
\[c_m = \frac{(\xi + K) - \sqrt{(\xi - K)^2 + 4\xi^2}}{2},\]
\[c_M = \frac{(\xi + K) + \sqrt{(\xi - K)^2 + 4\xi^2}}{2}.\]
3 Uniqueness result

In this section we study the uniqueness of the solution of the problem \( \mathcal{P} \) [125].

**Theorem 3.1** Assume that \( \rho_0^s, \rho_0^f, C_0 \) are positive constants and the internal energy density is a semipositive quadratic form in corresponding terms, then two solutions \( \{ u_i^s, \phi_i^s, \phi_i^f, \pi_i^f, T \} \) and \( \{ 2u_i^s, 2\phi_i^s, 2\phi_i^f, 2\pi_i^f, 2T \} \) two solution of the problem \( \mathcal{P} \) corresponding to the some loads are so that

\[
1u_i^s = 2u_i^s, \quad 1\phi_i^s = 2\phi_i^s, \quad 1T = 2T, \quad 1\pi_i^f = 2\pi_i^f + \pi_i^f,
\]

where \( \pi_i^f \) satisfies the relation

\[
\text{grad}\ \pi_i^f = 0. \tag{3.2}
\]

**Proof.** We denote by \( \{ u_i^s, u_i^f, \phi_i^s, \phi_i^f, \pi_i^f, T \} \) the difference between these two solutions. Then \( \{ u_i^s, u_i^f, \phi_i^s, \phi_i^f, \pi_i^f, T \} \) is solution of the problem \( \mathcal{P}_0 \) defined by the system

\[
\rho_0^s u_i^{ss} = (\lambda^s + \mu^s)u_i^{s,j,j} + (\mu^s + k^s)u_i^{s,j} + k^s \varepsilon_{ijk} \phi_k^{s,j} - \xi (u_i^{s} - u_i^{f}) - \left( \frac{\zeta}{T_0} + \beta_0 \right) T_i^s,
\]

\[
\rho_0^f u_i^{sf} = (\alpha^s + \beta^s)\phi_i^{s,j,j} + \gamma^s \phi_i^{s,j} + k^s (\varepsilon_{ijk} u_i^{s,j} - 2 \phi_i^{s}) - \omega (\phi_i^{s} - \phi_i^{f}),
\]

\[
\rho_0^f u_i^{ff} = -\pi_i^s + (\mu^f + k^f)u_i^{f,j,j} + k^f \varepsilon_{ijk} \phi_k^{f,j} + \xi (u_i^{s} - u_i^{f}) + \frac{\zeta}{T_0} T_i^f,
\]

\[
\rho_0^f \phi_i^{ff} = (\alpha^f + \beta^f)\phi_i^{f,j,j} + \gamma^f \phi_i^{f,j} + k^f (\varepsilon_{ijk} u_i^{f,j} - 2 \phi_i^{f}) + \omega (\phi_i^{s} - \phi_i^{f}),
\]

\[
0 = -C_0 T^f + (-\beta_0 T_0 + \zeta)u_i^{s} + \frac{K}{T_0} T_i^{ii},
\]

\[
u_i^{f,j} = 0,
\]

by the boundary conditions

\[
u_i^s = 0, \quad \phi_i^s = 0, \quad T = 0 \quad \text{on} \quad \partial B \times [0, \infty), \quad (\sigma = s, f) \tag{3.4}
\]

and by the initial conditions

\[
u_i^s(x, 0) = 0, \quad \nu_i^s(x, 0) = 0, \quad T(x, 0) = 0, \quad \phi_i^s(x, 0) = 0, \quad \phi_i^s(x, 0) = 0, \quad x \in \bar{B}. \tag{3.5}
\]

Let remark that \( \{ u_i^s, u_i^f, \phi_i^s, \phi_i^f, \pi_i^f, T \} \) is solution of the problem \( \mathcal{P} \) with the loads \( \mathcal{I} = \{ 0, 0, 0; 0, 0, 0; 0, 0, 0 \} \).

Corresponding to these given data we have

\[
P(t) = 0, \quad E(0) = 0,
\]

and thus, in view of the Lemma 2.1, we deduce

\[
E(t) = 0, \quad t \in [0, \infty). \tag{3.6}
\]
The local form of the Clausius-Duhem inequality implies that the dissipation energy density is non-negative. Since $E$ is a positive semidefinite quadratic form, we get
\[
\frac{1}{2} \int_B \left[ \sum_{\sigma=s,f} (\rho_0 u_i^\sigma u_i^\sigma + \rho_0 j_i^\sigma \phi_i^\sigma \phi_i^\sigma) + \frac{C_0 T^2}{T_0} \right] dv = 0, \quad (3.7)
\]
and since $\rho_0^s > 0$, $\rho_0^f > 0$, $C_0 > 0$, we deduce that
\[
u_i^s = 0, \quad \phi_i^s = 0, \quad T = 0, \quad (\sigma = s, f). \quad (3.8)
\]
Using the initial conditions we have
\[
u_i^s = 0, \quad \phi_i^s = 0, \quad T = 0, \quad (\sigma = s, f). \quad (3.9)
\]
By using the relations in (3.3), we deduce the relation (3.2) and the proof is complete.

4 Continuous dependence of solutions

Throughout this section we study the continuous dependence of solution of the problem $(P)$ with respect to the initial and the body loads [125]. To establish an estimate describing the continuous dependence upon the initial data we shall assume that $\{u_i^s, u_i^f, \phi_i^s, \phi_i^f, \pi^f, T\}$ is the solution of the problem $(P)$ with null boundary data and null body loads. For this type of external data system, using Lemma 2.1, we deduce the following result.

**Theorem 4.1** (Continuous dependence upon initial data) Suppose that $\rho_0^s, \rho_0^f, C_0$ are positive constants and the internal energy density is semipositiv quadratic form in corresponding terms. Let $\{u_i^s, u_i^f, \phi_i^s, \phi_i^f, \pi^f, T\}$ be a solution of the problem $(P)$ with the external data system $I = \{0,0,0; 0,0,0; 0,0,0,0\}$. Then, for all $t \in I$ we have
\[
E(t) = E(0). \quad (4.1)
\]

Now we study the continuous data dependence of solution upon the supply terms $\{f_i^s, l_i^s, h\}$.

**Theorem 4.2** (Continuous dependence upon the supply terms) Assume that $\rho_0^s, \rho_0^f, C_0$ are positive constants and the internal energy density is a positive semidefined quadratic form in the corresponding terms. Let $\{u_i^s, u_i^f, \phi_i^s, \phi_i^f, \pi^f, T\}$ be solution of the problem $P$ corresponding to external data system $I = \{f_i^s, l_i^s, h; 0,0,0,0,0,0\}$. Then, for all $t \in I$ we have
\[
[E(t)]^{1/2} \leq \frac{1}{2} \int_0^t g(\tau) d\tau, \quad (4.2)
\]
where
\[
g(s) = \left\{ \int_B \left[ \sum_{\sigma=s,f} (\rho_0^s f_i^\sigma f_i^\sigma + \rho_0^f l_i^\sigma l_i^\sigma) + \frac{\rho_0^2}{C_0 T_0} h^2 \right] dv \right\}^{1/2}. \quad (4.3)
\]
Proof. Under the hypothesis of the theorem, the Lemma 2.1 implies

\[ E(t) = \int_0^t \int_B \left[ \sum_{\sigma=s,f} \left( \rho_0^\sigma f_i^\sigma u_i^\sigma + \rho_0^\sigma f_i^\sigma \phi_i^\sigma \right) + \rho_0^\sigma T \frac{T}{T_0} \right] dv, \quad \forall \ t \geq 0. \]  

(4.4)

By means of the Schwarz inequality we obtain

\[ E(t) \leq \int_0^t \int_B \left\{ \sum_{\sigma=s,f} \left( \rho_0^\sigma u_i^\sigma u_i^\sigma + \rho_0^\sigma \phi_i^\sigma \phi_i^\sigma \right) + \frac{C_0 T^2}{T_0} \right\} \frac{1}{2} g(\tau) d\tau, \quad \forall \ t \geq 0, \]  

(4.5)

where

\[ g(\tau) = \left\{ \int_B \left[ \sum_{\sigma=s,f} \left( \rho_0^\sigma f_i^\sigma f_i^\sigma + \rho_0^\sigma \ell_i^\sigma \ell_i^\sigma \right) + \frac{\rho_0^2}{C_0 T_0} h^2 \right] dv \right\}^{\frac{1}{2}}. \]  

(4.6)

We define the function

\[ \mathcal{Y}(t) = \left[ E(t) \right]^{\frac{1}{2}}. \]  

(4.7)

This function is well defined because the internal energy density and the dissipation potential are non-negative. The inequality (4.5) become

\[ \mathcal{Y}^2(t) \leq \int_0^t \mathcal{Y}(s) g(\tau) d\tau, \quad \forall \ t \geq 0. \]  

(4.8)

By the Brezis lemma (see [270], pp. 47) we deduce the inequality

\[ \mathcal{Y}(t) \leq \frac{1}{2} \int_0^t g(\tau) d\tau, \quad \forall \ t \geq 0, \]

and the proof is complete.

5 Existence and uniqueness

In this section we use the results of the semigroup theory of linear operators to establish an existence theorem for the solution of the problem \((\mathcal{P})\) when the boundary conditions are [126]

\[ u_i^* = 0, \ \phi_i^* = 0, \ T^* = 0. \]  

(5.1)

We suppose that \(B\) is a bounded domain of class \(C^2\) and the internal energy density and the dissipation potential are positive defined quadratic forms in corresponding terms. We denote this problem by \((\tilde{\mathcal{P}})\). Firstly, we will rewrite the initial boundary value problem \((\tilde{\mathcal{P}})\) in an abstract Cauchy problem in a Hilbert space [225, 269].

Let us define

\[ \mathbf{X} = \{ \mathbf{w} = (\mathbf{u}^s, \mathbf{v}^s, \mathbf{\phi}^s, \mathbf{\nu}^s, \mathbf{v}^f, \mathbf{\nu}^f, T); \ \mathbf{u}^s \in \mathbf{H}_0^1(B), \mathbf{v}^s \in \mathbf{L}^2(B), \mathbf{\phi}^s \in \mathbf{H}_0^1(B), \mathbf{\nu}^s \in \mathbf{L}^2(B), \mathbf{v}^f \in \mathbf{H}(B), \mathbf{\nu}^f \in \mathbf{L}^2(B), T \in \mathbf{L}^2(B) \}; \]  

(5.2)

where \(\mathbf{L}^2(B) = [\mathbf{L}^2(B)]^3, \mathbf{H}_0^1(B) = [\mathbf{H}_0^1(B)]^3, \mathbf{H}(B)\) is the closure of the space \(\mathbf{Y} = \{ \mathbf{u} \in \mathbf{C}^\infty_0(B); \ u_i = 0 \text{ in } B, u_n = 0 \text{ in } \partial B \} \) in \(\mathbf{L}^2(B)\) [255] and \(\mathbf{H}_0^1(B)\) is the well known Sobolev space [2].
Because $\Omega$ is open, bounded, connected of class $C^2$, we have (see [255])

$$L^2(B) = H \oplus H_1 \oplus H_2,$$

(5.3)

where $H$, $H_1$, $H_2$ are mutually orthogonal spaces

$$H_1 = \{ u \in L^2(B); \; u = \text{grad} \; p, \; p \in H^1(B), \Delta p = 0 \}$$

(5.4)

and

$$H_2 = \{ u \in L^2(B); \; u = \text{grad} \; q, \; q \in H^1_0(B) \}.$$  

(5.5)

Further, we introduce the operators

$$A^1_s w = v^s_i,$$

$$A^2_i w = \frac{1}{\rho_0} \left[ (\lambda^s + \mu^s) u^s_{i,j} + (\mu^s + k^s) u^s_{i,j} + k^s \varepsilon_{i,j,k} \phi^s_{k,j} - \xi (v^f_i - v^f_j) - \left( \frac{\zeta}{T_0} + \beta_0 \right) T_i \right],$$

$$A^3_i w = \nu^s_i,$$

$$A^4_i w = \frac{1}{\rho_0 j^s} [ (\alpha^s + \beta^s) \phi^s_{i,j} + \gamma^s \phi^s_{i,j} + k^s (\varepsilon_{i,j,k} u^s_{k,j} - 2 \phi^s_i) - w (v^s_i - v^s_j)],$$

(5.6)

$$A^1_i w = \frac{1}{\rho_0} P \left[ (\mu^f + k^f) \nu^f_{i,j} + k^f \varepsilon_{i,j,k} \nu^f_{k,j} + \xi (v^s_i - v^s_j) + \frac{\zeta}{T_0} T_i \right],$$

$$A^2_i w = \frac{1}{\rho_0 j^f} [ \alpha^f \nu^f_{i,j} + \beta^f \nu^f_{i,j} + \gamma^f \nu^f_{i,j} + k^f (\varepsilon_{i,j,k} \nu^f_{k,j} - \nu^f_i) + w (v^s_i - v^s_j)],$$

$$A^3_i w = \frac{T_0}{C_0} \left[ - \beta_0 + \frac{\zeta}{T_0} \right] v^s_{i,i} + \frac{K}{T_0} T_{i,i},$$

where $P : L^2(B) \rightarrow H(B)$ is the Leray Projector.

Let $A$ be the operator

$$A = (A^{s1}, A^{s2}, A^{s3}, A^{s4}, A^{f1}, A^{f2}, A^T)$$

(5.7)

with the domain

$$D(A) = \{ w = (u^s, v^s, \phi^s, \nu^s, v^f, \nu^f, T) \in X; A w \in X, \; v^f = 0, \; \nu^f = 0, \; T = 0 \text{ pe } \partial B \}.$$  

(5.8)

We note that $C^\infty_0(B) \times C^\infty_0(B) \times C^\infty_0(B) \times C^\infty_0(B) \times V(B) \times C^\infty_0(B) \times C^\infty_0(B)$, where $V(B) = H^1_0(B) \cap H^2(B) \cap H(B)$, is a dense subset of $X$ [255] which is contained in $D(A)$.

With the above definitions, the problem $(\tilde{P})$, can be transformed into the following abstract equation in the Hilbert space $X$

$$\frac{dw}{dt}(t) = A w(t) + F(t), \; w(0) = w_0,$$  

(5.9)
where

\[ \mathbf{F}(t) = \left( 0, f^s, 0, \frac{1}{j^s} \mathbf{l}^s, P(f^f), \frac{1}{j^f} \mathbf{l}^f, \frac{\rho_0 T_0}{C_0} \mathbf{h} \right) \]  \hspace{1cm} (5.10)

and

\[ \mathbf{w}_0 = (g^s, h^s, a^s, b^s, h^f, b^f, Q). \]  \hspace{1cm} (5.11)

On \( X \) we define the following inner product

\[
\langle (u^s, v^s, \phi^s, \nu^s, \nu^f, \nu^f, T), (\bar{u}^s, \bar{v}^s, \bar{\phi}^s, \bar{\nu}^s, \bar{\nu}^f, \bar{\nu}^f, \bar{T}) \rangle_X
= \int_B \left\{ \rho_0 v_i^s \bar{v}_i^s + \rho_0 j^s v_i^s \bar{v}_i^s + \rho_0 j^f v_i^f \bar{v}_i^f + \rho_0 j^f v_i^f \bar{v}_i^f + \frac{C_0}{T_0} \bar{T} \right. \\
+ \mathcal{W}_1^s(e_{ij}^s, \bar{e}_{ij}^s) + \mathcal{W}_2^s(\gamma_{ij}^s, \bar{\gamma}_{ij}^s) \right\} dv,
\]

where \( \mathcal{W}_1^s \) and \( \mathcal{W}_2^s \) are the bilinear forms defined by the relations (2.14).

Because the internal energy density is positive defined, we have

\[
\mathcal{M}(e_{ij}^s e_{ij}^s + j^s \gamma_{ij}^s \bar{\gamma}_{ij}^s) \geq \mathcal{E}(u^s, \phi^s) \geq \mathcal{M}(e_{ij}^s e_{ij}^s + j^s \gamma_{ij}^s \bar{\gamma}_{ij}^s),
\]

where

\[
\mathcal{M} = \frac{1}{2} \min \{ \sigma_m^s, \frac{\delta_m^s}{j^s} \}, \quad \mathcal{E} = \frac{1}{2} \max \{ \sigma_M^s, \frac{\delta_M^s}{j^s} \}. \]  \hspace{1cm} (5.14)

On the other hand, we have

\[
e_{ij}^s e_{ij}^s \leq 2u_{ij,i}^s u_{ij,i}^s + 3\phi_k^s \phi_k^s. \]  \hspace{1cm} (5.15)

Using the boundary conditions, as in [39], we deduce that

\[
\int_B e_{ij}^s e_{ij}^s dv \geq \frac{1}{2} \int_B (u_{ij,i}^s u_{ij,i}^s + u_{ij,i}^s u_{ij,i}^s) dv \geq \frac{1}{2} \int_B (u_{ij,i}^s u_{ij,i}^s + u_{ij,i}^s u_{ij,i}^s) dv.
\]  \hspace{1cm} (5.16)

The relations (5.13)-(5.16) give us the following inequalities

\[
a_M \int_B (u_{ij,i}^s u_{ij,i}^s + \phi_k^s \phi_k^s + j^s \phi_{ij,i}^s \phi_{ij,i}^s) dv \geq U(t)
\]

\[
U(t) \geq a_m \int_B (u_{ij,i}^s u_{ij,i}^s + j^s \phi_{ij,i}^s \phi_{ij,i}^s) dv,
\]

where

\[
a_M = 3\mathcal{M}, \quad a_m = \frac{\mathcal{M}}{2}. \]  \hspace{1cm} (5.18)

A direct consequence of the above inequalities is the fact that the norm induced by \( \mathcal{E} \) is equivalent with the usual norm on \( H_0^1(B) \times H_0^1(B) \). Hence \( \langle \ , \ \rangle_X \) defines a norm equivalent to the usual norm on \( X \).

**Lemma 5.1** If the internal energy density is positive defined, then the operator \( A \) is dissipative, i.e.

\[
\langle Aw, w \rangle_X \leq 0, \text{ for all } w \in D(A).
\]  \hspace{1cm} (5.19)
Proof. First of all we note that, in view of the relation (5.3), we can find \( p \in H^1(B) \) with \( \Delta p = 0 \) and \( q \in H^1_0(B) \) so that
\[
\begin{align*}
P \left[ (\mu^f + k^f) v^f_{i,j,j} + k^f \varepsilon_{ijk} v^f_{k,j} + \xi (v^s_i - v^f_i) + \frac{\zeta}{T_0} T_{,i} \right] \\
= (\mu^f + k^f) v^f_{i,j,j} + k^f \varepsilon_{ijk} v^f_{k,j} + \xi (v^s_i - v^f_i) + \frac{\zeta}{T_0} T_{,i} - p,_{i} - q,_{i}.
\end{align*}
\]
(5.20)

Using the divergence theorem and the boundary conditions we find that
\[
\langle A w, w \rangle_X = \int_B \left\{ v^s_i (t^s_{ji,j} + \widehat{p}^s_i) + v^s_i (m^s_{ji,j} + \varepsilon_{ijk} t^s_{jk} + \widehat{m}^s_i) \\
+ v^f_i (t^f_{ji,j} + \widehat{p}^f_i - p,_{i} - q,_{i}) + v^f_i (m^f_{ji,j} + \varepsilon_{ijk} t^f_{jk} + \widehat{m}^f_i) \\
+ T \left[ (-\beta_0 + \frac{\zeta}{T_0}) v^s_{i,i} + \frac{K}{T_0^2} T_{,ii} \right] + \mathcal{W}^s_1 (e_{ij}, a^s_{ij}) + \mathcal{W}^s_2 (\gamma^s_{ij}, b^s_{ij}) \right\} dv \\
= -\int_B \Phi(v^s, \nu^s, v^f, \nu^f, T) dv.
\]
(5.21)

Thus, using the Clausius-Duhem inequality we obtain the relation (5.19) and the proof is complete.

Lemma 5.2 If the internal energy density and the dissipation potential are positive defined, then the operator \( A \) satisfies the condition
\[
\text{Range}(I - A) = X.
\]
(5.22)

Proof. Let us consider \( w^* = (u^s, v^s, \phi^s, v^f, \nu^f, T^*) \in X \). We must show that the equation
\[
w - A w = w^*,
\]
(5.23)
has a solution in \( \mathcal{D}(A) \). From the definition of the operator \( A \) we obtain the system
\[
\begin{align*}
u^s_i - A^s_1 w &= u^s_i, \\
\phi^s_i - A^s_2 w &= \phi^s_i, \\
v^f_i - A^f_1 w &= \nu^f_i, \\
T - A^T_1 w &= T^*.
\end{align*}
\]
(5.24)

By eliminating the functions \( \nu^s_i \) and \( v^s_i \), we obtain for the determination of the functions
We study this system in the following Hilbert space

We introduce the bilinear form

\[
L_i^{s1} y = u_i^s - \frac{1}{\rho_0} \left[ (\lambda^s + \mu^s) u_{j,ij}^s + (\mu^s + \kappa^s) u_{i,jj}^s + \kappa^s \varepsilon_{ijk} \phi_{k,j}^s \right]
\]

\[-\xi (u_i^s - v_i^f) - \left( \frac{\zeta}{T_0} + \beta_0 \right) T_i] = g_i^{s1},
\]

\[
L_i^{s2} y = \phi_i^s - \frac{1}{\rho_0^j} \left[ (\alpha^s + \beta^s) \phi_{j,ij}^s + \gamma^s \phi_{i,jj}^s + \kappa^s \varepsilon_{ijk} u_k^s - 2\phi_i^s \right]
\]

\[-C_0 \chi(\phi_i^s - \nu_i^f) = g_i^{s2},
\]

\[
L_i^{f1} y = v_i^f - \frac{1}{\rho_0} \left[ -p_i - q_i + (\mu^f + \kappa^f) v_{i,jj}^s + \kappa^f \varepsilon_{ijk} v_{k,j}^s \right]
\]

\[+ \xi (u_i^s - v_i^f) + \frac{\zeta}{T_0} T_i] = g_i^{f1},
\]

\[
L_i^{f2} y = v_i^f - \frac{1}{\rho_0^j} \left[ (\alpha^f + \beta^f) v_{j,ij}^f + \gamma^f v_{i,jj}^f + \kappa^f \varepsilon_{ijk} v_{k,j}^f - 2\nu_i^f \right]
\]

\[+ C_0 \chi(\phi_i^s - \nu_i^f) = g_i^{f2},
\]

\[
L^T y = T - \frac{T_0}{C_0} \left[ \left( -\beta_0 + \frac{\zeta}{T_0} \right) u_{i,i} + \frac{K}{T_0^2} T_{i,i} \right] = g^{*T},
\]

where \( y = (u^s, \phi^s, v^f, \nu^f, T) \), \( p \in H^1(B) \) with \( \Delta p = 0 \), \( q \in H_0^1(B) \) and

\[
g_i^{s1} = u_{i,i}^s + \frac{\zeta}{\rho_0} u_i^s, \quad g_i^{s2} = u_i^s + \frac{C_0}{\rho_0^j} \phi_i^s,
\]

\[
g_i^{f1} = v_{i,j}^f + \frac{\zeta}{\rho_0} v_i^f, \quad g_i^{f2} = v_i^f + \frac{C_0}{\rho_0^j} \phi_i^s,
\]

\[
g^{*T} = T - \frac{T_0}{C_0} \left( -\beta_0 + \frac{\zeta}{T_0} \right) u_i^s.
\]

We study this system in the following Hilbert space

\[
Z = (H_0^1(B) \cap H^2(B)) \times (H_0^1(B) \cap H^2(B)) \times V(B) \times
\]

\[
\times (H_0^1(B) \cap H^2(B)) \times (H_0^1(B) \cap H^2(B)).
\]

We introduce the bilinear form \( B : Z \times Z \to \mathbb{R} \)

\[
B(y, \tilde{y}) = \left\langle (L_i^{s1} y, L_i^{s2} y, L_i^{f1} y, L_i^{f2} y, L_i^T y), \right.
\]

\[
\left. \left( \rho_0^s \tilde{u}_i^s, \rho_0^j \tilde{\phi}_i^s, \rho_0^f \tilde{v}_i^f, \rho_0^f \tilde{u}_i^f, \frac{C_0}{\rho_0^j} \tilde{T}_i \right) \right\rangle_{L^2 \times L^2 \times L^2 \times L^2 \times L^2}
\]

\[
= \int_B \left\{ \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left[ \rho_0^s u_i^s \tilde{u}_i^s + \rho_0^j \phi_i^s \tilde{\phi}_i^s + \rho_0^f v_i^f \tilde{v}_i^f + \rho_0^f \tilde{u}_i^f + \frac{C_0}{\rho_0^j} \tilde{T}_i \right] dv \right. \]

\[\left. + \mathcal{W} \left( \left( u_i^s - v_i^f, T_i \right), \left( \tilde{u}_i^s - \tilde{v}_i^f, \tilde{T}_i \right) \right) \right\} dv,
\]

\[
\mathcal{W}_1(a_i^j, \tilde{a}_i^j) + \mathcal{W}_2(b_i^j, \tilde{b}_i^j) + \mathcal{W}_1(e_i^s, \tilde{e}_i^s) + \mathcal{W}_2(\gamma_i^s, \tilde{\gamma}_i^s)
\]
where \( \tilde{y} = (\tilde{u}^s, \tilde{\phi}^s, \tilde{v}^f, \tilde{\nu}^f, \tilde{T}) \) and the linear operator \( l : \mathbf{Z} \rightarrow \mathbb{R} \)

\[
    l(\tilde{y}) = \left\langle (g^{*s1}, g^{*s2}, g^{*f1}, g^{*f2}, g^T), \left( \rho_0 \tilde{u}^s, \rho_0 \tilde{\phi}^s, \rho_0 \tilde{v}^f, \rho_0 \tilde{\nu}^f, \frac{C_0}{T_0} \tilde{T} \right) \right\rangle_{L^2 \times L^2 \times L^2 \times L^2 \times L^2}
\]

\[
= \int_B \left\{ \rho_0^s \left( v_{i}^{s*} + \frac{\xi}{\rho_0^s} u_{i}^{s*} \right) \tilde{v}_{i}^s + \rho_0^j \left( v_{i}^{j*} + \frac{\nu}{\rho_0^j} \phi_{i}^{j*} \right) \tilde{\phi}_{i}^s + \rho_0^f \left( v_{i}^{f*} - \frac{\nu}{\rho_0^f} \phi_{i}^{f*} \right) \tilde{\nu}_{i}^f \right\} dv + \rho_0^f \int_B \left\{ \frac{C_0}{T_0} \left[ T^* - \frac{T_0}{C_0} \left( -\beta_0 + \frac{\zeta}{T_0} u_{i}^s \right) \tilde{T} \right] \right\} dv.
\]

(5.30)

For the boundary conditions considered in this section, Chirita [39] (see the Eqs. (3.8) and (3.15) of this paper) proved that

\[
    \int_B a^f_{i,j} a^f_{i,j} dv \geq \frac{1}{2} \int_B v^f_{i,j} v^f_{i,j} dv,
\]

(5.31)

and because we have

\[
a^f_{i,j} a^f_{i,j} \leq 2v^f_{i,j} v^f_{i,j} + 3v^f_{i,j},
\]

(5.32)

we will have the inequalities

\[
b_M \int_B (v^f_{i,j} v^f_{i,j} + v^f_{i,j} v^f_{i,j} + j^f v^f_{i,j} v^f_{i,j}) dv \geq \int_B \{ W_1^f (a^f_{i,j}, a^f_{i,j}) + W_2^f (b^f_{i,j}, b^f_{i,j}) \} dv,
\]

(5.33)

\[
\int_B \{ W_1^f (a^f_{i,j}, a^f_{i,j}) + W_2^f (b^f_{i,j}, b^f_{i,j}) \} dv \geq b_m \int_B (v^f_{i,j} v^f_{i,j} + j^f v^f_{i,j} v^f_{i,j}) dv,
\]

where

\[
b_M = 3 \max \{ \sigma_M, \delta_M^f \}, \quad b_m = \frac{1}{2} \min \{ \sigma_m, \delta_m^f \}.
\]

(5.34)

Moreover, in view of (2.15), we have

\[
c_m^f[a(v^s_i - v^f_i)(v^s_i - v^f_i) + b \frac{T_i T_j}{T_0^2}] \leq W(\xi, \xi),
\]

(5.35)

\[
W(\xi, \xi) \leq c_M^f[a(v^s_i - v^f_i)(v^s_i - v^f_i) + b \frac{T_i T_j}{T_0^2}],
\]

where \( \xi = (a(v^s_i - v^f_i), b \frac{T_i}{T_0}) \), \( a = \sqrt{\frac{\mu^f}{K}} \), \( b = \frac{\sqrt{K \mu^f}}{\mu^s} \). Similar inequalities follows for the quantity \( W(\xi, \xi) \).

From the above relations, the Schwarz inequality and the Poincaré inequality we see that \( B \) is bounded, i.e.

\[
B(y, \tilde{y}) \leq M_1 \| y \|_Z \| \tilde{y} \|_Z, \quad M_1 = \text{positive constant}
\]

(5.36)
and coercive
\[ \mathcal{B}(\mathbf{y}, \mathbf{y}) \geq M_2 \| \mathbf{y} \|_{Z}^2, \quad M_2 = \text{positive constant} \] (5.37)
while the linear operator \( l \) is bounded
\[ l(\tilde{\mathbf{y}}) \leq M_3 \| \tilde{\mathbf{y}} \|_{Z}, \quad M_3 = \text{positive constant}. \] (5.38)

Using the Lax-Milgram theorem we prove the existence of solution of the system (5.24) in \( Z \) and, as a consequence, we have the solution of system (5.25). The proof is complete.

**Theorem 5.1** If the internal energy density and the dissipation potential are positive quadratic forms in the corresponding terms, then the operator \( A \) definite by (5.7) generates a \( C_0 \) contractive semigroup in \( X \).

The proof follows using the previous lemmas and the Lumer–Phillips corollary to Hille–Yosida theorem [225].

**Theorem 5.2** Assume that \( f_i^s, l_i^s, f_i^f, l_i^f, h \in C^1([0, t_1); L^2(B)), \ w_0 \in \mathcal{D}(A) \) and the internal energy density and the dissipation potential are positive defined quadratic forms in corresponding terms. Then, there exist a uniqueness solution \( w \in C^1((0, t_1); X) \cap C^0([0, t_1); \mathcal{D}(A)) \) of the Cauchy problem (5.9).

The proof result from the results concerning the abstract Cauchy problem [225], [269].

**Corollary 5.1** In the hypothesis of Theorem 5.2 we have the following estimate

\[
\| w(t) \|_X \leq \| w_0(t) \|_X + M_4 \int_0^t (\| f^s(\tau) \|_{L^2(B)} + \| l^s(\tau) \|_{L^2(B)}) \\
+ \| P f^f(\tau) \|_{L^2(B)} + \| h(\tau) \|_{L^2(B)}) d\tau,
\] (5.39)

where \( M_4 \) is a positive constant.

For the proof of this Corollary we use the fact that the semigroup generated by \( A \) is contractive.

### 6 Lagrange-Brun integral identities

In this section we establish some *identities of Lagrange-Brun type* [25] that are essential in investigation the problem of asymptotic partition of total energy.

The considered problem is the problem (\( P \)) in the absence of the body forces, the body couples and energy source. We denote this problem by (\( P_0 \)).

If \( f \) is a continuous function on \( B \times [0, \infty) \), then we denote by \( \bar{f} \) the function definite by

\[ \bar{f}(x, t) = \int_0^t f(x, s) ds, \quad (x, t) \in B \times [0, \infty). \] (6.1)
We associate with the solution of the problem \((\mathcal{P}_0)\) the function

\[
I(t) = \frac{1}{2} \int_B \sum_{\sigma=s,f} \left( \rho_0^\sigma u_i^\sigma(t) u_i^\sigma(t) + \rho_0^\sigma j^\sigma \phi_i^\sigma(t) \phi_i^\sigma(t) \right) dv
\]

\[
+ \int_0^t \int_B [\Phi(u^u, \phi^u, u^f, \phi^f, \tilde{T})(\tau) - 2 \frac{\zeta}{T_0} (u_i^s(\tau) - u_i^f(s)) \tilde{T}(\tau)] dv d\tau.
\]

In view of the conservation of the total energy (2.7) for the problem \((\mathcal{P}_0)\) we have

\[
E(t) = E(0) \text{ for all } t \in [0, \infty).
\]

Using the notation (6.1), we can write the energy equation in the form

\[
\rho_0 T_0 \eta = \tilde{q}_{i,i} + \rho_0 T_0 \eta(0),
\]

where

\[
\tilde{q}_i(t) = \zeta(u_i^s(t) - u_i^f(t)) + \frac{K}{T_0} \tilde{T}_i(t) - \zeta g_i^s.
\]

**Lemma 6.1** Let \(\{u^u, u^f, \phi^u, \phi^f, \pi^f, T\}\) be solution of the problem \((\mathcal{P}_0)\). Then, for all \(t \in [0, \infty)\), we have

\[
\frac{d}{dt} I(t) = \frac{1}{2} \int_B \left\{ \left[ \rho_0^u \dot{u}_i^u(2t) g_i^u + \sum_{\sigma=s,f} \rho_0^\sigma u_i^\sigma(2t) h_i^\sigma \right.ight.
\]

\[
+ \rho_0^u j^u \dot{\phi}_i^u(2t) a_i^u + \sum_{\sigma=s,f} \rho_0^\sigma j^\sigma \phi_i^\sigma(2t) b_i^\sigma \left. \right] + \mathcal{W}(g_i^u, 0, (u_i^u(2t) - u_i^f(2t), 0)) + \omega a_i^u(\phi_i^u(2t) - \phi_i^f(2t)) \right\} dv
\]

\[
+ \frac{1}{2} \int_0^t \int_B \left\{ 2 \frac{\zeta}{T_0} \left[ T_i(t + \tau)(u_i^s(t - \tau) - u_i^f(t - \tau))ight.ight.
\]

\[
- T_i(t - \tau)(u_i^s(t + \tau) - u_i^f(t + \tau)) \left. \right] - \rho_0 \eta(0)(T(t + \tau) - T(t - \tau))
\]

\[
+ \frac{\zeta}{T_0} g_i^u(T_i(t - \tau) - T_i(t + \tau)) \right\} dv d\tau.
\]

**Proof.** Let us introduce the notation

\[
L(t, \tau) = \sum_{\sigma=s,f} \left[ t_{\gamma_i}^\sigma(t) u_i^\gamma(\tau) + m_{\gamma_i}^\sigma(t) \phi_i^\gamma(\tau) \right.
\]

\[
- \tilde{p}_i^\gamma(t) u_i^\gamma(\tau) - \varepsilon_{ijk} t_{\gamma_j}^\sigma(t) \phi_i^\gamma(\tau) - \tilde{m}_i^\gamma(t) \phi_i^\gamma(\tau) - \rho_0 \eta(t) T(\tau). \]
Using the constitutive equations, we have

\[
L(t + \tau, t - \tau) - L(t - \tau, t + \tau) \\
= \mathcal{W}_1^f((v^f_{i,j}(t + \tau), v^f_i(t + \tau)), (u^f_{i,j}(t - \tau), \phi^f_i(t - \tau))) \\
- \mathcal{W}_1^f((v^f_{i,j}(t - \tau), v^f_i(t - \tau)), (u^f_{i,j}(t + \tau), \phi^f_i(t + \tau))) \\
+ \mathcal{W}_2^f(v^f_{i,j}(t + \tau), \phi^f_i(t - \tau)) - \mathcal{W}_2^f(v^f_{i,j}(t - \tau), \phi^f_i(t + \tau)) \\
+ \mathcal{W}((v^f_i - v^f_i)(t + \tau), T_i(t + \tau)), (u^f_i - u^f_i)(t - \tau), 0) \\
- \mathcal{W}((v^f_i - v^f_i)(t - \tau), T_i(t - \tau)), (u^f_i - u^f_i)(t + \tau), 0)) \\
+ \mathcal{W}((v^s_i - v^s_i)(t + \tau)(\phi^s_i - \phi^f_i)(t - \tau) \\
- (v^s_i - v^s_i)(t - \tau)(\phi^s_i - \phi^f_i)(t + \tau)] \\
- (\pi^f(t + \tau)u^f_{i,i}(t - \tau) - \pi^f(t - \tau)u^f_{i,i}(t + \tau)).
\]

(6.8)

On the other hand, in view of the equations of motion, we obtain

\[
L(t + \tau, t - \tau) - L(t - \tau, t + \tau) \\
= \sum_{\sigma = s, f} \left( t^\sigma_{j,i}(t + \tau)u^\sigma_i(t - \tau) - t^\sigma_{j,i}(t - \tau)u^\sigma_i(t + \tau) \\
+ m^\sigma_{j,i}(t + \tau)\phi^\sigma_i(t - \tau) - m^\sigma_{j,i}(t - \tau)\phi^\sigma_i(t + \tau) \right)_j \\
- \left( \tilde{q}_i(t + \tau)\frac{T}{T_0}(t - \tau) - \tilde{q}_j(t - s)\frac{T}{T_0}(t + \tau) \right)_j \\
- \sum_{\sigma = s, f} \left( \rho_0^\sigma u^\sigma_{i,i}(t + \tau)u^\sigma_i(t - \tau) - \rho_0^\sigma u^\sigma_{i,i}(t - \tau)u^\sigma_i(t + \tau) \\
+ \rho_0^\sigma \phi^\sigma_i(t + \tau)\phi^\sigma_i(t - \tau) - \rho_0^\sigma \phi^\sigma_i(t - \tau)\phi^\sigma_i(t + \tau) \right) \\
+ \zeta(u^\sigma_i(t + \tau) - u^\sigma_i(t + \tau))\frac{T_i}{T_0}(t - \tau) \\
- \zeta(u^\sigma_i(t - \tau) - u^\sigma_i(t - \tau))\frac{T_i}{T_0}(t + \tau) \\
+ \frac{K}{T^2}(\tilde{T}_i(t + \tau)T_i(t - \tau) - \tilde{T}_i(t - \tau)T_i(t + \tau)) \\
- \rho_0\eta(0)(T(t - \tau) - T(t + \tau)).
\]

(6.9)

Let remark that

\[
u^f_{i,j}(0) = 0 \quad \text{and} \quad \phi^f_{i,j}(0) = 0.
\]

(6.10)

We can verify that for every two functions \(f, h \in C^2([0, \infty))\), the following relations hold

\[
\int_0^t f''(t + \tau)h(t - \tau)d\tau = f'(2t)h(0) - f(t)'h(t) + \int_0^t h'(t - \tau)f'(t + \tau)d\tau,
\]

\[
\int_0^t f(t + \tau)h'(t - \tau)d\tau = -h(0)f(2t) + f(t)h(t) + \int_0^t f'(t + \tau)h(t - \tau)d\tau.
\]

Using the above relations, by an integration over \(B \times [0, t]\), and then by using the divergence theorem and the above relations, we obtain the identity (6.6) and the proof is complete.
Lemma 6.2 Suppose that \( \{u^s, u^f, \phi^s, \phi^f, \pi^f, T\} \) is solution of the problem \((P_0)\). Then for every \( t \in [0, \infty) \), the following identity holds

\[
\frac{dI}{dt}(t) = \frac{dI}{dt}(0) + 2 \int_0^t (K(\tau) - U(\tau) - S(\tau)) d\tau \\
+ \int_0^t \int_B \left[ \frac{\zeta}{T_0} g_i^s T_i(\tau) - 2 \frac{\zeta}{T_0} (u_i^s(\tau) - u_i^f(\tau)) T_i(\tau) + \eta_0 T(\tau) \right] dv d\tau.
\]

(6.11)

Proof. We introduce the following notation

\[
H(t) = \sum_{\sigma=s,f} \left( t_{ij}^\sigma(t) u_{ij}^\sigma(t) + m_{ij}^\sigma(t) \phi_{ij}^\sigma(t) \right) - \tilde{p}_i^\sigma(t) u_i^\sigma(t) - \varepsilon_{ijk} t_{jk}^\sigma(t) \phi_{ij}^\sigma(t) - \tilde{m}_i^\sigma(t) \phi_i^\sigma(t) + \rho_0 \eta(t) T(t).
\]

In view of constitutive equations, we deduce that

\[
H(t) = 2E(t) + C_0 T^2(t) + \frac{1}{2} \frac{d}{dt} \Phi(u^s, \phi^s, u^f, \phi^f, 0) \\
+ \frac{\zeta}{T_0} T_i(t)(u_i^s(t) - u_i^f(t)) - \pi^f u_{i,i}^f(t).
\]

(6.13)

By using the equations of motion, the incompressibility condition and the energy equation, we obtain

\[
H(t) = \sum_{\sigma=s,f} \left( t_{ij}^\sigma(t) u_{ij}^\sigma(t) + m_{ij}^\sigma(t) \phi_{ij}^\sigma(t) + \tilde{q}_j(t) \frac{T}{T_0}(t) \right)_{,j} \\
+ \sum_{\sigma=s,f} \left( \rho_0^s v_i^\sigma(t) v_i^\sigma(t) + \rho_0^s j^\sigma v_i^\sigma(t) \nu_i^\sigma(t) \right) \\
- \frac{d}{dt} \sum_{\sigma=s,f} \left( \rho_0^s u_i^\sigma(t) v_i^\sigma(t) + \rho_0^s j^\sigma \phi_i^\sigma(t) \nu_i^\sigma(t) \right) \\
- \frac{\zeta}{T_0} (u_i^s(t) - u_i^f(t)) T_i(t) - \frac{K}{T_0^2} \tilde{T}_i(t) T_i(t) \\
+ \frac{\zeta}{T_0} g_i^s T_i(t) + \rho_0 \eta(0) T(t).
\]

(6.14)

Thus, in view of the relations (6.13) and (6.14), using the divergence theorem we have

\[
\frac{d}{dt} \int_B \left\{ \sum_{\sigma=s,f} \left( \rho_0^s v_i^\sigma(t) u_i^\sigma(t) + \rho_0^s j^\sigma \nu_i^\sigma(t) \phi_i^\sigma(t) \right) \\
+ \frac{1}{2} \Phi(u^s, \phi^s, u^f, \phi^f, 0) + \frac{1}{2} \frac{K}{T_0^2} \tilde{T}_i(t) \tilde{T}_i(t) \right\} dv \\
= 2K(t) - 2U(t) - 2S(t) \\
- \int_B \left[ 2 \frac{\zeta}{T_0} (u_i^s(t) - u_i^f(t)) T_i(t) - \frac{\zeta}{T_0} g_i^s T_i(t) \\
- \rho_0 \eta(0) T(t) - \pi^f u_{i,i}^f(t) \right] dv,
\]

(6.15)

and using the incompressibility condition, the proof is complete.
7 The asymptotic partition

In this section we study the asymptotic partition of energy for the solution of the problem \((P_0)\) [129]. If \(\{u^s, u^f, \phi^s, \phi^f, \pi^f, T\}\) is the solution of the problem \((P_0)\), for the different energies, we introduce the following Cesàro means

\[
K_c(t) = \frac{1}{t} \int_0^t K(\tau)d\tau, \quad U_c(t) = \frac{1}{t} \int_0^t U(\tau)d\tau,
\]
\[
D_c(t) = \frac{1}{t} \int_0^t D(\tau)d\tau, \quad S_c(t) = \frac{1}{t} \int_0^t S(\tau)d\tau.
\]

In what follow, using the identities (6.3), (6.6) and (6.11), we establish some relations that describe the asymptotic behavior of these Cesàro mean energies.

**Theorem 7.1** Let \(\{u^s, u^f, \phi^s, \phi^f, \pi^f, T\}\) be solution of the initial-boundary value problem \((P_0)\). For all choices of initial data so that \((g^s, h^s, a^s, b^s, h^f, b^f, Q) \in D(A)\), where \(A\) is the operator defined by (5.7), we have

\[
\lim_{t \to \infty} S_c(t) = 0,
\]
\[
\lim_{t \to \infty} K_c(t) = \lim_{t \to \infty} U_c(t),
\]
\[
\lim_{t \to \infty} D_c(t) = E(0) - 2 \lim_{t \to \infty} K_c(t).
\]

**Proof.** From (6.3), because \(\{u^s, u^f, \phi^s, \phi^f, \pi^f, T\}\) is the solution of the problem \((P_0)\), we have

\[
K_c(t) + U_c(t) + D_c(t) + S_c(t) = E(0), \quad t \geq 0.
\]

On the other hand, the relations (6.6) and (6.11) imply

\[
K_c(t) - U_c(t) - S_c(t) = -\frac{1}{2t} \frac{dI}{dt}(0) + \frac{1}{4t} \int_B \left\{ \rho_0^s u^s_i(2t) g_i^s + \sum_{\sigma=s,f} \rho_0^\sigma \mu^\sigma_i(2t) h_i^\sigma + \rho_0^s \phi_i^s(2t) a_i^s + \sum_{\sigma=s,f} \rho_0^\sigma j^\sigma_i(2t) b_i^\sigma + W((g_i^s,0),(u_i^s(2t) - u_i^f(2t),0)) + \Box a_i^s(\phi_i^s(2t) - \phi_i^f(2t)) \right\} dv + \frac{1}{4t} \int_0^t \int_B \left\{ 2 \frac{\zeta}{T_0} \left[ T_i(t+\tau)(u_i^s(t+\tau) - u_i^f(t+\tau)) - T_i(t)(u_i^s(t) - u_i^f(t)) \right] - \rho_0 \eta(0)(T(t+\tau) - T(t) + 2T(\tau)) - \rho_0 \eta(0)(T(t+\tau) - T(t) + 2T(\tau)) \right\} dv d\tau.
\]
Thus, in view of relations (7.6)–(7.8), we have

Moreover, we have

When \( t \) goes to infinity, we deduce the relation (7.2)\(_1\).

From the Schwarz’s inequality, the Poincaré inequality and the relations (5.35), (2.7), we deduce

Thus, in view of relations (7.6)–(7.8), we have

In a similar way, we can prove that

Further, using the Schwarz’s inequality for an appropriate bilinear form defined by the relations (2.14), we have

\[
\int_B \left\{ -2 \frac{dI}{dt}(0) \right\} \, dv 
= \int_0^{2t} \int_B \{ (\mathcal{W}(g_i^s, 0), (u_i^s(2t) - u_i^f(2t), 0)) + \nabla a_i^s(\phi_i^s(2t) - \phi_i^f(2t)) \} \, dv \\
-2 \int_B (\rho_0^s h_i^s + \rho_0^s h_i^s b_i^s) \, dv \\
\leq (2t \int_B \Phi(g_i^s, a_i^s, 0, 0, 0) \, dv + 2E^{1/2}(0) \left( \int_B \rho_0^s a_i^s a_i^s \, dv \right)^{1/2} + 2E^{1/2}(0) \left( \int_B \rho_0^s a_i^s a_i^s \, dv \right)^{1/2} 
\]
\[
E^{1/2}(0) \left\{ (2t)^{1/2} \left[ \xi \int_B (h^s_i - h^f_i)(h^s_i - h^f_i) dv + \varpi \int_B (b^s_i - b^f_i)(b^s_i - b^f_i) dv \right] \right. \\
+ 2 \left[ \left( \int_B \rho_0 g^s_i g^s_i dv \right)^{1/2} + \left( \int_B \rho_0 g^s_i g^s_i dv \right)^{1/2} \right].
\]

We have used that the incompressibility condition implies
\[
u_{ij}^l(t) = 0 \tag{7.11}
\]
and in consequence, using the divergence theorem, the Schwarz’s inequality, the Poincaré inequality, and the inequality
\[(a + b + c)^2 \leq 3(a^2 + b^2 + c^2) \text{ for all } a, b, c \in \mathbb{R}, \tag{7.12}\]
we obtain
\[
\int_0^t \int_B 2 \frac{\zeta}{T_0} T_{ij}(t + \tau)(u^s_i(t + \tau) - u^f_i(t + \tau)) dv d\tau \\
\leq 2 \frac{\zeta}{T_0} \left( \int_0^t \int_B T^2(\tau) dv d\tau \right)^{1/2} \left( \int_0^t \int_B u^s_i(t + \tau) u^s_{j} j (t + \tau) dv d\tau \right)^{1/2} \tag{7.13}
\]
where
\[
M = 2\zeta \left( \frac{3c_1}{b a_m} \right)^{1/2} E(0). \tag{7.14}
\]
Similarly, we obtain the estimates
\[
\int_0^t \int_B 2 \frac{\zeta}{T_0} T_{ij}(t - \tau)(u^s_i(t - \tau) - u^f_i(t - \tau)) dv d\tau \leq Mt^{1/2}, \\
\int_0^t \int_B 2 \frac{\zeta}{T_0} T_{ij}(t - \tau)(u^s_i(t - \tau) - u^f_i(t - \tau)) dv d\tau \leq Mt^{1/2}, \\
\frac{\zeta}{T_0} \int_0^t \int_B g^s_i T_{ij}(t - \tau) - T_{ij}(t + \tau) - 2T_{ij}(\tau) dv ds \\
\leq 4 \frac{\zeta}{b^{1/2}} E^{1/2}(0) t^{1/2} \left( \int_B g^s_i g^s_i dv \right)^{1/2} \tag{7.15}
\]
and
\[
\int_0^t \int_B \rho_0 \eta(0) (T(t + \tau) - T(t - \tau) + 2T(\tau)) dv d\tau \\
\leq 4t^{1/2} E(0) \text{vol}(B) \sup_{x \in B} |\rho_0 \eta(0)|^{1/2} T_0 c_1^{1/2} B^{1/2}. \tag{7.16}
\]
Using the above estimates we deduce (7.2)2,3, and the proof is complete.

8 Stability of the thermodynamical process

Let us consider two solutions \( \mathbf{w}^* = (\mathbf{u}^{**}, \phi^{**}, \mathbf{v}^{**}, \nu^{**}, T^{**}, \pi^{**}) \) and \( \mathbf{w}^* + \mathbf{w} = (\mathbf{u}^{**} + \mathbf{u}^*, \phi^{**} + \phi^*, \mathbf{v}^{**} + \mathbf{v}^*, \nu^{**} + \nu^*, T^{**} + T, \pi^{**} + \pi^*) \) of the problem \((\mathcal{P})\), corresponding to the same boundary data, the same body forces, body couples and heat source and the following initial data \( \mathbf{g}^{**}, \mathbf{h}^{**}, \mathbf{a}^{**}, \mathbf{b}^{**}, r^* \) and \( \mathbf{g}^{**} + \mathbf{g}^*, \mathbf{h}^{**} + \mathbf{h}^*, \mathbf{a}^{**} + \mathbf{a}^*, \mathbf{b}^{**} + \mathbf{b}^*, r^* + r \), respectively.
Because the both admissible processes are solutions of the equations (1.77), by substrac-
tion, we deduce that the perturbation \( \mathbf{w} = (u^s, \phi^s, v^f, \nu^f, T, \pi^f) \) is solution of the problem

\[
\mathbf{w}_0 = (g^s, a^s, h^s, b^s, h^f, b^f, r).
\]

Let remark that the norm induced by the inner product (5.12), defined on the Hilbert space

\[ X \]

and it is a equivalent norm with the usual norm on the space \( X \).

**Definition 8.1** [118] We say that the thermodynamical process is energetic stable if

\[
\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 : \| \mathbf{w}_0 \|_X < \delta(\varepsilon) \Rightarrow \| \mathbf{w} \|_X < \varepsilon. \quad (8.2)
\]

**Definition 8.2** [118] We say that the thermodynamical process is energetic asymptotic stable

if it is energetic stable and

\[
\exists \gamma \in (0, \infty] : \| \mathbf{w}_0 \|_X < \gamma \Rightarrow \lim_{t \to \infty} \| \mathbf{w} \|_X = 0. \quad (8.3)
\]

If \( \gamma = \infty \) then we say that the thermodynamical process is unconditionally energetic asymptotic stable.

**Remark 8.1** The conservation of the total energy given by the Lemma 2.1, proves that the
thermodynamical process is energetic stable in terms of the norm defined above.

The main result of this section proves that the thermodynamical process is in fact uncondition-
ally energetic asymptotic stable [131].

From (5.33) and (5.35) we have

\[
\int_B \Phi dv \geq \int_B \left\{ b_m(v_{i,j}^f v_{i,j}^f + j^f v_{i,j}^f v_{i,j}^f) \\
+ c_m [a(v_i^s - v_i^f)(v_i^s - v_i^f) + b T_i T_i^2] \right\} dv. \quad (8.4)
\]

The following theorem give us supplementary information about the time bahavior of the
Cesáro mean of the kinetic energy and thermal energy associated to the perturbation \( \mathbf{w} \).

**Theorem 8.1** Let \( \mathbf{w} \) be solution of the problem \((\mathcal{P}_0)\) corresponding to the loads \( g^s, h^\sigma, a^\sigma, b^\sigma \in H^1(B), r \in H^1(B) \). If the internal energy density and the dissipation potential are positive defined quadratic forms, then we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t (K(\tau) + S(\tau))d\tau = 0. \quad (8.5)
\]

**Proof.** It is easy to remark that (2.7) implies

\[
\frac{d}{dt}(K(t) + U(t) + S(t)) + \int_B \Phi(t) dv = 0. \quad (8.6)
\]
By direct integration of the above relation and using the inequality (8.4), we obtain

\[
K(t) + S(t) + \int_0^t \int_B \left\{ b_m (v_i^f v_i^f + j^f v_i^f \nu_i^f) + c_m \left[a(v_i^s - v_i^f)(v_i^s - v_i^f) + b \frac{T_i T_i}{T_0^2}\right] + \omega (v_i^s - v_i^f)(v_i^s - v_i^f) \right\} dvds 
\leq K(0) + U(0) + S(0). 
\]

(8.7)

Moreover, using the arithmetic–geometric means inequality we have

\[
K(t) + S(t) + \int_0^t \int_B \left\{ \frac{b_m}{cp^f} + \frac{c_m a}{\rho^f} \left(1 - \frac{1}{\varepsilon}\right) \right\} \rho^f j^f v_i^f 
+ \frac{b_m}{cp^f} + \frac{\omega}{\rho^f j^f} \left(1 - \frac{1}{\varepsilon^*}\right) \right\} \rho^f j^f v_i^f 
+ \left(\frac{c_m a}{\rho^s} (1 - \varepsilon) \right) \rho^s j^s v_i^s v_i^s + \frac{\omega}{\rho^s j^s} \left(1 - \varepsilon^*\right) \rho^s j^s v_i^s + \frac{c_m b}{cT_0^2} \right\} dvds 
\leq K(0) + U(0) + S(0), 
\]

for all constants \(\varepsilon, \varepsilon^* > 0\), where \(c > 0\) is the constant from the Poincaré inequality.

Now, we choose the constants \(\varepsilon\) and \(\varepsilon^*\) so that

\[
\frac{c_m ac}{c_m ac + b_m} < \varepsilon < 1, \quad \frac{\omega c}{\omega c j^f + b_m} < \varepsilon^* < 1, 
\]

(8.9)

and we deduce

\[
K(t) + S(t) + \kappa \int_0^t (K(\tau) + S(\tau)) d\tau \leq K(0) + U(0) + S(0), 
\]

(8.10)

where

\[
\kappa = 2 \min \left\{ \frac{b_m}{cp^f} + \frac{c_m a}{\rho^f} \left(1 - \frac{1}{\varepsilon}\right), \frac{b_m}{cp^f} + \frac{\omega}{\rho^f j^f} \left(1 - \frac{1}{\varepsilon^*}\right), \frac{c_m a}{\rho^s} (1 - \varepsilon), \frac{\omega}{\rho^s j^s} (1 - \varepsilon^*), \frac{c_m b}{cT_0^2} \right\}.
\]

(8.11)

Thus, we obtain the inequality

\[
\int_0^t (K(\tau) + S(\tau)) d\tau \leq \frac{1}{\kappa} [K(0) + S(0) + U(0)] (1 - e^{-\kappa t}) 
\]

(8.12)

which gives us the relation (8.5).

Using the above theorem we can prove the following result which describes the time behavior of the energies associated to the perturbation \(w\).

**Theorem 8.2** Let \(U\) be solution of the problem \((P_0)\). If the internal energy density and the dissipation potential are positive defined quadratic forms, then for all choices of initial data \(g^s, h^s, a^s, b^s \in H^1(B), r \in H^1(B)\), we have

\[
\lim_{t \to \infty} K(t) = 0, \quad \lim_{t \to \infty} U(t) = 0, 
\]

\[
\lim_{t \to \infty} S(t) = 0, \quad \lim_{t \to \infty} D(t) = E(0). 
\]

(8.13)
Proof. We combine the result given by the previous theorem and the results given by the Theorem 7.1, and we deduce
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t U(s) ds = 0 \tag{8.14}
\]
and, thus,
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t D(s) ds = E(0). \tag{8.15}
\]
Because the dissipation potential is positive defined, we obtain that the norm \(||w(t)||_X\) is a decreasing function as function of time. Let \(m\) be so that
\[
\lim_{t \to \infty} (K(t) + U(t) + S(t)) = m, \tag{8.16}
\]
where
\[
m = \inf_{t \in [0, \infty)} (K(t) + U(t) + S(t)). \tag{8.17}
\]
But
\[
\frac{1}{t} \int_0^t (K(\tau) + U(\tau)) d\tau \geq m \tag{8.18}
\]
and, if we take \(t \to \infty\), using the relation (8.14), we deduce that it is mandatory to have \(m = 0\).

Because \(K(\cdot), U(\cdot)\) and \(S(\cdot)\) are positive functions the relations (8.13)_{1-4} results.

Moreover, from the conservation law (2.7) it follows (8.13)_5.

Concluding, we have

**Corollary 8.1** If the internal energy and the dissipation potential are positive defined quadratic forms, then the thermodynamical process is unconditionally energetic asymptotic stable.

### 9 Vibrations in micropolar mixtures

#### 9.1 Non heat conducting mixtures

In the case of mixtures which as as constituents a micropolar elastic solid and a non-heat conducting micropolar incompressible fluid, according with the classical theories (see [53, 165, 85]), we assume that the constitutive functionals for the solid do not depend on temperature and the constitutive functionals for the fluid are independent by the gradient of the temperature.

We will have that the heat flux is zero. Because the fluid is incompressible, the pressure is a unknown of the problem and thus we can see that the energy equation is not couplet with the motion equations. In this section we study the purely mechanical behavior of the mixture.

For isotropic elastic materials, we have the following constitutive equations of the linear theory

\[
\begin{align*}
t_{ji}^s &= \lambda^s u_{k,k}^s \delta_{ji} + \mu^s (u_{j,i}^s + u_{i,j}^s) + k^s (u_{i,j}^s + \varepsilon_{ijk} \phi_k^s), \\
m_{ji}^s &= \alpha^s \phi_{k,k} \delta_{ji} + \beta^s \phi_{j,i} + \gamma^s \phi_{i,j}, \\
t_{ji}^f &= -\pi^f \delta_{ji} + \mu^f (v_{j,i}^f + v_{i,j}^f) + k^f (v_{i,j}^f + \varepsilon_{ijk} \nu_k^f), \\
m_{ji}^f &= \alpha^f \nu_{k,k} \delta_{ji} + \beta^f \nu_{j,i} + \gamma^f \nu_{i,j}, \\
p_t^s &= -\tilde{p}_i^f = -\xi (u_i^s - v_i^f), \\
\tilde{m}_i^s &= -\tilde{m}_i^f = -\varpi (u_i^s - v_i^f).
\end{align*}
\tag{9.19}
\]
The quantities have the same significations as in the previous sections.

The equations of motion are

\[
\begin{align*}
\rho_0^s u''^s_i &= t^s_{ji,j} + F_i^s + \hat{p}_i^s, \\
\rho_0^f v''^f_i &= t^f_{ji,j} + F_i^f + \hat{p}_i^f, \\
\rho_0^s j^s \phi''^{ms} &= m_{j}^{ji,j} + \varepsilon_{ijk} t_{jk}^s + L_i^s + \hat{m}_i^s, \\
\rho_0^f j^f d_i^f &= m_{j}^{ji,j} + \varepsilon_{ijk} t_{jk}^f + L_i^f + \hat{m}_i^f, \\
v_{i,i}^{f} &= 0,
\end{align*}
\]  

(9.20)

where \( F_i^s = \rho_0^s f_i^s \) and \( L_i^s = \rho_0^f l_i^s \).

Now, the dissipation potential is given by

\[
\begin{align*}
\Phi(\mathbf{v}^s, \mathbf{v}^f, \mathbf{u}^f) &= \mu^f a_{ij}^f a_{ij}^f + (\mu^f + k^f) a_{ij}^f a_{ij}^f + \alpha^f b_{ik}^f b_{ik}^f + \beta^f b_{ij}^f b_{ij}^f \\
+ \gamma^f b_{ij}^f b_{ij}^f + \xi(v_i^s - v_i^f)(v_i^s - v_i^f) + \omega(v_i^s - v_i^f)(v_i^s - v_i^f).
\end{align*}
\]

(9.21)

We introduce the constitutive equations in the equation of motion and we obtain the following system of partial differential equations

\[
\begin{align*}
(\lambda^s + \mu^s) u_{j,i,j}^s + (\mu^s + k^s) u_{i,jj}^s + k^s \varepsilon_{ijk} \phi_{k,j}^s - \xi(v_i^s - v_i^f) + F_i^s &= \rho_0^s u'', \\
(\alpha^s + \beta^s) \phi_{j,i,j}^s + \gamma^s \phi_{i,jj}^s + k^s (\varepsilon_{ijk} u_{k,j}^s - 2 \phi_{k,j}^s) - \omega (v_i^s - v_i^f) + L_i^s &= \rho_0^s s \phi''^s, \\
-\pi_i^s + (\mu^f + k^f) v_{j,i,jj}^f + k^f \varepsilon_{ijk} v_{k,j}^f + \xi(v_i^s - v_i^f) + F_i^f &= \rho_0^f v''_i^f, \\
(\alpha^f + \beta^f) v_{j,i,j}^f + \gamma^f v_{i,jj}^f + k^f (\varepsilon_{ijk} v_{k,j}^f - 2 v_i^f) + \omega (v_i^s - v_i^f) + L_i^f &= \rho_0^f l_i^f v''_i^f, \\
v_{i,i}^{f} &= 0.
\end{align*}
\]  

(9.22)

To these equations we adjoin boundary conditions

\[
\begin{align*}
u_i^s(x, t) &= u_i^{ss}(x, t), \quad \phi_i^s(x, t) = \phi_i^{ss}(x, t), \\
v_i^f(x, t) &= v_i^{sf}(x, t), \quad \nu_i^f(x, t) = \nu_i^{sf}(x, t) \quad \text{on} \quad \partial B \times I,
\end{align*}
\]  

(9.23)

and initial conditions

\[
\begin{align*}
u_i^s(x, 0) &= g_i^s(x), \quad v_i^{sf}(x, 0) = h_i^s(x), \\
\phi_i^s(x, 0) &= a_i^s(x), \quad \nu_i^{sf}(x, 0) = b_i^s(x) \quad \text{in} \ B,
\end{align*}
\]

(9.24)

where \( u_i^{ss}, v_i^{sf}, \phi_i^{ss}, \nu_i^{sf}, g_i^s, h_i^s, a_i^s \) and \( b_i^s \) are prescribed functions

\[
u_i^{ss} n_i = 0, \quad v_i^{sf} n_i = 0.
\]

(9.25)

We denote by \((\hat{P})\) the initial boundary value problem defined by the relations (9.22)–(9.24).

We say that \((u_i^s, v_i^f, \phi_i^s, v_i^{sf}, \pi^f)\) is an admissible process in \( B \times I \) if:

(a) \( u_i^s \) and \( \phi_i^s \) are of class \( C^2 \) in \( B \times I \);

(b) \( v_i^f \) and \( \nu_i^f \) are of class \( C^{2,1} \) in \( B \times I \);

(c) \( u_i^s, v_i^f, \phi_i^s, \nu_i^f \) are of class \( C^0 \) on \( \partial B \times I \);

(d) \( \pi^f \) are of class \( C^{1,0} \) in \( B \times I \).

By solution of the problem \((\hat{P})\) we mean an admissible process which satisfy the equations (9.22), the initial conditions (9.24) and the boundary conditions (9.23).
9.2 Complete solutions of the dynamical system

In this section we establish a Galerkin type representation of the solution of the field equations (9.22).

Let us introduce the differential operators:

\[ Q_1^s(\Delta) = (\mu^s + k^s)\Delta - \xi \frac{\partial}{\partial t} - \rho^s \frac{\partial^2}{\partial t^2}, \]
\[ Q_2^s(\Delta) = \gamma^s \Delta - 2k^s - \omega \frac{\partial}{\partial t} - \rho^s \frac{\partial^2}{\partial t^2}, \]
\[ Q_3^s(\Delta) = (\lambda^s + \mu^s)\Delta + Q_1^s, \quad Q_4^s(\Delta) = (\alpha^s + \beta^s)\Delta + Q_2^s, \]
\[ Q_5^s(\Delta) = (k^s)^2\Delta + Q_1^s Q_2^s, \quad (9.26) \]
\[ Q_6^s(\Delta) = (\mu^f + k^f)\Delta - \xi - \rho^f \frac{\partial}{\partial t}, \quad Q_2^f(\Delta) = \gamma^f \Delta - 2k^f - \omega - \rho^f j^f \frac{\partial}{\partial t}, \]
\[ Q_3^f(\Delta) = (\alpha^f + \beta^f)\Delta + Q_4^f, \quad Q_4^f(\Delta) = (k^f)^2\Delta + Q_1^f Q_2^f, \]
\[ P_1(\Delta) = \omega^2 \frac{\partial}{\partial t} - Q_2^s Q_3^s, \quad P_2(\Delta) = \xi^2 \frac{\partial}{\partial t} - Q_1^s Q_4^s. \]

In the above quantities and in the following, if \( Q_n, \ n = 1, 2, \ldots, m \) are differential operators and \( G \) is, for example, a \( C^\infty(\mathbb{R}) \) function, we use the notation

\[ Q_1 Q_2 \ldots Q_n G \equiv Q_1(Q_2(\ldots(Q_n(G))\ldots)). \]

With the help of these operators, we also define the following differential operators:

\[ D_1(\Delta) = -\omega^2 \frac{\partial}{\partial t} + Q_3^f Q_4^f, \]
\[ D_2(\Delta) = Q_5^s Q_4^f + 2\omega k^s k^f \Delta \frac{\partial}{\partial t} - (\omega^2 Q_1^f Q_4^f + \xi^2 Q_2^s Q_2^f) \frac{\partial}{\partial t} + \xi^2 \omega^2 \frac{\partial^2}{\partial t^2}, \]
\[ D_3(\Delta) = D_2 - (Q_4^f Q_2^s - \frac{\partial}{\partial t} \omega^2 Q_1^f)Q_5^s, \quad (9.27) \]
\[ D_4(\Delta) = Q_4^s Q_3^f[(k^s)^2 - (\alpha^s + \beta^s)Q_1^s] + 2\omega \omega k^s k^f \Delta \frac{\partial}{\partial t} \]
\[ + \omega^2 Q_1^f[(k^s)^2 - (\alpha^f + \beta^f)Q_1^f] \frac{\partial}{\partial t} + \xi^2 \omega^2 Q_2^s Q_2^f \frac{\partial}{\partial t} + (\alpha^f + \beta^f) \xi^2 \omega^2 \frac{\partial^2}{\partial t^2}, \]
\[ D_5(\Delta) = Q_5 \omega (k^f)^2 + \xi k^s k^f \left( \omega^2 \frac{\partial}{\partial t} + Q_4^f Q_2^f \right) + \omega \{(\alpha^s + \beta^s) Q_1^f \}
\[ + (\alpha^f + \beta^f) Q_2^f + (\alpha^f + \beta^f) (\alpha^s + \beta^s) \Delta \}] P_2 + (k^s)^2 Q_1^f Q_2^f, \]
\[ D_6(\Delta) = Q_5^s Q_4^f[(k^f)^2 - (\alpha^f + \beta^f)Q_1^f] + 2\omega \omega k^s k^f Q_4^s \frac{\partial}{\partial t} \]
\[ + \omega^2 Q_1^f[(k^s)^2 - (\alpha^s + \beta^s)Q_1^s] \frac{\partial}{\partial t} + \xi^2 (\alpha^f + \beta^f) Q_2^s Q_2^f \frac{\partial}{\partial t} + \xi^2 \omega^2 (\alpha^s + \beta^s) \frac{\partial^2}{\partial t^2}. \]

Using the method introduced by Moisil [210], we construct a representation of Galerkin type for the solution of the dynamical problem [132].
Theorem 9.1 Let consider

$$u^s_r = \Delta Q_3^s D_1 (\omega^2 \frac{\partial}{\partial t} Q_1^s + Q_4^s Q_2^s) G^s_i + D_1 D_3 G^s_{j,rj}
$$

\[ + \varepsilon_{rjk} \Delta Q_3^s D_1 (k^s Q_4^f) + \xi \omega k^f \frac{\partial}{\partial t} H^s_{j,k} \]

\[ + \Delta Q_5^s D_1 (\omega k^s k^f \Delta + \xi P_1^s) G^f_r - Q_5^s D_1 (\omega k^s k^f \Delta + \xi P_1^s) G^f_{j,rj} \]

\[-\varepsilon_{ijk} \Delta Q_3^s D_1 (\omega k^s k^f + \xi k^f) H^f_{j,k} - \xi P_r, \]

\[ \phi^s_r = \varepsilon_{rjk} \Delta Q_3^s D_1 (k^s Q_4^f + \xi \omega k^f \frac{\partial}{\partial t} G^s_{j,k} \]

\[ + \Delta Q_5^s D_1 [(k^f)^2 \Delta Q_1^s - P_2 Q_5^s] H^s_r + \Delta Q_3^s D_4 H^s_{j,rj} \]

\[-\varepsilon_{ijk} \Delta Q_3^s D_1 (\omega k^s Q_1^f + \xi k^f Q_2^f) G^s_{j,k} \]

\[ + \Delta Q_5^s D_1 (\xi k^s k^f \Delta + \omega P_2) H^f_j - \Delta Q_5^s D_5 H^f_{j,rj}, \]

\[ v^f_r = \frac{\partial}{\partial t} \Delta Q_3^s D_1 (\omega k^s k^f \Delta + \xi P_1^s) G^s_r - \frac{\partial}{\partial t} Q_5^s D_1 (\omega k^s k^f \Delta + \xi P_1^s) G^s_{j,rj} \]

\[-\varepsilon_{ijk} \frac{\partial}{\partial t} \Delta Q_3^s D_1 (\omega k^s Q_1^f + \xi k^f Q_2^f) H^s_{j,k} \]

\[ + \Delta Q_5^s D_1 (\omega^2 \frac{\partial}{\partial t} Q_1^s + Q_5^s Q_2^s) G^f_r - Q_5^s D_1 (\omega^2 \frac{\partial}{\partial t} Q_1^s + Q_5^s Q_2^s) G^f_{j,rj} \]

\[ + \varepsilon_{ijk} \Delta Q_3^s D_1 (\xi \omega k^s \frac{\partial}{\partial t} + k^f Q_5^s) H^f_{j,k} + Q_5^s P_r, \]

\[ \nu^f_r = -\varepsilon_{ijk} \frac{\partial}{\partial t} Q_3^s D_1 (\omega k^s Q_1^f + \xi k^f Q_2^f) G^s_{j,k} \]

\[ + \frac{\partial}{\partial t} \Delta Q_3^s D_1 (\xi k^s k^f \Delta + \omega P_2) H^s_r - \frac{\partial}{\partial t} \Delta Q_3^s D_5 H^s_{j,rj} \]

\[ + \varepsilon_{ijk} \Delta Q_3^s D_1 (\xi \omega k^s \frac{\partial}{\partial t} + k^f Q_5^s) G^f_{j,k} \]

\[ + \Delta Q_5^s D_1 [(k^f)^2 \Delta Q_1^s - P_2 Q_5^s] H^f_j + \Delta Q_5^s D_6 H^f_{j,rj}, \]

\[ \pi^f = \xi \frac{\partial}{\partial t} D_1 D_2 G^s_{r,r} - Q_3^s D_1 D_2 G^f_{r,r} - (\xi^2 \frac{\partial}{\partial t} - Q_1^s Q_3^s) P, \]

where $G^s_r, G^f_r, H^s_r, H^f_r$ and $P$ satisfy

\[ \Delta Q_5^s D_1 D_2 G^s_r = -F^s_r, \]

\[ \Delta Q_5^s D_1 D_2 H^s_r = -L^s_r, \]

\[ \Delta Q_5^s P = 0 \ (s = s, f). \]

Then $u^s_r, v^f_r, \phi^s_r, \nu^f_r$ and $\pi^f$ satisfy the equations (9.22).

Proof. It is easy to see that for every $A \in \mathcal{C}^3 (\mathbb{R}^3)$ we have

\[ \varepsilon_{rjk} A_{j,kr} = 0, \quad \varepsilon_{jmn} A_{m,nrj} = 0, \quad \varepsilon_{rjk} A_{m,kmj} = 0, \]

and

\[ \varepsilon_{krj} \varepsilon_{kmn} = \delta_{rm} \delta_{jn} - \delta_{rn} \delta_{jm}. \]
To prove that \( u^f_r \) satisfies the incompressibility condition (9.22)\(_5\), we use the relations (9.29)\(_5\), (9.31) and the relations (9.30) for \( A = H^s \) and \( A = H^f \).

By a direct substitution of the relations (9.28) in (9.22) and using the relations (9.29), (9.30), (9.31) and the identities

\[
Q^s_3 D_5 = \frac{\varpi}{\varrho} \frac{\partial}{\partial t} D_6 = D_1 \{ k^s (\varrho k^s Q^l_4 + \xi \varpi k^f \frac{\partial}{\partial t}) - (\alpha^s + \beta^s) (k^f)^2 \Delta Q^l_1 - P_2 Q^l_2 \} ,
\]
\[
Q^f_3 D_5 = \frac{\varpi}{\varrho} D_6 = D_1 \{ k^f (\varrho k^s Q^l_1 + \xi \varpi Q^l_2) + (\alpha^s + \beta^s) (k^f)^2 \Delta + \varpi P_2 \}
\]
\[
Q^l_3 D_5 = \frac{\varpi}{\varrho} D_4 = D_1 \{ (\alpha^f + \beta^f) (k^f k^s \Delta + \varpi P_1) + k^f (\varrho k^f Q^l_1 + \xi k^s Q^l_2) \},
\]
\[
Q^l_5 D_6 = \frac{\varpi}{\varrho} \frac{\partial}{\partial t} D_5 = D_1 \{ k^f (\varrho k^s \frac{\partial}{\partial t} + k^f Q^l_5) - (\alpha^f + \beta^f) [(k^s)^2 \Delta Q^l_1 - P_2 Q^l_2] \},
\]
\[
\varepsilon_{ijk} P_{k,j} = 0,
\]
we have that \( u^s_r, \ u^f_r, \ \phi^s_r, \ \nu^f_r \) and \( \pi^f \) are solutions of the basic system of equations (9.22), and the proof is complete.

9.3 Fundamental solution for steady-state vibration problem

In this section we use the representation described in the previous section in order to determine the fundamental solution of equations of motion for the case of steady vibrations. We assume that the internal energy density and the dissipation potential are positive defined quadratic forms in the corresponding terms.

We suppose that the body loads are of the type

\[
F^\sigma_r = \text{Re} \left[ F_r^{*\sigma}(x) e^{-i\omega t} \right], \quad L^\sigma_r = \text{Re} \left[ L_r^{*\sigma}(x) e^{-i\omega t} \right],
\]

where \( i^2 = -1 \).

Using the substitutions (1.66), we can rewrite the system of equations (9.22) in terms of the unknowns \( u^\sigma_r, \phi^\sigma_r \) and we search a solution of the form

\[
u^\sigma_r = \text{Re} \left[ u^\sigma_r(x) e^{-i\omega t} \right], \quad \phi^\sigma_r = \text{Re} \left[ \phi^\sigma_r(x) e^{-i\omega t} \right],
\]

\[
\pi^f = \text{Re} \left[ \pi^{*f}(x) e^{-i\omega t} \right], \quad (\sigma = s, f).
\]

It is easy to see that, if we know \( u^f_r \) and \( \phi^f_r \) (see (1.66)), we can find \( u^s_r \) and \( \nu^f_r \) from

\[
\nu^f_r = u^f_r, \quad \nu^f_r = \phi^f_r.
\]

Let us introduce the differential operators:

\[
Q^s_1 (\Delta) = (\mu^s + k^s) \Delta + i \omega \xi + \rho^s \omega^2, \quad Q^s_2 (\Delta) = \gamma^s \Delta - 2k^s + i \omega \varpi + \rho^s \omega^2 j^s, \]
\[
Q^s_3 (\Delta) = (\lambda^s + \mu^s) \Delta + Q^s_1, \quad Q^s_4 (\Delta) = (\alpha^s + \beta^s) \Delta + Q^s_2, \]
\[
Q^s_5 (\Delta) = (k^s)^2 \Delta + Q^s_1 Q^s_2, \quad Q^f_1 (\Delta) = (\mu^f + k^f) \Delta - \xi + i \rho^f \omega, \]
\[
Q^f_2 (\Delta) = \gamma^f \Delta - 2k^f - \varpi + i \rho^f \omega j^f, \quad Q^f_3 (\Delta) = (\alpha^f + \beta^f) \Delta + Q^f_2,
\]

40
\[ Q_{ij}^{sf}(\Delta) = (k^f)^2 \Delta + Q_{ij}^{sf} Q_{ij}^{sf}, 
\] 
\[ P_i(\Delta) = -i \omega \varpi^2 - Q_{2f}^{sf} Q_{2f}^{sf}, 
\] 
\[ P_2(\Delta) = -i \omega \xi^2 - Q_{ij}^{sf} Q_{ij}^{sf}, 
\] 
\[ D_1^f(\Delta) = i \omega \varpi^2 + Q_{3f}^{sf} Q_{3f}^{sf}, 
\] 
\[ D_2(\Delta) = Q_5^{sf} Q_4^{sf} - 2i \omega \xi \varpi k^s k^f \Delta + i \omega (\varpi^2 Q_1^{sf} Q_2^{sf} + \xi^2 Q_2^{sf} Q_2^{sf}) - \xi^2 \omega^2 \varpi^2, 
\] 
\[ D_3^f(\Delta) = D_2^f - (Q_4^{sf} Q_2^{sf} + i \omega \varpi Q_1^{sf}) Q_3^{sf}, 
\] 
\[ D_4^f(\Delta) = Q_4^{sf} Q_3^{sf} [(k^s)^2 - (\alpha^s + \beta^s) Q_1^{sf}] - 2i \omega \xi \varpi k^s k^f Q_3^{sf} 
\] 
\[ - i \omega \varpi^2 Q_1^{sf} [(k^f)^2 - (\alpha^f + \beta^f) Q_1^{sf}] - i \omega \xi^2 (\alpha^s + \beta^s) Q_2^{sf} Q_3^{sf} - (\alpha^f + \beta^f) \xi^2 \omega^2 \varpi^2, 
\] 
\[ D_5^f(\Delta) = Q_5^{sf} k^f \frac{\partial}{\partial x} + \xi k^s k^f (-i \omega \varpi^2 + Q_3^{sf} Q_1^{sf} + \varpi \{[(\alpha^s + \beta^s) Q_2^{sf} + (\alpha^f + \beta^f) (\alpha^s + \beta^s)] P_2^f + (k^s)^2 Q_1^{sf} Q_2^{sf} \}, 
\] 
\[ D_6^f(\Delta) = Q_5^{sf} Q_4^{sf} [(k^f)^2 - (\alpha^f + \beta^f) Q_1^{sf}] - 2i \omega \xi \varpi k^s k^f Q_3^{sf} 
\] 
\[ - i \omega \varpi^2 Q_1^{sf} [(k^f)^2 - (\alpha^s + \beta^s) Q_1^{sf}] - i \omega \xi (\alpha^f + \beta^f) Q_2^{sf} Q_4^{sf} - \omega^2 \xi^2 \varpi^2 (\alpha^s + \beta^s). 
\] 

We introduce the differential matrix operator 
\[ D \left( \frac{\partial}{\partial x} \right) = \left\| D_{mn} \left( \frac{\partial}{\partial x} \right) \right\|_{13 \times 13}, \] 
where 
\[ D_{rj} = Q_1^{sf} \delta_{rj} + (\lambda^s + \mu^s) \frac{\partial^2}{\partial x_r \partial x_j}, \]
\[ D_{r:3+j} = -D_{3+j:r} = k^s \varepsilon_{rjk} \frac{\partial}{\partial x_k}, \]
\[ D_{r:j+6} = D_{j+6:r} = -i \omega \xi, \]
\[ D_{r:j+9} = D_{j+9:r} = D_{r:13} = D_{13:r} = 0, \]
\[ D_{r:3:j+6} = D_{3:j+6} = D_{r:3+13} = D_{3+13:r} = 0, \]
\[ D_{r:3+3:j+3} = Q_2^{sf} \delta_{rj} + (\alpha^s + \beta^s) \frac{\partial^2}{\partial x_r \partial x_j}, \]
\[ D_{r:3+3:j+9} = D_{3+3:j+9} = -i \omega \varpi, \]
\[ D_{r:6:j+6} = -i \omega Q_1^{sf} \delta_{rj}, \]
\[ D_{r:6:j+9} = -D_{j+9:r+6} = -i \omega k^f \varepsilon_{rjk} \frac{\partial}{\partial x_k}, \]
\[ D_{r:9+j+9} = -i \omega Q_2^{sf} \delta_{rj} - i \omega (\alpha^f + \beta^f) \frac{\partial^2}{\partial x_r \partial x_j}; \]
\[ D_{r:6+13} = -\frac{1}{\chi} D_{13:r+6} = -\frac{\partial}{\partial x_r}, \]
\[ \chi = (\lambda^s + 2\mu^s + k^s)^{-1} \prod_{\sigma=s,f} [(\alpha^\sigma + \beta^\sigma + \gamma^\sigma \gamma^\sigma (k^\sigma + \mu^\sigma)]^{-1}. \]

If we introduce the thirteen-dimensional vectors \( \mathbf{U} = (u^{s s}, \phi^{s s}, u^{s f}, \phi^{s f}, \pi^{s f}) \) and \( \mathbf{F} = (F^{s s}, L^{s s}, F^{s f}, L^{s f}, 0) \), it is easy to see that the basic system of equations which describes the behavior of the amplitudes \( \mathbf{U} \) of the steady vibrations can be written in the form 
\[ D \left( \frac{\partial}{\partial x} \right) \mathbf{U} = -\mathbf{F}. \]
We denote by $t_{tr}^{s*}$ and $m_{tr}^{s*}$ the amplitudes of the components of stress tensor and of the couple stress tensor given by

\[
\begin{align*}
t_{tr}^{s*} &= \lambda^s u_{k,k}^{s*} \delta_{lr} + \mu^s (u_{l}^{s*} + u_{r}^{s*}) + k^s (u_{r,l}^{s*} + \varepsilon_{rlk} \phi_k^{s*}), \\
m_{tr}^{s*} &= \alpha^s \phi_{k,k}^{s*} \delta_{lr} + \beta^s \phi_{l}^{s*} + \gamma^s \phi_{r}^{s*}, \\
t_{tr}^{sf} &= -\pi^s \delta_{lr} - i \omega \mu^f (u_{l}^{sf} + u_{r}^{sf}) - i \omega k^f (u_{r,l}^{sf} + \varepsilon_{rlk} \phi_k^{sf}), \\
m_{tr}^{sf} &= -i \omega \alpha^f \phi_{k,k}^{sf} \delta_{lr} - i \omega \beta^f \phi_{l}^{sf} - i \omega \gamma^f \phi_{r}^{sf},
\end{align*}
\]  
(9.39)

Thus, the amplitudes of the surface forces and of the surface couple in a point $x$ of the boundary $\partial B$ of the domain $B \subset \mathbb{R}^3$ are

\[
t_r^{s*} = t_{tr}^{s*} n_t, \quad m_r^{s*} = m_{tr}^{s*} n_t, \quad (9.40)
\]

where $n_t$ are the components of the outward unit normal vector $n$ to the surface.

Let us introduce the matricial differential operator

\[
H \left( \frac{\partial}{\partial x}, n(x) \right) = \left\| H_{mn} \left( \frac{\partial}{\partial x}, n(x) \right) \right\|_{12 \times 13},
\]  
(9.41)

where

\[
\begin{align*}
H_{r:l} &= (\mu^s + k^s) \delta_{lr} \frac{\partial}{\partial n} + \lambda^s n_r \frac{\partial}{\partial x_l} + \mu^s n_l \frac{\partial}{\partial x_r}, \\
H_{r:l+3} &= k^s \varepsilon_{rkl} n_k, \\
H_{r:l+6} &= H_{r:l+9} = H_{r:13} = 0, \\
H_{3+r:l} &= H_{3+r:6+l} = H_{3+r:9+l} = H_{3+r:13} = 0, \\
H_{3+r:3+l} &= \alpha^s n_r \frac{\partial}{\partial x_l} + \beta^s n_l \frac{\partial}{\partial x_r} + \gamma^s \delta_{rl} \frac{\partial}{\partial n}, \\
H_{r+6;l} &= H_{r+9;l} = H_{6+r:3+l} = H_{r+9:6+l} = H_{r+9:l+3} = H_{r+9:13} = 0; \\
H_{6+r:6+l} &= -i \omega (\mu^f + k^f) \delta_{rl} \frac{\partial}{\partial n} - i \omega \mu^f n_l \frac{\partial}{\partial x_r}, \\
H_{6+r:9+l} &= -i \omega k^f \varepsilon_{rkl} n_k, \quad H_{6+r:13} = -n_r, \\
H_{r+9;l+9} &= -i \omega \alpha^f n_r \frac{\partial}{\partial x_l} - i \omega \beta^f n_l \frac{\partial}{\partial x_r} - i \omega \gamma^f \delta_{rl} \frac{\partial}{\partial n}.
\end{align*}
\]  
(9.42)

From (9.40), we can write the twelve dimensional vector $P \equiv (t^{s*}, m^{s*}, t^{sf}, m^{sf})$ in the form

\[
P = H \left( \frac{\partial}{\partial x}, n(x) \right) U.
\]  
(9.43)

**Definition 9.1** Let be $y \in \mathbb{E}^3$. A fundamental solution of the system (9.38) is a matrix $\Gamma(x, y; \omega) = \|\Gamma_{rj}\|_{13 \times 13}$ which satisfies the condition [157]

\[
D \left( \frac{\partial}{\partial x} \right) \Gamma(x, y; \omega) = -\delta(x - y) I, \quad x \in \mathbb{E}^3,
\]  
(9.44)

where $\delta(\cdot)$ is the Dirac delta and $I = \|\delta_{rj}\|_{13 \times 13}$ is the unit matrix.
According to the general theory of the fundamental solutions of the differential operators [191], we have to say that a fundamental solution is unique up to a matrix which has as columns solutions of the homogeneous system

$$
D \left( \frac{\partial}{\partial x} \right) U = 0. \tag{9.45}
$$

As a consequence of the Theorem 9.1, we obtain the following result:

**Theorem 9.2** Let consider

$$u^s_r = \Delta Q^s_3 D^*_1 (i \omega \varpi^2 Q^s_1 + Q^s_4 Q^*_2) G^s_r + D^*_1 D^*_3 G^s_{r,j,rj}
+ \varepsilon_{rjk} \Delta Q^s_3 D^*_1 (k^s Q^s_4 - i \omega \xi \varpi k^j) H^s_{j,k}
+ \Delta Q^s_3 D^*_1 (\varpi k^s k_j^f \Delta + \xi P^*_1) G^s_r - Q^s_3 D^*_1 (\varpi k^s k_j^f \Delta + \xi P^*_1) G^s_{r,j,rj}
- \varepsilon_{rjk} \Delta Q^s_3 D^*_1 (i k^s Q^s_1 f + \xi k^j Q^s_2) H^s_{j,k} - \xi P^*_r, \tag{9.46}
$$

$$\phi^s_r = \varepsilon_{rjk} \Delta Q^s_3 D^*_1 (k^s Q^s_4 - i \omega \xi \varpi k^j) G^s_{j,k}
+ \Delta Q^s_3 D^*_1 [(k^j)^2 \Delta Q^s_1 - P^*_2 Q^s_2] H^s_r + \Delta Q^s_3 D^*_4 H^s_{r,j,rj}
- \varepsilon_{rjk} \Delta Q^s_3 (\omega k^j Q^s_1 + \xi k^s Q^s_2) G^s_{j,k}
+ \Delta Q^s_3 D^*_1 (\xi k^s k^f \Delta + \varpi P^*_1) H^s_r - \Delta Q^s_3 D^*_5 H^s_{r,j,rj},
$$

$$u^f_r = \Delta Q^s_3 D^*_1 (\varpi k^s k^f \Delta + \xi P^*_1) G^s_r - Q^s_3 D^*_1 (\varpi k^s k^f \Delta + \xi P^*_1) G^s_{r,j,rj}
- \varepsilon_{rjk} \Delta Q^s_3 D^*_1 (i \omega \varpi k^s + k^j Q^s_5) H^s_{j,k} + \frac{i}{\omega} Q^s_3 P^*_r, \tag{9.47}
$$

$$\phi^f_r = - \varepsilon_{rjk} \Delta Q^s_3 D^*_1 (i \omega \varpi^2 Q^s_1 + Q^s_5 Q^*_2) G^s_{j,k}
+ \Delta Q^s_3 D^*_1 (\xi k^s k^f \Delta + \varpi P^*_1) H^s_r - \Delta Q^s_3 D^*_5 H^s_{r,j,rj}
+ \frac{i}{\omega} \varepsilon_{rjk} \Delta Q^s_3 D^*_1 (-i \omega \varpi k^s + k^j Q^s_5) G^s_{j,k}
+ \frac{i}{\omega} \Delta Q^s_3 D^*_1 [(k^s)^2 \Delta Q^s_1 - P^*_2 Q^s_2] H^s_r + \frac{i}{\omega} \Delta Q^s_3 D^*_6 H^s_{r,j,rj},
$$

$$\pi^f = - i \omega \xi D^*_1 D^*_2 G^s_{r,r} - Q^s_3 D^*_1 D^*_3 G^s_{r,r} + (i \xi^2 \omega + Q^s_1 Q^s_3) P^*,
$$

where $G^s_r, G^s_r, H^s_r, H^s_r$ and $P^s$ are solutions of the equations

$$
\Delta Q^s_3 D^*_1 D^*_s G^s_{r\sigma} = - F^s_{r\sigma}, \quad \Delta Q^s_3 D^*_1 D^*_s H^s_{r\sigma} = - L^*_{r\sigma},
\Delta Q^s_3 P^* = 0 \quad (\sigma = s, f).
\tag{9.47}
$$

Then $u^s_r, u^f_r, \phi^s_r, \phi^f_r$ and $\pi^f$ satisfy the equations (9.38).
We denote by \( k^2_n, n = 1, 2 \) and respectively, by \( k^2_m, m = 3, 4, 5, 6 \) the roots of the equations

\[
D^*_1(-k) = 0, \quad D^*_2(-k) = 0. \tag{9.48}
\]

It is convenient to write

\[
Q^*_s(\Delta) = (\lambda^s + 2\mu^s + k^s)(\Delta + k^2_7), \tag{9.49}
\]

where \( k_7 \) is the complex number defined by

\[
k^2_7 = -\frac{1}{\lambda^s + 2\mu^s + k^s}(\rho^s \omega^2 - i\omega \xi). \tag{9.50}
\]

We assume that \( k^2_n \neq k^2_m \), for \( n \neq m, \ n, m = 1, 2, ..., 7 \), and we choose the complex number \( k_n \) such that \( \text{Im}[k_n] \geq 0 \), for \( n = 1, 2, ..., 7 \).

With the help of these quantities, we can rewrite the equations (9.47) in the following form

\[
\Delta \prod_{n=1}^{7}(\Delta + k^2_n)G^*_{r\sigma} = -\chi F^*_{r\sigma},
\]

\[
\Delta \prod_{n=1}^{7}(\Delta + k^2_n)H^*_{r\sigma} = -\chi L^*_{r\sigma},
\]

\[
\Delta(\Delta + k^2_7)P^* = 0 \quad (\sigma = s, f). \tag{9.51}
\]

**Proposition 9.1** Assume that \( F^*_{r\sigma} = \delta_{rk}\delta(x-y), \ L^*_{r\sigma} = 0, \ F^*_{r\sigma} = 0 \) and \( L^*_{r\sigma} = 0 \). Then, the equations (9.51) have the solution \( G^*_{r\sigma} = \delta_{rk}E(x,y;\omega), \ H^*_{r\sigma} = 0, \ G^*_{r\sigma} = 0, \ H^*_{r\sigma} = 0, \ P^* = 0 \), where

\[
E(x,y;\omega) = \sum_{n=1}^{7} c_n E_n(x,y;\omega),
\]

\[
E_n = \frac{\chi}{4\pi \rho k^2_n}(1 - e^{ik_n x}), \tag{9.52}
\]

\[
c^{-1}_n = \prod_{m=1, m \neq n}^{7} (k^2_n - k^2_m) \ n = 1, 2, ..., 7,
\]

\[
\rho^2 = (x_r - y_r)(x_r - y_r).
\]

**Proof.** First of all, we remark that

\[
\Delta(\Delta + k^2_n)E_n = -\chi \delta(x-y). \tag{9.53}
\]

Taking into account the relations

\[
\sum_{n=1}^{7} c_n = 0, \sum_{n=m}^{7} c_n \prod_{l=1}^{m-1} (k^2_l - k^2_n) = 0 \ \text{for} \ m = 2, 3, ..., 6 \tag{9.54}
\]

\[
\Delta(\Delta + k^2_n)E_m = \chi \delta(x-y) + (k^2_n - k^2_m) \Delta E_m \ \text{for} \ n, m = 1, 2, ..., 7,
\]
and the method presented in the paper [249], we have
\[ \Delta \prod_{n=1}^{7} (\Delta + k_n^2) E(x, y; \omega) = -\chi \delta(x - y), \] (9.55)
and the proof is complete.

We denote by \((u_r^{s(k)}, \phi_r^{s(k)}, u_r^{f(k)}, \phi_r^{f(k)})\) the amplitudes of displacements caused by the concentrated loads \(F_r^{s} = \delta_{rk}\delta(x - y), \ L_r^{s} = 0, \ F_r^{f} = 0\) and \(L_r^{f} = 0\). In view of the relations given by the previous theorem, we have
\[ u_r^{s(k)} = \Delta Q_3^{s} D_1^{*} (i \omega \omega^2 Q_1^{f} + Q_4^{f} Q_2^{s}) \delta_{rk} E + D_3^{*} E_{r,k}, \]
\[ \phi_r^{s(k)} = \varepsilon_{rkl} \Delta Q_3^{s} D_1^{*} (k^{s} Q_4^{f} - i \omega \xi \omega k^{f}) E_{l}, \]
\[ u_r^{f(k)} = \Delta Q_3^{s} D_1^{*} (i \omega \omega^2 Q_1^{f} + \xi k^{f} Q_2^{s}) \delta_{rk} E - Q_3^{s} D_1^{*} (\omega k^{s} k^{f} \Delta + \xi P_1^{*}) E_{r,k}, \]
\[ \phi_r^{f(k)} = -\varepsilon_{rkl} \Delta Q_3^{s} D_1^{*} (\omega k^{s} Q_1^{f} + \xi k^{f} Q_2^{s}) E_{l}, \]
\[ \pi^{(k)} = -i \omega \xi D_1^{*} D_2^{*} E_{k}. \]

Correspondingly to the concentrated loads, \(F_r^{s} = 0, \ L_r^{s} = 0, \ F_r^{f} = \delta_{rk}\delta(x - y), \ L_r^{f} = 0\), we have the following amplitude of displacement, denoted by \((u_r^{s(3+k)}, \phi_r^{s(3+k)}, u_r^{f(3+k)}, \phi_r^{f(3+k)})\),
\[ u_r^{s(3+k)} = \varepsilon_{rkl} \Delta Q_3^{s} D_1^{*} (k^{s} Q_4^{f} - i \omega \xi \omega k^{f}) E_{l}, \]
\[ \phi_r^{s(3+k)} = \Delta Q_3^{s} D_1^{*} [(k^{f})^{2} \Delta Q_1^{s} - P_2^{s} Q_2^{f}] \delta_{rk} E + \Delta Q_3^{s} D_1^{*} E_{r,k}, \]
\[ u_r^{f(3+k)} = -\varepsilon_{rkl} \Delta Q_3^{s} D_1^{*} (\omega k^{s} Q_1^{f} + \xi k^{s} Q_2^{f}) E_{l}, \]
\[ \phi_r^{f(3+k)} = \Delta Q_3^{s} D_1^{*} (\xi k^{s} k^{f} \Delta + \omega P_2^{s}) \delta_{rk} E - \Delta Q_3^{s} D_1^{*} E_{r,k}, \]
\[ \pi^{(3+k)} = 0. \]

If \(F_r^{s} = 0, \ L_r^{s} = 0, \ F_r^{f} = \delta_{rk}\delta(x - y) \) and \(L_r^{f} = 0\), then the corresponding displacement vectors denoted by \((u_r^{s(6+k)}, \phi_r^{s(6+k)}, u_r^{f(6+k)}, \phi_r^{f(6+k)})\) are
\[ u_r^{s(6+k)} = \Delta Q_3^{s} D_1^{*} (\omega k^{s} k^{f} \Delta + \xi P_1^{*}) \delta_{rk} E - Q_3^{s} D_1^{*} (\omega k^{s} k^{f} \Delta + \xi P_1^{*}) E_{r,k}, \]
\[ \phi_r^{s(6+k)} = -\varepsilon_{rkl} D_1^{*} \Delta Q_3^{s} (\omega k^{s} Q_1^{f} + \xi k^{s} Q_2^{f}) E_{l}, \]
\[ u_r^{f(6+k)} = \frac{i}{\omega} \Delta Q_3^{s} D_1^{*} (i \omega \omega^2 Q_1^{s} + Q_5^{s} Q_2^{f}) \delta_{rk} E \]
\[ -\frac{i}{\omega} Q_3^{s} D_1^{*} (i \omega \omega^2 Q_1^{s} + Q_5^{s} Q_2^{f}) E_{r,k}, \]
\[ \phi_r^{f(6+k)} = \frac{i}{\omega} \varepsilon_{rkl} \Delta Q_3^{s} D_1^{*} (\omega k^{s} k^{f} + k^{f} Q_5^{s}) E_{l}, \]
\[ \pi^{(6+k)} = -Q_3^{s} D_1^{*} D_2^{*} E_{k}. \]

Finally, if \(F_r^{s} = 0, \ L_r^{s} = 0, \ F_r^{f} = 0\) and \(L_r^{f} = \delta_{rk}\delta(x - y)\), then we have for the
displacement vectors \((u^*_r (9+k), \phi^*_r (9+k), u^*_r (9+k), \phi^*_r (9+k))\) the expressions

\[
\begin{align*}
\varepsilon_r_{kl} \Delta Q^*_{3} D^*_1 (\varepsilon k^* Q^*_1 + \xi k^* Q^*_2) E, \\
\phi^*_r (9+k) = \Delta Q^*_{3} D^*_1 (\xi k^* k^* \Delta + \varepsilon P^*_2 \delta_r k) E - \Delta Q^*_{3} D^*_5 E, \\
\varepsilon_r_{kl} \Delta Q^*_{3} D^*_1 (\varepsilon k^* k^* \Delta + \varepsilon P^*_2 \delta_r k) E, \\
\phi^*_r (9+k) = \Delta Q^*_{3} D^*_1 (\xi k^* k^* \Delta + \varepsilon P^*_2 \delta_r k) E, \\
\pi^*_r (9+k) &= 0.
\end{align*}
\]

From the above discussion we can conclude:

**Theorem 9.3** The matrix \(\Gamma(x, y; \omega)\) defined by

\[
\begin{align*}
\Gamma_r (k) &= u^*_r (k), \quad \Gamma_{3+r} (k) = \phi^*_r (k), \quad \Gamma_{6+r} (k) = u^*_r (k), \quad \Gamma_{9+r} (k) = \phi^*_r (k), \\
\Gamma_{r} (3+k) &= u^*_r (3+k), \quad \Gamma_{3+r} (3+k) = \phi^*_r (3+k), \quad \Gamma_{6+r} (3+k) = u^*_r (3+k), \\
\Gamma_{9+r} (3+k) &= \phi^*_r (3+k), \quad \Gamma_{6+r} (6+k) = u^*_r (6+k), \quad \Gamma_{3+r} (6+k) = \phi^*_r (6+k), \\
\Gamma_{9+r} (6+k) &= \phi^*_r (6+k), \quad \Gamma_{9+r} (9+k) = u^*_r (9+k), \\
\Gamma_{3+r} (9+k) &= \phi^*_r (9+k), \quad \Gamma_{6+r} (9+k) = u^*_r (9+k), \quad \Gamma_{9+r} (9+k) = \phi^*_r (9+k), \\
-k \Gamma_{k} (13) &= \pi^*_r (13), \quad -k \Gamma_{6+k} (13) = \Gamma_{13} (6+k) = \pi^*_r (6+k), \\
\Gamma_{3+k} (13) &= \Gamma_{13} (3+k) = \Gamma_{9+k} (13) = \Gamma_{13} (9+k) = 0, \\
\Gamma_{13} (13) &= -i \omega (i \xi^2 + Q^*_1 Q^*_2) D^*_1 D^*_2,
\end{align*}
\]

is a fundamental solution of the system (9.38).

### 9.4 Basic properties of the matrix \(\Gamma(x, y; \omega)\)

In this section we point out some basic properties of the fundamental solution constructed in the previous section. These basic properties of fundamental matrix are useful if we want to apply the potential method for the framework theory.

Let us first note that

**Proposition 9.2** The fundamental matrix \(\Gamma(x, y; \omega)\) is so that

(i) \(\Gamma(x, y; \omega) = \Gamma^T(y, x; \omega)\);

(ii) If \(x \neq y\), then each column \(\Gamma^{(m)}(x, y; \omega), (m = 1, 2, ..., 13)\) of the matrix \(\Gamma\) satisfies at \(x\) the homogeneous system

\[
D \left( \frac{\partial}{\partial x} \right) \Gamma^{(m)}(x, y; \omega) = 0.\tag{9.61}
\]

**Lemma 9.1** The function \(E\) has the following properties:

(i) \(\frac{\partial E}{\partial x_1^s \partial x_2^s \partial x_3^s} = O(\epsilon) \quad (\epsilon \to 0), \quad \text{for all even } s \leq 11;\)
\[
\frac{\partial E}{\partial x^s_1 \partial x^s_2 \partial x^s_3} = \text{const} + O(\varrho^2) \ (\varrho \to 0), \text{ for all } s \leq 11;
\]

\[
\frac{\partial E}{\partial x^s_1 \partial x^s_2 \partial x^s_3} = O(\varrho^{13-s}) \ (\varrho \to 0), \text{ for all } s \geq 12;
\]

where \(s_1, s_2, s_3 \in \mathbb{N}^*\) and \(s = s_1 + s_2 + s_3\).

\textbf{Proof.} It is easy to see that in the neighborhood of \(y\), we have

\[
1 - e^{ik_n \varrho} = \frac{1}{k_n^2 \varrho} + \sum_{m=0}^\infty \frac{(ik_n)^m}{(m + 2)!} \varrho^{m+1}.
\]

On the other hand, we can deduce that

\[
\sum_{n=1}^7 c_n k_n^{2p} = 0, \text{ for } p = 0, 1, \ldots, 5 \text{ and } \sum_{n=1}^7 c_n k_n^{12} = 1.
\]

Thus, we obtain

\[
E = \frac{\chi}{4\pi} \left[ -i \sum_{n=1}^7 \frac{c_n}{k_n} + \sum_{n=1}^7 \sum_{m=1}^6 c_n (ik_n)^{2m-1} \frac{\varrho^{2m}}{2m+1} + \frac{1}{14!} \varrho^{13} \right.
\]

\[
\left. + \sum_{n=1}^7 \sum_{m=13}^\infty c_n (ik_n)^m \frac{\varrho^{m+1}}{(m + 2)!} \right].
\]

Using this relation we obtain the conclusions ofLemma.

Let us introduce the matrix \(\Pi\) defined by

\[
\Pi_{r;k} = \frac{1}{4\pi} \left( \frac{1}{\mu^s + k^s} - \frac{1}{2a^s} \right) \delta_{rk} \frac{1}{\varrho} + \frac{1}{8\pi a^s} \frac{x_r x_k}{\varrho^3},
\]

\[
\Pi_{3+r;3+k} = \frac{1}{4\pi} \left( \frac{1}{\gamma^s} - \frac{1}{2b^s} \right) \delta_{rk} \frac{1}{\varrho} + \frac{1}{8\pi b^s} \frac{x_r x_k}{\varrho^3},
\]

\[
\Pi_{r;3+k} = \Pi_{6+r;3+k} = \Pi_{9+r;3+k} = \Pi_{13;r;3+k} = 0,
\]

\[
\Pi_{6+r;6+k} = \frac{i}{8\pi \omega (\mu^f + k^f)} \left( \delta_{rk} \frac{1}{\varrho} + \frac{x_r x_k}{\varrho^3} \right),
\]

\[
\Pi_{r;6+k} = \Pi_{3+r;6+k} = \Pi_{9+r;6+k} = 0,
\]

\[
\Pi_{13;6+k} = \Pi_{3+r;13} = \Pi_{6+r;13} = \Pi_{9+r;13} = 0,
\]

\[
\Pi_{9+r;9+k} = \frac{i}{4\pi \omega} \left( \frac{1}{\gamma^s} - \frac{1}{2b^s} \right) \delta_{rk} \frac{1}{\varrho} + \frac{i}{8\pi \omega b^s} \frac{x_r x_k}{\varrho^3},
\]

\[
\Pi_{r;9+k} = \Pi_{3+r;9+k} = \Pi_{6+r;9+k} = \Pi_{13;9+k} = 0,
\]

\[
\Pi_{k;13} = \Pi_{3+k;13} = \Pi_{9+k;13} = 0,
\]

\[
\Pi_{13;13} = \omega (\xi + \rho^f \Omega) \frac{1}{4\pi} \frac{1}{\varrho},
\]

47
where
\[
\alpha^s = \frac{(\mu^s + k^s)(\lambda^s + 2\mu^s + k^s)}{\lambda^s + \mu^s} \quad \lambda^s = \frac{\gamma^s (\alpha^s + \beta^s + \gamma^s)}{\alpha^s + \beta^s}.
\]  
(9.66)

We can observe that
\[
\Pi(x, y; \omega) = \Pi(y, x; \omega), \quad \Pi(x, y; \omega) = \Pi^T(x, y; \omega).
\]  
(9.67)

We denote by \(\Pi^{(m)}(x, y; \omega)\), \((m = 1, 2, ..., 13)\) the columns of the matrix \(\Pi(x, y; \omega)\).

In view of Lemma 9.1 we have

**Proposition 9.3** The differences
\[
G^{(m)}(x, y; \omega) = \Gamma^{(m)}(x, y; \omega) - \Pi^{(m)}(x, y; \omega),
\]  
(9.68)

remain bounded when \(x = y\) and the first derivations of these differences have only a pole of the first order for \(x = y\).

In the subsections 9.6 we will introduce the notion of regular vectors. From the established results we will can remark that the matrix \(\Gamma(x, y; \omega)\) is the unique fundamental solution, up to a rearrangement of the columns, for which the columns are regular vectors in \(\mathbb{R}^3\).

### 9.5 Solution for point loads problem in cylindrical coordinates

We consider an infinite micropolar solid-fluid mixture and a point \(y\) in the mixture. A concentrated force \(F^{ss}(x) = \delta(x - y) m\) is applied to the mixture, where \(m\) is an unit vector. Based on the general solution described in previous section, we give the solution of the problem corresponding to this point force.

We choose a system of the Cartesian axes such that the origin \(O\) is in the point \(y\) and the direction of \(Ox_3\) is given by the unit vector \(m\). In the Cartesian coordinates \((x_1, x_2, x_3)\) we have \(F^{ss}(x) = \delta(x) e_3\).

Using (9.56) we find that the displacement of the solid is given by
\[
u^{ss} = \Delta Q_3^{ss} D_1^* (i\omega \varpi^2 Q_1^{sf} + Q_4^{sf} Q_2^{ss}) E e_3 + D_1^* D_3^* \text{grad} E_3,
\]  
(9.69)

where \(e_k\) are the unit vectors of the Cartesian axes.

In the cylindrical coordinates \((r, \theta, z)\), because \(E\) is independent by \(\theta\), the components of the displacement of the solids are
\[
u_r^{ss} = D_1^* D_3^* \frac{z r}{\varrho^2} \left( \frac{\partial^2 E}{\partial \varrho^2} - \frac{1}{\varrho} \frac{\partial E}{\partial \varrho} \right),
\]
\[
u_{\theta}^{ss} = 0,
\]
\[
u_z^{ss} = \Delta Q_3^{ss} D_1^* (i\omega \varpi^2 Q_1^{sf} + Q_4^{sf} Q_2^{ss}) E + D_1^* D_3^* \frac{1}{\varrho^2} \left( \frac{\partial^2 E}{\partial \varrho^2} z^2 + \frac{r^2}{\varrho} \frac{\partial E}{\partial \varrho} \right),
\]  
(9.70)

where we use that \(\varrho^2 = r^2 + z^2\).
Similarly, we find

\[ u^s_r = -Q^s_3 D^*_1 (\varpi k^s k^f \Delta + \xi P^*_1) z r \left( \frac{\partial^2 E}{\partial \varrho^2} - \frac{1}{\varrho} \frac{\partial E}{\partial \varrho} \right), \]

\[ u^s_\theta = 0, \]

\[ u^s_z = \Delta Q^s_3 D^*_1 (\varpi k^s k^f \Delta + \xi P^*_1) \]

\[ -Q^s_3 D^*_1 (\varpi k^s k^f \Delta + \xi P^*_1) \frac{1}{\varrho^2} \left( \frac{\partial^2 E}{\partial \varrho^2} z^2 + \frac{r^2}{\varrho} \frac{\partial E}{\partial \varrho} \right) \]

and

\[ \pi^s_f = -i \omega \xi D^*_1 D^*_2 \frac{\partial E}{\partial z}. \]

(9.71)

On the other hand, the microrotations are

\[ \phi^s_r = -\Delta Q^s_3 D^*_1 (k^s Q^s_4 f - i \omega \varpi k^f) \text{curl}(0,0,E), \]

\[ \phi^s_\theta = \Delta Q^s_3 D^*_1 (\varpi k^s Q^s_1 f + \xi k^f Q^s_2) \text{curl}(0,0,E), \]

(9.72)

and thus, we have

\[ \phi^s_r = \phi^s_\theta = 0, \]

\[ \phi^s_\theta = \Delta Q^s_3 D^*_1 (k^s Q^s_4 f - i \omega \varpi k^f) \frac{r}{\varrho} \frac{\partial E}{\partial \varrho}, \]

\[ \phi^s_f = \phi^s_z = 0, \]

\[ \phi^s_f = -\Delta Q^s_3 D^*_1 (\varpi k^s Q^s_1 f + \xi k^f Q^s_2) \frac{r}{\varrho} \frac{\partial E}{\partial \varrho}, \]

(9.73)

and the solution of the point force problem is complete.

Let now consider that the concentrated couple \( \mathbf{L}^s(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y}) \mathbf{m} \) is applied to the mixture.

As above, we will find that the corresponding solution is

\[ u^s_r = u^s_\theta = 0, \]

\[ u^s_z = \Delta Q^s_3 D^*_1 (k^s Q^s_4 f - i \omega \varpi k^f) \frac{r}{\varrho} \frac{\partial E}{\partial \varrho}, \]

\[ \phi^s_r = \Delta Q^s_3 D^*_1 \frac{z r}{\varrho^2} \left( \frac{\partial^2 E}{\partial \varrho^2} - \frac{1}{\varrho} \frac{\partial E}{\partial \varrho} \right), \]

\[ \phi^s_\theta = 0, \]

\[ \phi^s_z = \Delta Q^s_3 D^*_1 [(k^f)^2 \Delta Q^s_1 - P^s_2 Q^s_4 f] E + \Delta Q^s_3 D^*_4 \frac{1}{\varrho^2} \left( \frac{\partial^2 E}{\partial \varrho^2} z^2 + \frac{r^2}{\varrho} \frac{\partial E}{\partial \varrho} \right), \]

(9.74)

\[ u^f_r = u^f_\theta = 0, \]

\[ u^f_z = -\Delta Q^s_3 D^*_1 (\varpi k^f Q^s_1 + \xi k^f Q^s_2) \frac{r}{\varrho} \frac{\partial E}{\partial \varrho}, \]

\[ \phi^f_r = \Delta Q^s_3 D^*_2 \frac{z r}{\varrho^2} \left( \frac{\partial^2 E}{\partial \varrho^2} - \frac{1}{\varrho} \frac{\partial E}{\partial \varrho} \right), \]

(9.75)

\[ \phi^f_\theta = \frac{\partial E}{\partial \varrho}. \]
\[ \phi^s_0 = 0, \]
\[ \phi^s_z = \Delta Q^s_3 D^s_1(\xi k^s k^f \Delta + \pi P^s_2) E - \Delta Q^s_3 D^s_3 \frac{1}{\varrho} \left( \frac{\partial^2 E}{\partial \varrho^2} z^2 + \frac{\varrho^2}{\varrho} \frac{\partial E}{\partial \varrho} \right), \]
\[ \pi^f = 0. \]

Similarly we can find the solutions which correspond to the concentrated loads \( F^*(x) = \delta(x - y)m \) and \( L^*(x) = \delta(x - y)m \) and the problem is solved.

### 9.6 Somigliana type representation

In this section we establish an integral representation formula for the solution of the equation (9.22). We denote by \( \mathbf{V} \) the vector obtained from a vector \( \mathbf{V} \) by cutting the last component, by \( \mathbf{A} \) the matrix obtained from a matrix \( \mathbf{A} \) by cutting the last row, by \( \mathbf{A} \) the matrix obtained from a matrix \( \mathbf{A} \) by cutting the last column while the matrix \( \mathbf{A} \) represents the matrix \( \mathbf{A} \) without the last column and the last row. Let us consider a finite domain \( B^+ \subset \mathbb{R}^3 \), bounded by the Liapunov surface \( \partial B, B^+ = B^+ \cup \partial B \) and \( B^- = \mathbb{R}^3 \setminus \bar{B}^+ \) the exterior of \( B^+ \). We use the notation \( B^- = B^- \cap \Sigma(0, R) \), where \( \Sigma(0, R) \) is the sphere with its center at \( 0 \) and radius \( R \). The twelve-dimensional vector function \( \mathbf{U} \) is called regular vector in \( B^+ \) if \( u^{**}, \phi^{**}, u^{*f}, \phi^{*f} \in C^1(B^+) \cap C^2(B^+) \) and \( \pi^f \in C(B^+) \cap C^1(B^+) \). If \( \mathbf{U} \) is defined in \( B^+ \), then \( \mathbf{U} \) is called regular vector in \( B^- \) if \( u^{**}, \phi^{**}, u^{*f}, \phi^{*f} \in C^1(B^-) \cap C^2(B^-) \), \( \pi^f \in C(B^-) \cap C^1(B^-) \), \( u^{**}, \phi^{**}, u^{*f}, \phi^{*f}, \sigma^f \) are regular vectors in \( B^+ \), then the following relations hold

\[
\begin{align*}
\int_{B^+} \mathbf{V} D \left( \frac{\partial}{\partial x} \right) U d\mathbf{x} \\
= \int_{\partial B^+} \mathbf{V} H \left( \frac{\partial}{\partial x}, \mathbf{n}(x) \right) U d\mathbf{a}_x + \int_{B^+} \mathbf{C}(U, \mathbf{V}) d\mathbf{x} \\
- \int_{B^+} \left[ E(U, \mathbf{V}) - \sum_{\sigma=s,f} \rho_0^\sigma \omega^2 (u^{**}_\sigma v^{**}_\sigma + j^{**}_\sigma \phi^{**}_\sigma \delta^{**}_\sigma) \right] d\mathbf{x},
\end{align*}
\]

\[
\begin{align*}
\int_{B^+} \mathbf{V} D \left( \frac{\partial}{\partial x} \right) \mathbf{U} d\mathbf{x} \\
= \int_{\partial B^+} \mathbf{V} H \left( \frac{\partial}{\partial x}, \mathbf{n}(x) \right) \mathbf{U} d\mathbf{a}_x - \int_{B^+} \mathbf{C}(U, \mathbf{V}) d\mathbf{x} \\
- \int_{B^+} \left[ E(U, \mathbf{V}) - \sum_{\sigma=s,f} \rho_0^\sigma \omega^2 (u^{**}_\sigma v^{**}_\sigma + j^{**}_\sigma \phi^{**}_\sigma \delta^{**}_\sigma) \right] d\mathbf{x},
\end{align*}
\]

### Theorem 9.4

If \( \mathbf{U} = (u^{**}, \phi^{**}, u^{*f}, \phi^{*f}, \pi^f) \) and \( \mathbf{V} = (v^{**}, \psi^{**}, v^{*f}, \psi^{*f}, \partial^f) \) are regular vectors in \( B^+ \), then the following relations hold.

\[ (9.76) \]

\[ (9.77) \]

\[ (9.78) \]
where the superposed bar denotes complex conjugate, and
\[
E(\tilde{U}, \tilde{V}) = \lambda^s u^s_{k,k} v^s_{r,r} + \mu^s u^s_{t,t} v^s_{r,r} + (\mu^s + k^s) u^s_{r,r} v^s_{r,r} + 2k^s \epsilon \tau k (u^s_{j,j} \phi^s_k + v^s_{r,r} \phi^s_k) + 2k^s \phi^s_{r,r} \psi^s_{r,r} + \alpha^s \phi^s_{k,k} v^s_{r,r} + \beta^s \phi^s_{t,t} v^s_{r,r} + \gamma^s \phi^s_{r,r} \psi^s_{r,r},
\]
\[
C(\tilde{U}, \tilde{V}) = i \omega [\mu^f u^f_{t,t} v^f_{r,r} + (\mu^f + k^f) u^f_{r,r} v^f_{r,r}] + 2k^f \epsilon \tau k (u^f_{r,r} \psi^f_k + v^f_{r,r} \psi^f_k) + 2k^f \phi^f_{r,r} \psi^f_{r,r} + \alpha^f \phi^f_{k,k} v^f_{r,r} + \beta^f \phi^f_{t,t} v^f_{r,r} + \gamma^f \phi^f_{r,r} \psi^f_{r,r} + \xi (u^s_{r,r} - u^s_{r,r}) (v^s_{r,r} - v^s_{r,r}) + \omega (\phi^s_{r,r} - \phi^s_{r,r}) (\psi^s_{r,r} - \psi^s_{r,r})].
\]

**Proof.** The proof of this theorem results using the divergence theorem and relations (9.40).

Since
\[
E(\tilde{U}, \tilde{V}) = E(\tilde{V}, \tilde{U}), \quad C(\tilde{U}, \tilde{V}) = C(\tilde{V}, \tilde{U}),
\]
using (9.77) we obtain

**Theorem 9.5** If \( U \) and \( V \) are regular vectors in \( B^+ \) then
\[
\int_{B^+} \left[ V D \left( \frac{\partial}{\partial x} \right) U - U D \left( \frac{\partial}{\partial x} \right) V \right] dx \]
\[
= \int_{\partial B} \tilde{V} H \left( \frac{\partial}{\partial x}, n(x) \right) U - \tilde{U} H \left( \frac{\partial}{\partial x}, n(x) \right) V \right] dx.
\]

Because a regular vector verifies the conditions (9.76), we can also establish a similar result for \( B^- \).

We denote by \( \Lambda(x, y; \omega) \) the matrix obtained from \( H \left( \frac{\partial}{\partial y}, n(x) \right) \Gamma(x, y; \omega) \) by replacing \( x \) by \( y \) and interchanging the rows and columns
\[
\Lambda(x, y; \omega) = \left[ H \left( \frac{\partial}{\partial y}, n(y) \right) \Gamma(y, x; \omega) \right]^T
\]
\[
= \left[ H \left( \frac{\partial}{\partial y}, n(y) \right) \Gamma(x, y; \omega) \right]^T.
\]

**Theorem 9.6** If \( x \neq y \), then each column of the matrix \( \Lambda(x, y; \omega) \) satisfies at \( x \) the system
\[
D \left( \frac{\partial}{\partial x} \right) \Lambda(x, y; \omega) = 0, \quad x \neq y.
\]

**Proof.** From (9.82), we have
\[
\Lambda_{mn} = H_{np} \left( \frac{\partial}{\partial y}, n(y) \right) \Gamma_{pm}(y, x; \omega),
\]
where \( \Lambda_{mn} \) are the elements of the matrix \( \Lambda(x, y; \omega) \). Using (9.61) and (9.82), we get
\[
\left[ D \left( \frac{\partial}{\partial x} \right) \Lambda(x, y; \omega) \right]_{mn} = D_{mp} \left( \frac{\partial}{\partial x} \right) H_{np} \left( \frac{\partial}{\partial y}, n(y) \right) \Gamma_{pq}(y, x; \omega)
\]
\[
= H_{np} \left( \frac{\partial}{\partial y}, n(y) \right) D_{mp} \left( \frac{\partial}{\partial x} \right) \Gamma_{pq}(x, y; \omega) = 0,
\]

51
and the proof is complete.

We also introduce the matrix

$$M(x, y; \omega) = H \left( \frac{\partial}{\partial x}, n(x) \right) \tilde{\Pi}(x, y; \omega). \quad (9.85)$$

From (9.42) and (9.65) we have that

$$M_{jr} = \frac{1}{4\pi} \left[ -\left(1 - \frac{2\mu^s + k^s}{2\alpha^s}\right) \frac{n_k(x)(x_k - y_k)}{\varrho^3} \delta_{jr} - \frac{3}{2} \frac{2\mu^s + k^s}{2\alpha^s} \frac{n_k(x)(x_k - y_k)(x_j - y_j)(x_r - y_r)}{\varrho^2} + \delta_1 \left( \frac{n_j(x)(x_r - y_r)}{\varrho^3} - \frac{n_r(x)(x_j - y_j)}{\varrho^3} \right) \right], \quad (9.86)$$

$$M_{3+j; 3+r} = \frac{1}{4\pi} \left[ -\left(1 - \frac{\beta^s + \gamma^s}{2\alpha^s}\right) \frac{n_k(x)(x_k - y_k)}{\varrho^3} \delta_{jr} - \frac{3}{2} \frac{\beta^s + \gamma^s}{\alpha^s} \frac{n_k(x)(x_k - y_k)(x_j - y_j)(x_r - y_r)}{\varrho^2} + \delta_2 \left( \frac{n_j(x)(x_r - y_r)}{\varrho^3} - \frac{n_r(x)(x_j - y_j)}{\varrho^3} \right) \right],$$

$$M_{6+j; 6+r} = \frac{1}{8\pi} \left[ -\left(1 - \frac{\mu^f}{\mu^f + k^f}\right) \frac{n_k(x)(x_k - y_k)}{\varrho^3} \delta_{jr} - \frac{3}{2} \frac{\mu^f}{\mu^f + k^f} \frac{n_k(x)(x_k - y_k)(x_j - y_j)(x_r - y_r)}{\varrho^2} + 2\delta_3 \left( \frac{n_j(x)(x_r - y_r)}{\varrho^3} - \frac{n_r(x)(x_j - y_j)}{\varrho^3} \right) \right],$$

$$M_{9+j; 9+r} = \frac{1}{4\pi} \left[ -\left(1 - \frac{\beta^f + \gamma^f}{2\alpha^f}\right) \frac{n_k(x)(x_k - y_k)}{\varrho^3} \delta_{jr} - \frac{3}{2} \frac{\beta^f + \gamma^f}{\alpha^f} \frac{n_k(x)(x_k - y_k)(x_j - y_j)(x_r - y_r)}{\varrho^2} + \delta_4 \left( \frac{n_j(x)(x_r - y_r)}{\varrho^3} - \frac{n_r(x)(x_j - y_j)}{\varrho^3} \right) \right],$$

where

$$\delta_1 = \frac{2\mu^s(k^s + \mu^s) - k^s(\lambda^s + \mu^s)}{2(\mu^s + k^s)(\lambda^s + 2\mu^s + k^s)},$$

$$\delta_2 = \frac{\beta^s(\beta^s + \alpha^s) + \gamma^s(\beta^s - \alpha^s)}{2\gamma^s(\alpha^s + \beta^s + \gamma^s)},$$

$$\delta_3 = \frac{k^f}{2(\mu^f + k^f)}, \quad \delta_4 = \frac{\beta^f(\beta^f + \alpha^f) + \gamma^f(\beta^f - \alpha^f)}{2\gamma^f(\alpha^f + \beta^f + \gamma^f)} \quad (9.87)$$

and the other components of $M(x, y; \omega)$ are zero.

We see that the difference

$$L(x, y; \omega) = H \left( \frac{\partial}{\partial x}, n(x) \right) \tilde{\Gamma}(x, y; \omega) - M(x, y; \omega), \quad (9.88)$$
has, for $x = y$, only a pole of the first order.

In the book [189] we can find the following Lemma:

**Lemma 9.2** Let consider

$$k(x, y) = \|k_{pq}(x, y)\|_{\nu \times \nu}, \quad \nu \in \mathbb{N}^* \quad (9.89)$$

where

$$k_{pq}(x, y) = \sum_{j, r=1}^{3} a_{pq}^{jr} \eta_{jr}(x, y) + \sum_{j, r=0}^{n} \left[ b_{pq}^{jr}(x_1 - y_1)^j (x_2 - y_2)^r \right. \left. + c_{pq}^{jr}(x_1 - y_1)^j (x_3 - y_3)^r + d_{pq}^{jr}(x_2 - y_2)^j (x_3 - y_3)^r \right] \eta(x, y) r^{-j-r-2},$$

$$p, q = 1, 2, ..., \nu, \quad n \in \mathbb{N}^*,$$

$$\eta_{jr}(x, y) = \frac{n_r(x)(x_j - y_j) - n_j(x)(x_r - y_r)}{r}, \quad j, r = 1, 2, 3,$$

$$\eta(x, y) = \frac{n(x) \cdot (x - y)}{r}.$$

If $B^+$ is a finite domain in $\mathbb{R}^3$ bounded by the Lyapunov surface $\partial B$, $\overline{B}^+ = B^+ \cup \partial B$ and $B^- = \mathbb{R}^3 \setminus \overline{B}^+$, then

$$\int_{\partial B} k(x, y) da_x = \begin{cases} \Upsilon, & \text{if } y \in B^+ \\ \frac{\Upsilon}{2}, & \text{if } y \in \partial B \\ 0, & \text{if } y \in B^- \end{cases} \quad (9.90)$$

where

$$\Upsilon = \|\Upsilon_{pq}\|_{\nu \times \nu} \quad (9.91)$$

$$\Upsilon_{pq} = 4\pi \sum_{j, r=0}^{n} \left( b_{pq}^{jr} + c_{pq}^{jr} + d_{pq}^{jr} \right) \frac{(j - 1)!! (r - 1)!!}{(j + r + 1)!!}, \quad (9.92)$$

the last sum is only upon even $j$ and $r$.

Using this Lemma, we have

$$\int_{\partial B} H \left( \frac{\partial}{\partial y}, n(y) \right) \tilde{\Pi}(x, y; \omega) da_y = -\Psi(x) I, \quad (9.93)$$

where $I = \|\delta_{mn}\|_{12 \times 12}$, and

$$\Psi(x) = \begin{cases} 1, & \text{if } x \in B^+ \\ \frac{1}{2}, & \text{if } x \in \partial B \\ 0, & \text{if } x \in B^- \end{cases} \quad (9.94)$$

Obviously, the matrix

$$N(x, y; \omega) = \tilde{\Lambda}(x, y; \omega) - \Omega(x, y; \omega), \quad (9.95)$$
where
\[ \Omega(x, y; \omega) = M^T(y, x; \omega) \] (9.96)
has for \( x = y \) only a pole of the first order, and

\[ \int_{\partial B} \Omega(x, y; \omega) da_y = -\Psi(x) \] (9.97)

**Theorem 9.7** If \( U \) is a regular vector in \( B^+ \), solution of the system (9.38), then

\[ \hat{U}(x) = \int_{\partial B} \left[ \tilde{\Gamma}(x, y; \omega) H\left( \frac{\partial}{\partial y}, n(y) \right) U(y) - \hat{\Lambda}(x, y; \omega) \hat{U}(y) \right] da_y \]

\[ - \int_{B^+} \tilde{\Gamma}(x, y; \omega) \hat{F}(y) dy, \]

\[ U_{13}(x) = \sum_{p=1}^{12} \left\{ \sum_{k=1}^{13} \int_{\partial B} \left[ \Gamma_{13,p}(x, y; \omega) H_{pk}\left( \frac{\partial}{\partial y}, n(y) \right) U_k(y) - U_p(y) H_{pk}\left( \frac{\partial}{\partial y}, n(y) \right) \Gamma_{13,k}(x, y; \omega) \right] da_y \right. \]

\[ - \left. \int_{B^+} \Gamma_{13,p}(x, y; \omega)(x, y; \omega) F_p(y) dy \right\}. \]

**Proof.** Let \( \Sigma(y, \varepsilon) \) be a sphere with its center at \( y \), radius \( \varepsilon \) and surface \( \partial \Sigma(y, \varepsilon) \). We consider \( y \in B^+ \) and let \( \varepsilon \) be so small that \( \Sigma(y, \varepsilon) \subseteq B^+ \). We apply the Theorem 9.5, for the domain \( B^+ \setminus \Sigma(y, \varepsilon) \), to the regular vector \( U(x) \) and to the vector \( V(x) = \Gamma^{(m)}(x, y; \omega), (m = 1, 2, \ldots, 12) \). Thus, we have

\[ \int_{B^+ \setminus \Sigma(y, \varepsilon)} \left[ \Gamma^{(m)}(x, y; \omega) D\left( \frac{\partial}{\partial x} \right) U(x) \right] dx \]

\[ = \int_{\partial B^+} \left[ \hat{\Gamma}^{(m)}(x, y; \omega) H\left( \frac{\partial}{\partial x}, n(x) \right) U(x) \right] dx \]

\[ - \hat{U}(x) H\left( \frac{\partial}{\partial x}, n(x) \right) \Gamma^{(m)}(x, y; \omega) dx \]

\[ + \int_{\partial \Sigma(y, \varepsilon)} \left[ \hat{\Gamma}^{(m)}(x, y; \omega) H\left( \frac{\partial}{\partial x}, n(x) \right) U(x) \Gamma^{(m)}(x, y; \omega) \right] dx \]

\[ - \hat{U}(x) H\left( \frac{\partial}{\partial x}, n(x) \right) \Gamma^{(m)}(x, y; \omega) \right] dx. \]

We observe that the last element of the vector \( D\left( \frac{\partial}{\partial x} \right) U(x) \) is zero. In consequence, we have

\[ \lim_{\varepsilon \to 0} \int_{B^+ \setminus \Sigma(y, \varepsilon)} \left[ \Gamma^{(m)}(x, y; \omega) D\left( \frac{\partial}{\partial x} \right) U(x) \right] dx \]

\[ = \int_{B^+} \left[ \hat{\Gamma}^{(m)}(x, y; \omega) D\left( \frac{\partial}{\partial x} \right) U(x) \right] dx, \] (9.100)
We also have
\[ \hat{\Gamma}^{(m)}(x, y; \omega) \leq \text{const} \varrho^{-1}, \quad m = 1, 2, ..., 12. \]  
(9.101)

Because \( L(x, y; \omega) \) has for \( x = y \) only a pole of the first order, if we use the above relation we obtain (see also [189])
\[ \lim_{\varepsilon \to 0} \int_{\partial \Sigma(y, \varepsilon)} \left[ \hat{\Gamma}^{(m)}(x, y; \omega) H \left( \frac{\partial}{\partial x}, n(x) \right) U(x) \right] \mathrm{d}a_x = 0 \]  
(9.102)
\[ \lim_{\varepsilon \to 0} \int_{\partial \Sigma(y, \varepsilon)} \left[ \hat{U}(x) H \left( \frac{\partial}{\partial x}, n(x) \right) \Gamma^{(m)}(x, y; \omega) \right] \mathrm{d}a_x = \begin{cases} 
u^s_l(y), & \text{if } m = l \\
\phi^s_l(y), & \text{if } m = 3 + l \\
u^f_l(y), & \text{if } m = 6 + l \\
\phi^f_l(y), & \text{if } m = 9 + l.
\end{cases} \]  
(9.103)

Using the relations (9.100)–(9.103), from (9.99) we get
\[ \hat{U}(y) = \int_{\partial B} \left\{ \hat{\Gamma}^T(x, y; \omega) H \left( \frac{\partial}{\partial x}, n(x) \right) U(x) \right\} \mathrm{d}a_x \]  
(9.104)
\[ - \left[ H \left( \frac{\partial}{\partial x}, n(x) \right) \hat{\Gamma} \left( x, y; \omega \right) \right]^T \hat{U}(x) \right\} \mathrm{d}a_x \]  
- \int_{B^+} \hat{\Gamma}^T(x, y; \omega) D \left( \frac{\partial}{\partial x} \right) U(x) \mathrm{d}v_x. \]

Taking into account the propriety of \( \Gamma(x, y; \omega) \), given by the Proposition 9.2, we obtain \((9.98)_1\). We substitute now \((9.98)_1\) in the system (9.38) and using the Proposition 9.2 we deduce \((9.98)_2\) and the proof is complete.

**Corollary 9.1** If \( U \) is a regular vector in \( B^+ \), solution of the homogeneous equation
\[ D \left( \frac{\partial}{\partial x} \right) U = 0, \]  
(9.105)
then
\[ \hat{U}(x) = \int_{\partial B} \left[ \hat{\Gamma}(x, y; \omega) H \left( \frac{\partial}{\partial y}, n(y) \right) U(y) - \hat{\Lambda}(x, y; \omega) \hat{U}(y) \right] \mathrm{d}a_y, \]
\[ U_{13}(x) = \sum_{p=1}^{12} \sum_{k=1}^{13} \int_{\partial B} \left[ \Gamma_{13,p}(x, y; \omega) H_{pk} \left( \frac{\partial}{\partial y}, n(y) \right) U_k(y) \right. \]  
(9.106)
\[ - U_p(y) H_{pk} \left( \frac{\partial}{\partial y}, n(y) \right) \Gamma_{13,k}(x, y; \omega) \right] \mathrm{d}a_y. \]

We observe that the vectors \( \Gamma^{(m)}(x, y; \omega), \quad (m = 1, 2, ..., 12) \) satisfy the conditions (9.76). Thus, the same procedure can be used in order to obtain representation formulas in \( B^- \), but under the supplementary assumption (9.76).
Corollary 9.2 If $U$ is a regular vector in $B^-$, solution of the equation (9.105), then

$$
\hat{U}(x) = \int_{\partial B} \left[ \tilde{\Gamma}(x, y; \omega) H \left( \frac{\partial}{\partial y}, n(y) \right) U(y) - \tilde{\Lambda}(x, y; \omega) \hat{U}(y) \right] da_y,
$$

$$
U_{13}(x) = \sum_{p=1}^{12} \sum_{k=1}^{13} \int_{\partial B} \left[ \Gamma_{13,p}(x, y; \omega) H_{pk} \left( \frac{\partial}{\partial y}, n(y) \right) U_k(y) - U_p(y) H_{pk} \left( \frac{\partial}{\partial y}, n(y) \right) \Gamma_{13,k}(x, y; \omega) \right] da_y,
$$

(9.107)

9.7 Reduction of the boundary value problems to integral equations

Let us introduce the potential of a single layer

$$
W^{(0)}(x; \omega; \varphi) = \int_{\partial B} \tilde{\Gamma}(x, y; \omega) \varphi(y) da_y,
$$

(9.108)

and the potential of a double layer

$$
W^{(1)}(x; \omega; \varphi) = \int_{\partial B} \Lambda(x, y; \omega) \varphi(y) da_y.
$$

(9.109)

where $\varphi$ are twelve–dimensional vectors which are Hölder continuous on $\partial B$.

As in classical theories [189, 192], these potentials are suggested by the representation formula (9.98) and we have that the potential of a single-layer and the potential of a double layer satisfy the homogeneous system (9.105). Moreover, the first 12 components of the potential of a single-layer are continuous on the whole space, including the surface $\partial B$, while the last component of the potential of a single-layer and the potential of a double layer are not continuous in passing through $\partial B$.

In view of the above remarks and in view of the results obtained in classical theories [189, 192], in the following we consider for study the system

$$
D \left( \frac{\partial}{\partial x} \right) U = 0,
$$

(9.110)

and the following boundary value problems:

**Interior problem** (I): to find the solution of the system (9.110) which is a regular vector in $B^+$, satisfying the condition

$$
\lim_{x \to z} \hat{U}(x) = G(z),
$$

(9.111)

where $x \in B^+$, $z \in \partial B$ and $G$ is a given vector satisfying Hölder’s condition.

**Exterior problem** (E): to find the solution of the system (9.110) which is a regular vector in $B^-$, satisfying the condition

$$
\lim_{x \to z} \hat{U}(x) = G(z),
$$

(9.112)

where $x \in B^-$, $z \in \partial B$ and $G$ is a given vector satisfying Hölder’s condition.

We denote by $(I_0)$ and $(E_0)$ the homogeneous problems corresponding to $(I)$ and $(E)$. If we proceed as in [189, 192] (see also [268, 136, 5]) we obtain the results:
Theorem 9.8 The potential of a double layer tends to finite limits when the point \( x \) tends to \( z \in \partial B \), both from within and from without, and these limits are respectively equal to

\[
\begin{align*}
\left[ \hat{W}^{(1)}(x; \omega; \varphi) \right]^+ &= -\frac{1}{2} \varphi(z) + \int_{\partial B} \hat{\Lambda}(z, y; \omega) \varphi(y) da_y \\
\left[ \hat{W}^{(1)}(x; \omega; \varphi) \right]^- &= \frac{1}{2} \varphi(z) + \int_{\partial B} \hat{\Lambda}(z, y; \omega) \varphi(y) da_y,
\end{align*}
\]

(9.113)

the integrals on the right hand side should be considered as principal values.

The proof of the theorem is based on the representation of \( \hat{W}^{(1)}(z; \omega; \varphi) \) in the form

\[
\hat{W}^{(1)}(x; \omega; \varphi) = \int_{\partial B} \hat{\Lambda}(x; \omega; \varphi) \left[ \varphi(y) - \varphi(z) \right] da_y \\
+ \int_{\partial B} \left[ \hat{\Lambda}(x; \omega; \varphi) - \Omega(x; \omega; \varphi) \right] \varphi(z) da_y \\
+ \int_{\partial B} \Omega(x; \omega; \varphi) \varphi(z) da_y.
\]

(9.114)

Because \( \varphi \) satisfies the Hölder condition and the difference (9.95) has for \( x = y \) have only a pole of the first order, it follows that the first two integrals are continuous. Using the relation (9.97), we obtain the desired results.

Theorem 9.9 The \( \mathbf{H} \left( \frac{\partial}{\partial x}, n(x) \right) \) operator of the single layer potential \( W^{(0)}(x; \omega; \varphi) \), tends to finite limits, as the point \( x \) tends to the boundary point \( z \in \partial B \) from within or from without, and these limits are respectively equal to

\[
\begin{align*}
\left[ \mathbf{H} \left( \frac{\partial}{\partial x}, n(z) \right) W^{(0)}(z; \omega; \varphi) \right]^+ &= \frac{1}{2} \varphi(z) + \int_{\partial B} \mathbf{H} \left( \frac{\partial}{\partial x}, n(z) \right) \hat{\Gamma}(z, y; \omega) \varphi(y) da_y \\
\left[ \mathbf{H} \left( \frac{\partial}{\partial z}, n(z) \right) W^{(0)}(z; \omega; \varphi) \right]^- &= -\frac{1}{2} \varphi(z) + \int_{\partial B} \mathbf{H} \left( \frac{\partial}{\partial z}, n(z) \right) \hat{\Gamma}(z, y; \omega) \varphi(y) da_y.
\end{align*}
\]

(9.115)

The proof of this theorem used the fact that

\[
\begin{align*}
\mathbf{H} \left( \frac{\partial}{\partial z}, n(z) \right) W^{(0)}(x; \omega; \varphi) &= \int_{\partial B} \mathbf{H} \left( \frac{\partial}{\partial x}, n(x) \right) \hat{\Gamma}(x, y; \omega) \\
+ \mathbf{H} \left( \frac{\partial}{\partial y}, n(y) \right) \hat{\Gamma}(y, x; \omega) \varphi(y) da_y \\
- \int_{\partial B} \mathbf{H} \left( \frac{\partial}{\partial y}, n(y) \right) \hat{\Gamma}(y, x; \omega) \varphi(y) da_y.
\end{align*}
\]

(9.116)
The bahavior of the last term of the right hand is given by the previous theorem. Using (9.88) we have that

\[
H\left(\frac{\partial}{\partial x}, n(x)\right)\tilde{\Gamma}(x, y; \omega) + H\left(\frac{\partial}{\partial y}, n(y)\right)\tilde{\Gamma}(y, x; \omega) = M(x, y; \omega) - M(y, x; \omega) + J(x, y; \omega),
\]

where \(J(x, y; \omega)\) is a matrix with mild singularity. Because the boundary is a Lyapunov surface, we can say that the term

\[
\int_{\partial B} [M(x, y; \omega) - M(y, x; \omega)] \varphi dy,
\]

in continuous. With this the proof is complete.

**Theorem 9.10** If there exists one of the limits \(H\left(\frac{\partial}{\partial z}, n(z)\right)W^{(1)}(z; \omega; \varphi)\) or \(H\left(\frac{\partial}{\partial z}, n(z)\right)W^{(1)}(z; \omega; \varphi)\) and it is continuous on \(\partial B\) then there exists the other limit and

\[
H\left(\frac{\partial}{\partial z}, n(z)\right)W^{(1)}(z; \omega; \varphi) = H\left(\frac{\partial}{\partial z}, n(z)\right)W^{(1)}(z; \omega; \varphi). \tag{9.119}
\]

We denote by \((I_0)\) and \((E_0)\) the homogeneous problems corresponding to the problems \((I)\) and \((E)\).

We seek the solution of the problem \((I)\) in the form of a double layer potential and the solution of the problem \((E)\) in the form

\[
U(x) = W^{(1)}(x; \omega; \varphi) + aW^{(0)}(x; \omega; \varphi) \tag{9.120}
\]

where \(a \in \mathbb{R}, a \neq 0\). On the basis of Theorem 9.8 we obtain for unknown densities, corresponding to the interior problem and respectively to the exterior problem, the following singular integral equations:

\[
-\frac{1}{2} \varphi(z) + \int_{\partial B} \tilde{\Lambda}(z, y; \omega) \varphi(y) dy = G(z), \tag{9.121}
\]

\[
\frac{1}{2} \varphi(z) + \int_{\partial B} \left[\tilde{\Lambda}(z, y; \omega) + a\tilde{\Gamma}(z, y; \omega)\right] \varphi(y) dy = G(z). \tag{9.122}
\]

We denote by \((\tilde{I})\) and \((\tilde{E})\) the above two-dimensional singular integral equations and by \((\tilde{I}_0)\) and \((\tilde{E}_0)\) the corresponding homogeneous equations, (for \(G(z) = 0\)).

In view of the relation

\[
\Delta(\Delta + k^2)E_m = \chi\delta(x - y) + (k^2 - k_m^2)\Delta E_m \quad (9.123)
\]

for \(m = 1, 2, \ldots, 7\) and \(k \in \mathbb{R}\) we can remark that because we choose the complex numbers \(k_n, n = 1, 2, \ldots, 7\) such that \(\text{Im}[k_n] \geq 0\), the solution considered for the exterior problem \((E)\) satisfies the conditions (9.76) (see [268]).
9.8 Existence theorems

Now, we show that the Fredholm theory is applicable to the singular integral equations introduced in the previous section.

Let us introduce the notation

$$k(z, y; \omega) = 2\Lambda(z, y; \omega)$$

(9.124)

and the following integral operator corresponding to the problem (I)

$$K'\varphi = -\varphi(z) + \int_{\partial B} k(z, y; \omega)\varphi(y)da_y.$$  

(9.125)

According to (9.95) and to the properties of Liapunov surfaces, we can write

$$k(z, y; \omega) = k_1(z, y; \omega) + k_2(z, y; \omega)$$

(9.126)

where the non-zero components of the matrix $k_1(z, y; \omega)$ are

$$k_{r_1}^{(1)} = \frac{\delta_1}{2\pi \rho^3}h_{r_1}, \quad k_{3+r_1}^{(1)} = \frac{\delta_2}{2\pi \rho^3}h_{r_1}, \quad k_{6+r_1}^{(1)} = \frac{\delta_3}{2\pi \rho^3}h_{r_1},$$

(9.127)

and the components of the matrix $k_2(z, y; \omega)$ satisfy

$$k_{m,n}^{(2)} = O(\rho^{-2}), \quad \{m, n\} \subset \{1, 2, \ldots, 12\}.$$  

(9.128)

We consider a local system of coordinates $\xi_1, \xi_2, \xi_3$ with origin at $z \in \partial B$ and the axis $\xi_3$ directed along $n(z)$.

In this local system of coordinates, the system (9.121) has the form

$$\varphi_{3m+\beta}(z) + \frac{\delta_{m+1}}{2\pi} \int_{\partial B} \frac{\xi_\beta}{\rho^3} \varphi_{3(m+1)}(y)da_y + T_{3m+\beta}(\varphi) = 2G_{3m+\beta},$$

$$\varphi_{3(m+1)}(z) - \frac{\delta_{m+1}}{2\pi} \int_{\partial B} \frac{\xi_\alpha}{\rho^3} \varphi_{3m+\alpha}(y)da_y + T_{3(m+1)}(\varphi) = 2G_{3(m+1)},$$

(9.129)

where the operators $T_n, n = 1, \ldots, 12$ are integral operators with mild singularities.

The characteristics [206] of the above singular integrals are $\xi_1 = \cos \theta$ and $\xi_2 = \sin \theta$; the symbols result from characteristics by multiplication by $2\pi i$ and the symbolic determinant $\delta$ of the last system is equal to

$$\delta = \prod_{m=1}^4 (1 - \delta_m^2).$$

(9.130)

We can observe that

$$1 - \delta_2^2 = \frac{[(\gamma^s - \beta^s)(\alpha^s + \gamma^s + \beta^s) + \gamma^s(\beta^s + 2\alpha^s + \gamma^s)]}{[2\gamma^s(\alpha^s + \beta^s + \gamma^s)]^2}.$$  

(9.131)
In view of the conditions (1.73) and (1.75), we deduce that the first factor of the above product is a sum of strictly positive numbers and the second factor is strictly positive. Thus, $1 - \delta^2 > 0$. Similarly we have that $1 - \frac{\delta^2}{2} > 0$. On the other hand

$$1 - \frac{\delta^2}{2} > 0.$$  

Similarly we have that $1 - \frac{\delta^2}{4} > 0$. On the other hand

$$1 - \delta^2 > 0.$$  

If we use the substitutions

$$\tilde{\alpha} = \lambda^s, \quad \tilde{\beta} = \mu^s, \quad \tilde{\gamma} = k^s + \mu^s,$$

using the above procedure, we can deduce that $1 - \delta^2 > 0$.

We obtain that $\delta \neq 0$ and in consequence the integral operator $K$ is of normal type. Proceeding in a similar way, we can show that the integral operator corresponding to the equation $(E)$ is of normal type. It is simple to observe that the symbolic matrix of the system (9.129) is Hermitian and in consequence the index of this system is equal to zero [206]. Thus, the operator $K$ is a Fredholm operator [274] in the space $(L^2(\partial B))^{12}$.

In order to apply the Fredholm’s theorems, we intend to study the homogeneous equations $(I_0)$ and $(E_0)$. For this, we study first the homogeneous equations $(I_0)$ and $(E_0)$.

**Theorem 9.11** The solution regular in $B^+$ of the problem $(I_0)$ is $\hat{U} = 0$, $U_{13} = P$, where $\text{grad } P = 0$.

**Proof.** If $U$ is a regular solution of the problem $(I_0)$, then in view of relations (9.79), we have

$$E(\hat{U}, \hat{U}) = E(\hat{U}, \hat{U}), C(\hat{U}, \hat{U}) = C(\hat{U}, \hat{U}).$$  

Using the relations (9.77) we can write

$$\int_{B^+} \left[ E(\hat{U}, \hat{U}) - \sum_{\sigma=s, f} \rho_0^\sigma \omega^2 (u_1 u_1^* \phi_1 \phi_1^* + j_1 \phi_1 \phi_1^*) \right] dx = \int_{B^+} C(\hat{U}, \hat{U}) dx,$$

$$\int_{B^+} \left[ E(\hat{U}, \hat{U}) - \sum_{\sigma=s, f} \rho_0^\sigma \omega^2 (u_1 u_1^* \phi_1 \phi_1^* + j_1 \phi_1 \phi_1^*) \right] dx = - \int_{B^+} C(\hat{U}, \hat{U}) dx.$$  

In consequence, we have

$$\int_{B^+} C(\hat{U}, \hat{U}) dx = 0.$$  

But $-\omega^{-1} C(\hat{U}, \hat{U})$ is a positive definite quadratic form, and thus, in view of the boundary conditions, we deduce

$$\hat{U}(x) = 0, \ x \in B^+$$  

and the proof is complete.

In a similar manner and using the asymptotic relations (9.76), we obtain

**Theorem 9.12** The solution regular in $B^-$ of the problem $(E_0)$ is identically zero.
Lemma 9.3 The integral equation \((\tilde{I})\) has solutions in \((L^2(\partial B))^2\) for all functions \(G\) which satisfy the condition

\[
\int_{\partial B} G_{6+r} n_r da = 0. \tag{9.137}
\]

Proof. To prove this result, we have to investigate the corresponding homogeneous adjoint equation of the integral equation. This is given by

\[
-\frac{1}{2} \psi(z) + \int_{\partial B} \left[ H \left( \frac{\partial}{\partial z}, n(z) \right) \Gamma(z, y; \omega) \right] \psi(y) dy = 0. \tag{9.138}
\]

We denote by \(\psi_0\) the solution of this equation and by \(_0 \tilde{W}^{(0)}(x; \omega; \psi_0)\) the potential of single layer with the density \(\psi_0\). Thus, in view of the Theorem 9.9, we have

\[
D \left( \frac{\partial}{\partial x} \right) _0 \tilde{W}^{(0)} = 0, \quad \left[ H \left( \frac{\partial}{\partial z}, n(z) \right) _0 \tilde{W}^{(0)}(z, \omega; \psi_0) \right]^- = 0. \tag{9.139}
\]

Proceeding in a similar way as in the proof of the Theorem 9.11, we can show that the above boundary problem has the solution

\[
_0 \tilde{W}^{(0)}(x, \omega; \psi_0) = 0, \quad _0 W^{(0)}_{13}(x, \omega; \psi_0) = \text{const. in } B^- . \tag{9.140}
\]

Because the single layer potential vanishes at infinity, we have

\[
_0 \tilde{W}^{(0)}(x, \omega; \psi_0) = 0 \quad \text{in } B^- . \tag{9.141}
\]

On the other hand, because the first 12 components of the single layer potential are continuous, we obtain

\[
_0 \tilde{W}^{(0)}(z, \omega; \psi_0) = 0, \quad \text{on } \partial B \tag{9.142}
\]

and

\[
D \left( \frac{\partial}{\partial x} \right) _0 \tilde{W}^{(0)} = 0, \quad \text{in } B^+. \tag{9.143}
\]

From the Theorem 9.11, we have

\[
_0 \tilde{W}^{(0)}(x, \omega; \psi_0) = 0, \quad _0 W^{(0)}_{13}(x, \omega; \psi_0) = \text{const. in } B^+. \tag{9.144}
\]

We can suppose that

\[
_0 W^{(0)}_{13}(x, \omega; \psi_0) = 1 \quad \text{in } B^+. \tag{9.145}
\]

Thus, we have

\[
\left[ H \left( \frac{\partial}{\partial z}, n(z) \right) _0 \tilde{W}^{(0)}(z, \omega; \psi_0) \right]^+ = (0, 0, 0, 0, 0, 0, -n_1(z), -n_2(z), -n_3(z), 0, 0, 0)^T. \tag{9.146}
\]

In view of the Theorem 9.10, we deduce

\[
\left[ \hat{H} \left( \frac{\partial}{\partial z}, n(z) \right) _0 W^{(0)}(z; \omega; \varphi_0) \right]^+ - \left[ \hat{H} \left( \frac{\partial}{\partial z}, n(z) \right) _0 W^{(0)}(z; \omega; \varphi_0) \right]^-= \psi(z). \tag{9.147}
\]

61
We can conclude that
\[ \psi(z) = (0, 0, 0, 0, 0, -n_1(z), -n_2(z), -n_3(z), 0, 0, 0)^T. \]  
(9.148)

In view of the Fredholm’s theory, for a solution of the integral equation \((\tilde{I})\) to exist, it is necessary and sufficient that the solution of the adjoint equation to be orthogonal to the free terms of the integral equation \((\tilde{I})\). With this, the proof is complete.

Moreover, as in [189], we can show that if \(G\) satisfies Hölder’s condition, then the solution of the system (9.129) satisfies also Hölder’s condition. Let remark that the relation (9.137) is in concordance with the boundary conditions from the section 1.2.

Thus, we can conclude that:

**Theorem 9.13** The problem \((I)\) has solution for any vector \(G(z)\) satisfying Hölder’s condition and condition (9.137). This solution can be expressed by a double layer potential and is unique excepting a constant pressure.

To study the existence of the solution for the exterior problem \((E)\), we establish the following result:

**Lemma 9.4** The homogeneous equations \((\tilde{E}_0)\) have only the trivial solution.

*Proof.* We denote by \(\varphi_0\), a solution of the equation \((\tilde{E}_0)\). The function
\[ \theta U(x) = W^{(1)}(x; \omega; \varphi_0) + a W^{(0)}(x; \omega; \varphi_0) \]  
(9.149)

satisfies the relation (9.76) and
\[ D \left( \frac{\partial}{\partial x} \right)_0 U = 0, \left[ \theta \hat{U}(z) \right]^- = 0. \]  
(9.150)

From the uniqueness theorem it follows
\[ \theta U(x) = 0, x \in B^- . \]  
(9.151)

On the other hand, according to Theorems 9.8 and 9.9, we obtain
\[ \left[ \theta \hat{U}(z) \right]^- - \left[ \theta \hat{U}(z) \right]^+ = \varphi_0(z), \]  
(9.152)

\[ \left[ \mathbf{H} \left( \frac{\partial}{\partial z}, n(z) \right)_0 U(z) \right]^- - \left[ \mathbf{H} \left( \frac{\partial}{\partial z}, n(z) \right)_0 U(z) \right]^+ = -a \varphi_0(z), \]  
and in consequence
\[ \left[ \mathbf{H} \left( \frac{\partial}{\partial z}, n(z) \right)_0 U(z) + a_0 \hat{U}(z) \right]^+ = 0. \]  
(9.153)

Hence, \(\theta U\) is a solution of the equation (9.150) in \(B^+\), that satisfies the above boundary condition. In view of the Theorem 9.4, we have
\[ \int_{B^+} \left[ E(\theta \hat{U}, \theta \hat{U}) - \sum_{\sigma=s, f} \rho_{0}^2 \omega^2 (u^*_{q \sigma} n_{q \sigma}^* + j^*_{q \sigma} n_{q \sigma} + \varphi^*_{q \sigma}) \right] dv_x \]  
= \[ \int_{B^+} C(\theta \hat{U}, \theta \hat{U}) dv_x + \int_{\partial B} \left[ \theta \hat{U}^+ \right] \left[ \mathbf{H} \left( \frac{\partial}{\partial x}, n(x) \right)_0 U \right]^+ da_x \]  
(9.154)
\[
\int_{B^+} \left[ E(0, \hat{U}, 0) - \sum_{\sigma=s,f} \rho^{\sigma}_0 \omega^2 (0 u_i^{\sigma} \partial u_i^{\sigma} + j^\sigma_0 \phi^{*\sigma}_i \partial \phi^{\sigma}_i) \right] dv_x \\
= - \int_{B^+} C(0, \hat{U}, 0) dv_x + \int_{\partial B} [0 \hat{U}]^+ \left[ \bar{H} \left( \frac{\partial}{\partial x}, n(x) \right)_0 \hat{U} \right]^+ da_x.
\]

(9.155)

Thus, keeping in mind that \(i \omega^{-1} C(0, \hat{U}, 0)\) is a positive definite quadratic form and in view of the relation (9.152), we obtain

\[
0 \hat{U}(x) = 0, \ x \in B^+.
\]

(9.156)

From Theorem 9.8, we have

\[
\varphi_0(z) = \left[0 \hat{U}(z)\right]^- - \left[0 \hat{U}(z)\right]^+ = 0, \ z \in \partial B
\]

(9.157)

and the proof is complete.

We can conclude that:

**Theorem 9.14** The problem \((E)\) has solution for any vector \(G(z)\) that satisfies Hölder’s condition. This solution is unique and can be expressed in the form of a combination of the double layer potential with some single layer potential.

### 10 Uniqueness for a nonlinearity

#### 10.1 Uniqueness for bounded domains

In this subsection, we consider a bounded regular region \(B\) of Euclidean three-dimensional space, whose boundary is the regular surface \(\partial B\). This domain is filled with a micropolar mixture of a micropolar elastic solid and a non-heat conducting incompressible fluid.

In the case of small deformation of the solid and of non-slow flow of the fluid, the equations of the theory of micropolar solid–fluid mixtures are

\[
\begin{align*}
(\lambda^s + \mu^s) u^s_{i,j} + (\mu^s + k^s) u^s_{i,j} + k^s \varepsilon_{ijk} \phi^s_{k,j} - \xi(v^s_i - v^s_j) \\
+ \rho^s f^s_i &= \rho^s u^s_i, \\
(\alpha^s + \beta^s) \phi^s_{i,j} + \gamma^s \phi^s_{i,j} + k^s (\varepsilon_{ijk} u^s_{k,j} - 2 \phi^s_i) - \varpi (v^s_i - v^s_j) \\
+ \rho^s \ell^s_i &= \rho^s j^s \phi^s_{i,j}, \\
- \pi^s_i + (\mu^f + k^f) v^f_{i,j} + k^f \varepsilon_{ijk} v^f_{k,j} + \xi (v^f_i - v^f_j) \\
+ \rho^f f^f_i &= \rho^f (v^f_i + v^f_{i,j}), \\
(\alpha^f + \beta^f) v^f_{i,j} + \gamma^f v^f_{i,j} + k^f (\varepsilon_{ijk} v^f_{k,j} - 2 v^f_i) + \varpi (v^f_i - v^f_j) \\
+ \rho^f \ell^f_i &= \rho^f j^f (v^f_i + v^f_{i,j})
\end{align*}
\]

in \(B \times I\).

We assume that \(\rho^s\) and \(j^s\) are positive constants and \(f^s_i, \ell^s_i\) are continuous functions on \(B \times I\).
To these equations we adjoin the following boundary conditions

\[ u^s_i(x, t) = 0, \quad \phi^s_i(x, t) = \phi^{s*}_i(x, t), \quad \nu^f_i(x, t) = \nu^{s*}_i(x, t) \] on \( \partial B \times I \), \( 10.2 \)

and the following initial conditions

\[ u^s_i(x, 0) = g^s_i(x), \quad \nu^s_i(x, 0) = h^s_i(x), \]
\[ \phi^s_i(x, 0) = a^s_i(x), \quad \nu^s_i(x, 0) = b^s_i(x) \] in \( B \), \( 10.3 \)

where \( \phi^{s*}_i, \nu^{s*}_i, g^s_i, h^s_i, a^s_i \) and \( b^s_i \) are prescribed continuous functions.

We omit to mention the explicit dependence of functions on their spatial argument and when there will be no ambiguities, we will omit to specify the dependence of functions on the time variable.

We denote by \( (P) \) the initial-boundary values problem defined by the equations \( 10.1 \), the initial conditions \( 10.3 \) and the boundary conditions \( 10.2 \). We say that \( (u^s_i, v^s_i, \phi^s_i, \nu^s_i, \pi^f_i) \) is an admissible process on \( B \times I \) provided:

(a) \( u^s_i \) and \( \phi^s_i \) are of class \( C^2 \) on \( B \times I \);
(b) \( v^f_i \) and \( \nu^f_i \) are of class \( C^{2,1} \) on \( B \times I \);
(c) \( u^s_i, v^s_i, \phi^s_i, \nu^s_i \) are of class \( C^0 \) on \( \partial B \times I \);
(d) \( \pi^f_i \) is of class \( C^{1,0} \) on \( B \times I \).

By a solution of the boundary-initial value problem \( (P) \) we mean an admissible process that satisfies the equations \( 10.1 \), the initial conditions \( 10.3 \) and the boundary conditions \( 10.2 \).

In this section we study the uniqueness and continuous dependence of classical solutions to the boundary-initial value problem \( (P) \) in the time interval \( I = [0, T] \), when \( B \) is a bounded domain.

Let be \( (\tilde{u}^s, \tilde{v}^f, \tilde{\phi}^s, \tilde{\nu}^f, \tilde{\pi}^f) \) and \( (\hat{u}^s, \hat{v}^f, \hat{\phi}^s, \hat{\nu}^f, \hat{\pi}^f) \) two solutions of the problem \( P \), corresponding to the same boundary data and to the initial conditions \( (\hat{g}^s, \hat{h}^s, \hat{a}^s, \hat{b}^s, \hat{h}^f, \hat{b}^f) \) and \( (\tilde{g}^s, \tilde{h}^s, \tilde{a}^s, \tilde{b}^s, \tilde{h}^f, \tilde{b}^f) \), respectively, and to the body loads \( (\hat{f}^s, \hat{\ell}^s) \) and \( (\tilde{f}^s, \tilde{\ell}^s) \), respectively.

In order to simplify the notation, we denote actually the difference of these solutions by \( (u^s, v^f, \phi^s, \nu^f, \pi^f) \). This is a solution of the equations

\[ (\lambda^s + \mu^s)u^s_{ij,jj} + (\mu^s + k^s)u^s_{i,jj} + k^s \varepsilon_{ijk}\phi^s_{k,ij} - \xi(v^s_i - v^f_i) + \rho^s(\tilde{f}^s_i - \tilde{f}^s_i) = \rho^s u^s_i, \]
\[ (\alpha^s + \beta^s)\phi^s_{ij,ij} + \gamma^s \phi^s_{i,ij} + k^s (\varepsilon_{ijk} u^s_{k,ij} - 2 \phi^s_{i,ij}) - \varpi(v^s_i - v^f_i) + \rho^s(\tilde{\ell}^s_i - \tilde{\ell}^s_i) = \rho^s j^s \phi^s_i, \]
\[ -\pi^s_i + (\mu^f + k^f)v^f_{i,ij,jj} + k^f \varepsilon_{ijk}\nu^f_{k,ij} + \xi(v^f_i - v^s_i) + \rho^f(\tilde{f}^f_i - \tilde{f}^f_i) = \rho^f (v^f_i + \tilde{v}^f_i v^f_i + v^f_{i,j} \tilde{v}^f_j), \]
\[ (\alpha^f + \beta^f)\nu^f_{ij,ij} + \gamma^f \nu^f_{i,ij} + k^f (\varepsilon_{ijk} v^f_{k,ij} - 2 \phi^f_i) + \varpi(v^s_i - v^f_i) + \rho^f(\tilde{\ell}^f_i - \tilde{\ell}^f_i) = \rho^f j^f (v^f_i + \tilde{v}^f_i v^f_i + v^f_{i,j} \tilde{v}^f_j) \]
in $B \times I$, with the boundary conditions
\[
\begin{align*}
  u_i^s(x,t) = 0, & \quad \phi_i^s(x,t) = 0, & \quad u_i^f(x,t) = 0, & \quad \nu_i^f(x,t) = 0 \quad \text{on } \partial B \times I, \\
\end{align*}
\]
and the initial conditions
\[
\begin{align*}
  u_i^s(x,0) = \tilde{g}_i^s(x) - \tilde{g}_i^s(x), & \quad v_i^o(x,0) = \tilde{h}_i^o(x) - \tilde{h}_i^o(x), \\
  \phi_i^s(x,0) = \tilde{a}_i^s(x) - \tilde{a}_i^s(x), & \quad \nu_i^o(x,0) = \tilde{b}_i^o(x) - \tilde{b}_i^o(x) \quad \text{in } B \\
\end{align*}
\]

We associate with an admissible process $(u^s, v^f, \phi^s, \nu^f, \pi^f)$, the kinetic energies
\[
K^s(t) = \frac{1}{2} \int_B (\rho^s \bar{v}_i^o(t) - \bar{v}_i^o(t) + \rho^s \bar{w}_i^o(t) \bar{v}_i^o(t)) dv, 
\]
the internal energy
\[
U(t) = \int_B E((u^s, \phi^s), (u^s, \phi^s)) dv, 
\]
the dissipation energy
\[
D(t) = \int_0^t \int_B \Phi(v^s, \nu^f, \nu^f) dvds, 
\]
and the total energy
\[
E(t) = K^s(t) + K^f(t) + U(t) + D(t). 
\]

**Theorem 10.1** (Continuous dependence) Let $(u^s, v^f, \phi^s, \nu^f, \pi^f)$ be a solution of the initial-boundary value problem defined by relations (10.4)–(10.6). If the internal energy and the dissipation potential are positive semidefinite forms, then we have
\[
\begin{align*}
  [K^s(t) + K^f(t) + U(t)]^{\frac{1}{2}} \leq e^{\sigma t} \left\{ [K^s(0) + K^f(0) + U(0)]^{\frac{1}{2}} \\
  + \int_0^t \int_B \left( \rho^s(f_i^\alpha - \bar{f}_i^\alpha)(\dot{\hat{e}}_i^\alpha - \bar{\hat{e}}_i^\alpha) + \rho^o(\dot{\hat{e}}_i^\alpha - \bar{\hat{e}}_i^\alpha)(\dot{\hat{e}}_i^\alpha - \bar{\hat{e}}_i^\alpha) \right) dvds \right\}, 
\end{align*}
\]
where
\[
\sigma^2 = \frac{9}{4} \sup_{B \times [0,T]} \left( \dot{v}_{i,j}^f \dot{v}_{i,j}^f + j^f \dot{v}_{i,j}^f \dot{v}_{i,j}^f \right).
\]

**Proof.** As a consequence of Eqs. (10.4)–(10.6), using the divergence theorem, we have
\[
\frac{d}{dt} E(t) = \sum_{\alpha = s,f} \int_B \left[ \rho^s(f_i^\alpha - \bar{f}_i^\alpha)v_i^\alpha + \rho^o(\dot{\hat{e}}_i^\alpha - \bar{\hat{e}}_i^\alpha)v_i^\alpha \right] dv \\
- \int_B \rho^f(\dot{v}_{i,j}^f v_i^f + j^f \dot{v}_{i,j}^f v_i^f) dv.
\]
Using the positive definiteness of the dissipation potential, by a direct integration we obtain
\[
\begin{align*}
  K^s(t) + K^f(t) + U(t) \leq K^s(0) + K^f(0) + U(0) \\
  - \int_0^t \int_B \rho^s(\dot{v}_{i,j}^f v_i^f + j^f \dot{v}_{i,j}^f v_i^f) dvds \\
  + \sum_{\alpha = s,f} \int_0^t \int_B [\rho^o(f_i^\alpha - \bar{f}_i^\alpha)v_i^\alpha + \rho^o(\dot{\hat{e}}_i^\alpha - \bar{\hat{e}}_i^\alpha)v_i^\alpha] dvds.
\end{align*}
\]
Let $\Omega$ be a regular region of the Euclidean three-dimensional space. We denote by $B_{10}$ the open ball of center $O$ and radius $10$.

Uniqueness for exterior domains

We consider a point $O \in \Omega$, and let $\Gamma(O; R)$ be the open ball of center $O$ and radius $R$ and boundary $S_R$. We consider a Cartesian coordinate system with the origin in the point $O$.

From the inequalities (10.13) and (10.14) we get

$$y(t) \leq y(0) + \int_0^t \left[ \frac{3M}{2} y(s) + h(s) y^{\frac{3}{2}}(s) \right] ds,$$

where

$$y(t) = K^*(t) + K^f(t) + U(t).$$

By the Gronwall type inequality established by Dafermos (see Lemma 4.1. from the paper [35]) we deduce the relation (10.11) and the proof is complete.

**Corollary 10.1** (Uniqueness result) Assume that

(i) $\rho^\alpha$, $j^\alpha$ are strictly positives constants;

(ii) the internal energy and the dissipation potential are positive semidefinite forms.

Then two solutions $(\hat{u}^s, \hat{v}^f, \hat{\phi}^s, \hat{\phi}^f, \hat{\pi}^f)$ and $(\tilde{u}^s, \tilde{v}^f, \tilde{\phi}^s, \tilde{\phi}^f, \tilde{\pi}^f)$ corresponding to the same given data are such that

$$\hat{u}^s = \tilde{u}^s, \quad \hat{v}^f = \tilde{v}^f, \quad \hat{\phi}^s = \tilde{\phi}^s, \quad \hat{\phi}^f = \tilde{\phi}^f, \quad \hat{\pi}^f = \tilde{\pi}^f + \pi^f,$$

where $\pi^f$ satisfies the relation

$$\text{grad } \pi^f = 0.$$

**Proof.** From the previous theorem we deduce the relations (10.17)1–4. If we use these relations in equations (10.4)3 we obtain relation (10.18) and the proof is complete.

10.2 Uniqueness for exterior domains

Let $\Omega$ be a regular region of the Euclidean three-dimensional space. We denote by $B$ the exterior of the fixed bounded region $\Omega$. We consider a point $O \in \Omega$, and let $\Gamma(O; R)$ be the open ball of center $O$ and radius $R$ and boundary $S_R$. We consider a Cartesian coordinate system with the origin in the point $O$. 

66
We introduce the notations: \( r = |x|, R_0 = \inf \{ R > 0; \Omega \subset \Gamma(O; R) \} \) and for \( R > R_0 \) we consider the set \( \Omega_R = \Gamma(O; R) \setminus \Omega \).

In this section we study the uniqueness problem in connection with the classical solutions of the boundary-initial value problem \( (P) \) [135]. By a solution of the boundary-initial value problem \( (P) \), corresponding to the external data system \( \{ f_i^s, f_i^s, u_i^s, \phi_i^s, \nu_i^s, g_i^s, h_i^s, a_i^s, b_i^s \} \), we mean an admissible process that satisfies the equations (10.1), the initial conditions (10.3) and the boundary conditions (10.2).

Let us introduce the quantities

\[
\mathcal{K}^\sigma = \frac{1}{2} (\rho^\sigma v_i^\sigma v_i^\sigma + \rho^\sigma j^\sigma v_i^\sigma v_i^\sigma), \quad \mathcal{K} = \sum_{\sigma=s,f} \mathcal{K}^\sigma. \tag{10.19}
\]

**Theorem 10.2** Let \( (u^{s(1)}, \phi^{s(1)}, v^{f(1)}, \nu^{f(1)}, \pi^{f(1)}) \) and \( (u^{s(2)}, \phi^{s(2)}, v^{f(2)}, \nu^{f(2)}, \pi^{f(2)}) \) be two solutions of the problem \( (P) \), corresponding to the same external data system \( \{ F_i^s, L_i^s, u_i^s, \phi_i^s, \nu_i^s, \nu_i^f, v_i^f, g_i^s, h_i^s, a_i^s, b_i^s \} \), and let us assume that

(i) \( \rho^\sigma \) and \( j^\sigma \) are positive constants;

(ii) The internal energy density and the dissipation potential are positive defined;

(iii) \( |v_i^{f(\alpha)}| \leq A_1, |v_i^{s(\alpha)}| \leq A_2 (\alpha = 1, 2) \) for each \( (x, t) \in B \times I \), where \( A_1 \) and \( A_2 \) are positive constants;

(iv) \( v_i^{s(\alpha)}, v_i^{s(\alpha)}, u_{i,j}^{s(\alpha)}, \phi_{i,j}^{s(\alpha)} \) and \( \nu_{i,j}^{f(\alpha)} \) may be unbounded but there exist the real numbers \( k_q \) and the positive real numbers, \( M_q, q = 1, ..., 6 \) and \( \bar{r} \) such that

\[
\begin{align*}
|v_i^{s(\alpha)}| &\leq M_1 r^{k_1}, \quad |v_i^{s(\alpha)}| \leq M_2 r^{k_2}, \quad |u_{i,j}^{s(\alpha)}| \leq M_3 r^{k_3}, \\
|v_i^{f(\alpha)}| &\leq M_4 r^{k_4}, \quad |\phi_{i,j}^{s(\alpha)}| \leq M_5 r^{k_5}, \quad |\nu_{i,j}^{f(\alpha)}| \leq M_6 r^{k_6}
\end{align*}
\tag{10.20}
\]

for all \( r > \bar{r} \);

(v) the pressure difference satisfies the Serrin’s condition\(^1\)

\[
|\pi^{f(1)} - \pi^{f(2)}| \leq A_3 r^{-\varepsilon-\frac{1}{2}} \tag{10.21}
\]

as \( r \to \infty \), for a positive constant \( A_3 \) and any preassigned \( \varepsilon \in (0, 1) \),

then

\[
\begin{align*}
u^{f(1)} &= u^{s(2)}, \quad \phi^{s(1)} = \phi^{s(2)}, \\
v^{f(1)} &= v^{f(2)}, \quad \nu^{f(1)} = \nu^{f(2)}, \quad \pi^{f(1)} = \pi^{f(2)} + \pi',
\end{align*}
\tag{10.22}
\]

where

\[
\pi_{i,j}' = 0. \tag{10.23}
\]

\(^1\)This condition was used by [90] at the Serrin’s suggestion
Proof. We denote by \((u^s, \phi^s, v^f, \nu^f, \pi^f)\) the difference between the two solutions. This difference is solution of the problem defined by the following equations

\[
(\lambda^s + \mu^s)u^s_{ij,t} + (\mu^s + k^s)u^s_{ij,j} + k^s\varepsilon_{ijk}\phi^s_{k,j} - \xi(v^s_i - v^f_i) = \rho^s \frac{\partial^2}{\partial t^2} u^s_i,
\]

\[
(\alpha^s + \beta^s)\phi^s_{ij,t} + \gamma^s\phi^s_{ij,j} + k^s(\varepsilon_{ijk}u^s_{k,j} - 2\phi^s_{ij}) - \varpi(v^s_i - v^f_i) = \rho^s \frac{\partial}{\partial t} \phi^s_{ij}.
\]

\[
-\pi^f_{ij} + (\mu^f + k^f)v^f_{ij,t} + k^f\varepsilon_{ijk}v^f_{k,j} + \xi(v^s_i - v^f_i) = \rho^f \left( \frac{\partial}{\partial t} v^f_{ij} + v^f_{ij}(v^f_{ij}) \right) + \varpi(v^s_i - v^f_i)
\]

\[
\frac{\partial}{\partial t} v^f_{ij} + v^f_{ij} = 0,
\]

with homogeneous boundary conditions on \(\partial \Omega\) and null initial data.

We introduce the function

\[
g(r) = e^{-dr}, \quad d > 0
\]

and with the help of this function we define the quantity

\[
G = \begin{bmatrix} t^s_{ij} - \xi(v^s_i - v^f_i) \\ m^s_{ij} + \varepsilon_{ijk} t^s_{k,j} - \varpi(v^s_i - v^f_i) \end{bmatrix} \text{gvij}^s + \begin{bmatrix} m^s_{ij} + \varepsilon_{ijk} t^s_{k,j} + \varpi(v^s_i - v^f_i) \end{bmatrix} \text{gvij}^f,
\]

where \(t^s_{ij}\) and \(m^s_{ij}\) correspond to the difference \((u^s, \phi^s, v^f, \nu^f, \pi^f)\).

Using the equations of motion \((10.24)\), we obtain

\[
G = \frac{\partial}{\partial t} K + \left( \frac{\partial}{\partial t} K \text{gvj}^{f(2)} \right)_{ij} - g_{ij} \text{Kf} v^{f(2)} + \rho^f g_{ij} \left( v_{ij}^{f(1)} v_{ij}^f + j^f v_{ij}^{f(1)} v_{ij}^f \right) \]

\[
- \rho^f g_{ij} \left( v_{ij}^{f(1)} v_{ij}^f + j^f v_{ij}^{f(1)} v_{ij}^f \right) v_{ij}^f.
\]

On the other hand, using the constitutive equations, we have

\[
G = Q_{ij} - g_{ij} \frac{\partial}{\partial t} E(u^s, \phi^s) - g\Phi(v^s, \nu^f, \nu^f) - \mathcal{W}_1^s(e^s_{ij}, g_{ij} v^f_{ij})
\]

\[
- \mathcal{W}_2^s(\gamma^s_{ij}, v^f_{ij}) - \mathcal{W}_2^f (a^f_{ij}, g_{ij} v^f_{ij}) - \mathcal{W}_2^f (b^f_{ij}, v^f_{ij} g_{ij}) + \pi^f g_{ij} v^f_{ij},
\]

where

\[
Q_{ij} = \lambda^s e_{ij} \text{gvj}^s + (\mu^s + k^s) e_{ij} v^s_i + \mu^s e_{ij} v^f_i
\]

\[
\alpha^s e_{ij} v^f_i + \beta^s e_{ij} v^s_i + \gamma^s e_{ij} v^f_i
\]

\[
- \pi^f v^f_i + (\mu^f + k^f) a^f_{ij} v^f_i + \mu^f a^f_{ij} v^f_i
\]

\[
+ \alpha^f b^f_{ij} v^f_i + \beta^f b^f_{ij} v^f_i + \gamma^f b^f_{ij} v^f_i.
\]
Let remark that, in view of the incompressibility condition (10.24), we have

\[(v_{j,i}^f g v_i^f)_{,j} = v_{j,i}^f g v_i^f + v_{j,i}^f g v_i^f + v_{j,i}^f g v_i^f)_{,j} = v_{j,i}^f g v_i^f + v_{j,i}^f g v_i^f + v_{j,i}^f g v_i^f + v_{j,i}^f g v_i^f\]

and thus, using the relations (10.27) and (10.28), we deduce

\[g \frac{\partial}{\partial t} [K(t) + E(u^s, \phi^s)] + g \Phi(v^s, \nu^s, \nu^f, \nu^f)\]

\[K_{j,j} - W_1^s(e_{ij}^s, g_i v_j^s) - W_2^s(\gamma_{ij}^s, \nu_i^s g_j)\]

\[-W_1^f(a_{ij}^f, g_i v_j^f) - W_2^f(b_{ij}^f, v_i^f g_j) + \pi^f g_j v_j^f + g_j K_v f v_j^{f(2)}\]

\[= \frac{\partial}{\partial t} \frac{\partial}{\partial t} \left\{ W_1^s(e_{ij}^s, g_i v_j^s) - W_2^s(\gamma_{ij}^s, \nu_i^s g_j)\right\}
\]

\[+ \frac{\partial}{\partial t} \left\{ W_1^f(a_{ij}^f, g_i v_j^f) - W_2^f(b_{ij}^f, v_i^f g_j) + \pi^f g_j v_j^f + g_j K_v f v_j^{f(2)}\right\}\]

where

\[K_j = Q_j - K_v f v_j^{f(2)} - \rho^f g_j (v_i^{f(1)} v_i^f + j f v_i^{f(1)} v_i^f) v_j^f - \varepsilon_1 v_{j,i}^f g v_i^f + \varepsilon_1 v_{j,i}^f g v_i^f\]

and \(\varepsilon_1\) is a positive constant.

We remark that

\[|g_i| \leq d g, \quad g_{i,i} \leq d^2 g, \quad g_{i,i} = d^2 g^2.\]

In the following we use the Schwarz’s inequality, the arithmetic geometric mean inequality and the relations (11.19) and (10.33) to obtain

\[W_1^s(e_{ij}^s, g_i v_j^s) \leq \frac{\varepsilon_2}{2} g W_1^s(e_{ij}^s, e_{ij}^s) + \frac{1}{2 \varepsilon_2 g} W_1^s(g_i v_j^s, g_i v_j^s)\]

\[\leq \frac{\varepsilon_2}{2} g W_1^s(e_{ij}^s, e_{ij}^s) + \frac{1}{2 \varepsilon_2} \sigma_M^2 d^2 g v_j^s v_j^s\]

for every \(\varepsilon_2 > 0\).

Moreover, in view of the hypotheses of the theorem, the Schwarz’s inequality and the arith-
metic geometric mean inequality, we get

\[ W_2^s(\gamma_s^i, \nu^s_{i,j}) \leq \frac{\varepsilon_2}{2} g W_2^s(\gamma_s^i, \gamma_s^i) + \frac{1}{2\varepsilon_2} \delta_M d^2 g v_j^s v_j^s, \]

\[ W_1^f(a_{i,j}, g, g_{i,j}) \leq \frac{\varepsilon_3}{2} g W_1^f(a_{i,j}, a_{i,j}) + \frac{1}{2\varepsilon_3} \sigma_M d^2 g v_j^f v_j^f, \]

\[ W_2^f(b_{i,j}, \nu^f_{i,j}) \leq \frac{\varepsilon_3}{2} g W_2^f(b_{i,j}, b_{i,j}) + \frac{1}{2\varepsilon_3} \delta_M d^2 g v_j^f v_j^f, \]

\[ g_{j} v_i^{(1)} v_j^f \leq \frac{1}{2} \left( \sqrt{3} A_1 g d g_{j} v_i^{f(1)} v_j v_j + \frac{gd}{\sqrt{3} A_1} v_i^{f(1)} v_i^{f(1)} v_j v_j \right) \leq \sqrt{3} A_1 g d v_i^f v_i^f, \]

\[ g_{j} v_j^{(2)} \leq 3d g A_1, \quad (10.35) \]

\[ g_{j} v_i^{(1)} v_j^f \leq \frac{dg}{2} (3\varepsilon_4 A_2^2 v_i^f v_i^f + \frac{1}{\varepsilon_4} v_i^f v_i^f), \]

\[ v_i^{(1)} v_i^{f(1)} v_j v_j \leq \frac{1}{2} \left( 3 A_1^2 \varepsilon_5 v_i^f v_i^f + \varepsilon_5 v_i v_i v_i v_i \right), \]

\[ v_i^{(1)} v_i^{f(1)} v_j v_j \leq \frac{1}{2} \left( 3 A_2^2 \varepsilon_5 v_i^f v_i^f + \varepsilon_5 v_i v_i v_i v_i \right), \]

\[ g_{j} v_j^f v_j^f \leq \frac{dg}{2} \left( \varepsilon_7 v_i v_j v_i v_j + \varepsilon_7 v_i v_i v_i v_i \right), \]

\[ g_i v_i^f v_i^f \pi^f \leq \frac{1}{2} \left[ \varepsilon_8 d^2 g(\pi^f)^2 + \frac{1}{\varepsilon_8} g v_i^f v_i^f \right]. \]

Thus, from (10.31), (10.34) and (10.35) we deduce

\[ g \frac{\partial}{\partial t} [\mathcal{K}(t) + \mathcal{E}(u^s, \phi^s)] + g \Phi(v^s, \nu^s, v^f, \nu^f) \]

\[ \leq K_{j,i} + \frac{\varepsilon_2 g}{2} \left( W_1^s(\varepsilon_s^i, \varepsilon_s^i) + W_2^s(\gamma_s^i, \gamma_s^i) \right) \]

\[ + \frac{\varepsilon_3 g}{2} \left( W_1^f(a_{i,j}, a_{i,j}) + W_2^f(b_{i,j}, b_{i,j}) \right) \]

\[ + \frac{gd^2}{2\varepsilon_2} \left( \sigma_M v_s^s v_j^s + \delta_M v_j^f v_j^f \right) + \frac{gd^2}{2\varepsilon_3} \left( \sigma_M v_j^f v_j^f + \delta_M v_j^f v_j^f \right) \]

\[ + \frac{1}{2} \varepsilon_8 d^2 g(\pi^f)^2 + \frac{1}{\varepsilon_8} g v_i^f v_i^f \] + 3gA_1 K^f + \sqrt{3}\rho^f g A_1 v_i^f v_i^f \]

\[ + \frac{\rho^f g}{2} \left( 3\varepsilon_4 A_2^2 v_i^f v_i^f + \frac{1}{\varepsilon_4} v_i^f v_i^f \right) + \frac{\rho^f g}{2} \left( 3 A_1^2 \varepsilon_5 v_i^f v_i^f + \varepsilon_5 v_i v_i v_i v_i \right) \]

\[ + \frac{\rho^f g}{2} \left( \varepsilon_7 v_i v_j v_i v_j + \varepsilon_7 v_i v_i v_i v_i \right) + \frac{\rho^f g}{2} \left( \varepsilon_7 v_i v_j v_i v_j + \varepsilon_7 v_i v_i v_i v_i \right), \]

\[ + \frac{\varepsilon_1 d^2}{2} \left( \frac{\varepsilon_7}{\rho^f} v_i v_j v_i v_j + \frac{\rho^f}{\varepsilon_7} v_i^f v_i^f \right). \]
Now, choosing
\[
\varepsilon_2 = \frac{\sqrt{c_M}}{R_0}, \quad 0 < \varepsilon_7 < \frac{2\rho_f}{d}, \quad \varepsilon_5 = \frac{2\varepsilon_1}{\rho_f}\left(1 - \frac{d\varepsilon_7}{2\rho_f}\right),
\]
\[
\varepsilon_4 = \frac{1}{\sqrt{3j^f A_2}}, \quad \varepsilon_6 = \frac{4\delta_m^f}{\sigma_m^f j^f \varepsilon_1},
\]
\[
\varepsilon_8 > 0, \quad \varepsilon_M^\sigma = \max \left\{ \frac{\sigma_M^\sigma}{\rho_m^\sigma}, \frac{\delta_m^\sigma}{\rho_m^\sigma} \right\}
\] (10.37)

and introducing the notation
\[
\tilde{a}(\varepsilon_1) = \frac{1}{\varepsilon_8 \rho_f} + 2\sqrt{3dA_1} + \frac{3A_1^2}{\varepsilon_5} + \frac{3j^f A_2^2}{\varepsilon_6} + \frac{d\varepsilon_1}{\varepsilon_7},
\] (10.38)

then, from (10.36) we obtain
\[
g \frac{\partial}{\partial t} \left[ \mathcal{K} + \mathcal{E}(\mathbf{u}^s, \phi^s) \right] + g\Phi(\mathbf{v}^s, \nu^s, \nu^f, \nu^f) \leq K_{j,j} + \frac{g\sqrt{c_M} R_0}{R_0} \mathcal{E}(\mathbf{u}^s, \phi^s) + gd^2 R_0 \sqrt{c_M^s} K^s \]
\[
+ \frac{\varepsilon_8 g}{2} \Phi(\mathbf{v}^s, \nu^s, \nu^f, \nu^f) + g \left[ \tilde{a}(\varepsilon_1) + \frac{d^2 c_M^f}{\varepsilon_3} + 3dA_1 \right] K^f \]
\[
+ \varepsilon_1 g (v_{i,j}^f v_{i,j}^f + v_{j,i}^f v_{j,i}^f) + \frac{2\delta_m^f g \varepsilon_1}{\sigma_m^f} v_{i,j}^f v_{i,j}^f + \frac{\varepsilon_8 d^2 g}{2} (\pi^f)^2,
\] (10.39)

Using the following inequality (see [39]):
\[
a_{ij}^f a_{ij}^f \geq \frac{1}{2} (v_{i,j}^f v_{i,j}^f + v_{j,i}^f v_{j,i}^f),
\] (10.40)

and the relations (11.19) we deduce
\[
g \left\{ \frac{\partial}{\partial t} \left[ \mathcal{K} + \mathcal{E}(\mathbf{u}^s, \phi^s) \right] + \left[ 1 - \left( \frac{\varepsilon_3}{2} + \frac{2\varepsilon_1}{\sigma_m^f} \right) \right] \Phi(\mathbf{v}^s, \nu^s, \nu^f, \nu^f) \right\} \]
\[
\leq K_{j,j} + \kappa g \left[ \mathcal{K} + \mathcal{E}(\mathbf{u}^s, \phi^s) \right] + \frac{\varepsilon_8 d^2 g}{2} (\pi^f)^2,
\] (10.41)

where
\[
\kappa = \kappa(\varepsilon_1, \varepsilon_3) = \max \left\{ \frac{\sqrt{c_M}}{R_0}, \quad d^2 R_0 \sqrt{c_M^s}, \quad \tilde{a}(\varepsilon_1) + \frac{d^2 c_M^f}{\varepsilon_3} + 3dA_1 \right\}.
\] (10.42)

Now we choose the arbitrary constants \( \varepsilon_1 \) and \( \varepsilon_3 \) such that
\[
0 < \varepsilon_1 < \frac{\sigma_m^f}{2} \quad \text{and} \quad 0 < \varepsilon_3 < 2 \left( 1 - \frac{\varepsilon_1}{\sigma_m^f} \right)
\] (10.43)
and thus, in view of the positivity of the dissipation potential, we deduce
\[ \int_{\Omega_R} g \frac{\partial}{\partial t} [K + \mathcal{E}(u^s, \phi^s)] \, dv \leq \int_{\partial \Omega_R} K \eta \eta \, da + \kappa \int_{\partial B} K j \cdot n_j \, da \]
\[ + \epsilon_8 d^2 \int_{\Omega_R} g(\pi^f)^2 \, dv, \]
where \( \partial \Omega_R = S_R \cup \partial B \).

In view of the boundary conditions on \( \partial B \) and the hypotheses (iii) and (iv) of the theorem, it follows that for \( R \to \infty \) the boundary integral vanishes, and thus one obtains
\[ \int_B g \frac{\partial}{\partial t} [K + \mathcal{E}(u^s, \phi^s)] \, dv \leq \kappa \int_B [K + \mathcal{E}(u^s, \phi^s)] \, dv \]
\[ + \epsilon_8 d^2 \int_B g(\pi^f)^2 \, dv. \]  
(10.45)

Now, we use the assumption (v) to obtain
\[ \frac{dE}{dt} \leq \kappa E + c d^{2\kappa}, \quad t \in I \]
(10.46)
where
\[ E(t) = \int_B g[K + \mathcal{E}(u^s, \phi^s)] \, dv, \quad c = 2\pi A^2 \epsilon_8 \int_0^\infty e^{-t^{1-2\kappa}} \, dt. \]  
(10.47)

By a direct integration of the relation (10.46), we deduce
\[ E(t) \leq \frac{c}{\kappa} d^{2\kappa} e^{\kappa t}, \quad t \in I. \]  
(10.48)

For \( R \geq R_0 \), we have
\[ E(t) \geq \left\{ \int_{\Omega_R} [K(t) + \mathcal{E}(u^s, \phi^s)] \, dv \right\} e^{-dR}, \quad t \in I. \]  
(10.49)

Thus, we have the following estimate
\[ \int_{\Omega_R} [K(t) + \mathcal{E}(u^s, \phi^s)] \, dv \leq \frac{c}{\kappa} d^{2\kappa} e^{\kappa t + dR}, \quad t \in I. \]  
(10.50)

In view of the relation (10.42), it is clear that \( \kappa \) do not tend to zero (or infinity) when \( d \to 0 \). Thus, allowing \( d \to 0 \), and taking into account the positivity of the internal energy density, we deduce that \( K(t) = 0 \), for all \( t \in I \), and in consequence we obtain the relations (10.22)_{1-4}. The relation (10.22)_5 results from (10.24)_3 and the proof is complete.

**Remark 10.1** Setting formally the coupling coefficients \( \xi \) and \( \varpi \) to be zero in the field equations and following the same strategy as in the above theorem, then one can obtain uniqueness results for exterior domains in the theory of micropolar elastic solids and respectively, in the theory of micropolar viscous fluids. Thus, the above result can also be interpreted as an alternative to the method used by [219] and [31] in the proof of uniqueness for the micropolar motions in unbounded regions.
11 Micropolar solid and compressible fluid

11.1 The equations. Auxiliary results

We start this section having in mind that the general equations presented in the Section 1.1 hold true for mixtures consisting in a micropolar solid and a micropolar compressible fluid [134].

Inspired by the works [60, 61], we introduce the specific volume \( \upsilon_f \) defined by

\[
\upsilon_f = \frac{1}{\rho_f}.
\]

(11.1)

In what follows we consider the linear theory appropriate to small departures from the natural state. Moreover, we consider the case of small deformation of the micropolar elastic solid. Consequently, the micropolar elastic solid is endowed with the linear strain tensor \( e_{ij}^s \) and a linear microrotation gradient \( \gamma_{ij}^s \) will be used a strain measure.

Within the linear approximation, we get

\[
\upsilon_f \approx \frac{1}{\rho_0} + \varphi_f
\]

(11.2)

with

\[
\varphi_f = -\frac{1}{(\rho_0^f)^2}(\rho^f - \rho_0^f).
\]

(11.3)

Thus, the conservation law of mass is

\[
\rho_0^f \varphi_f = \upsilon_i^f.
\]

(11.4)

In view of the theories developed in [104], in the case of isotropic solids, we have the following constitutive equations

\[
\psi^s = \hat{\psi}^s(\theta, e_{ij}^s, \gamma_{ij}^s), \quad t_{ji}^s = \rho^s \frac{\partial \hat{\psi}^s}{\partial e_{ij}^s}, \quad m_{ij}^s = \rho^s \frac{\partial \hat{\psi}^s}{\partial \gamma_{ij}^s}, \quad \eta = -\frac{\partial \hat{\psi}}{\partial \theta},
\]

\[
\psi^f = \hat{\psi}^f(\theta, \upsilon), \quad t_{ji}^f = -\pi^f \delta_{ij} + s_{ij}^f, \quad \pi^f = \frac{\partial \hat{\psi}^f}{\partial \upsilon},
\]

\[
s_{ji}^f = \lambda^f \alpha_{kk}^f \delta_{ij} + (\mu^f + \kappa^f) \alpha_{ji}^f + \mu^f \alpha_{ij}^f,
\]

\[
m_{ji}^f = \alpha^f \beta_{kk}^f \delta_{ji} + \beta^f \beta_{ji}^f + \gamma^f \beta_{ij}^f,
\]

\[
\hat{p}_i^s = -\hat{p}_i^f = -\xi'(v_i^s - v_i^f) - \zeta \frac{\theta_i}{\theta},
\]

\[
\hat{m}_i^s = -\hat{m}_i^f = -\omega'(v_i^s - v_i^f),
\]

\[
q_i = \zeta'(v_i^s - v_i^f) + K' \frac{\theta_i}{\theta},
\]

(11.5)

where \( \lambda^f, \mu^f, \kappa^f, \alpha^f, \beta^f, \gamma^f, \omega^f, \xi^f, \zeta^f \) and \( K' \) are constant prescribed coefficients functions of \( \theta \) and \( \upsilon_f \).
In a second order approximation we choose

\[
\rho_0^f \phi^f = \psi_0^f - p_0^f \phi^f - \eta_0^f T - \frac{C_0^f}{2T_0} T^2 - b^f T \phi^f - \frac{1}{2} a^f (\phi^f)^2,
\]

(11.6)

where \( \psi_0^f, p_0^f, \eta_0^f, a^f, b^f \) and \( C_0^f \) are prescribed constants and \( \psi^s \) is given by (1.59).

We can assume that

\[
\psi_0^f = 0 \quad p_0^f = 0, \quad \eta_0^f = 0.
\]

(11.7)

In the linearized approximation, the field equations of the theory of micropolar solid–fluid mixtures (see [104, 60, 61]) become

\[
t_{ji}^s = (-b^s T + \lambda^s e_{kk}^s) \delta_{ij} + (\mu^s + k^s) e_{ij}^s + \mu^s \varepsilon_{ij}^s,
\]

\[
m_{ji}^s = \alpha^s \gamma_{kk}^s \delta_{ij} + \beta^s \gamma_{ij}^s + \gamma^s \gamma_{ji}^s,
\]

\[
t_{ji}^f = -\left(\frac{b^f T + a^f}{\rho_0^f} \phi^f\right) \delta_{ij} + s_{ji}^f,
\]

\[
s_{ji}^f = \lambda^f a_{kk}^f \delta_{ij} + (\mu^f + k^f) a_{ij}^f + \mu^f a_{ji}^f,
\]

\[
m_{ji}^f = \alpha^f b_{kk}^f \delta_{ij} + \beta^f b_{ij}^f + \gamma^f b_{ji}^f,
\]

\[
\bar{p}_i^s = -\bar{p}_i^f = -\bar{\epsilon}(v_i^s - v_i^f) - \frac{\zeta}{T_0} T_{,i},
\]

\[
\bar{v}_i^s = -\bar{v}_i^f = -\bar{\omega}(v_i^s - v_i^f),
\]

\[
\rho_0 \eta = \frac{C_0}{T_0} T + b^s u_{,i}^s + b^f \phi_{,i}^f,
\]

\[
q_i = \bar{\zeta}(v_i^s - v_i^f) + \frac{K}{T_0} T_{,i}
\]

where now the constitutive coefficients are constants.

Moreover, the equation of motion and the energy equation become

\[
\rho_0^s \ddot{v}_{i}^s = (\lambda^s + \mu^s) u_{,ij}^s + (\mu^s + \kappa^s) u_{ij}^s + \kappa^s \varepsilon_{ij}^s \phi_{k,j}^s - \bar{\epsilon}(v_i^s - v_i^f) - \left(\frac{\zeta}{T_0} + b^s\right) T_{,i} + \rho_0^s f_i^s,
\]

\[
\rho_0^s j^s \dot{v}_i^s = (\alpha^s + \beta^s) \phi_{,ij}^s + \gamma^s \phi_{k,j}^s + \kappa^s (\varepsilon_{ij}^s u_{k,j}^s - 2 \phi_{k,j}^s) - \bar{\omega}(v_i^s - v_i^f) + \rho_0^s \ell_i^s,
\]

\[
\rho_0^f \dot{v}_{i}^f = (\lambda^f + \mu^f) v_{,ij}^f + (\mu^f + \kappa^f) v_{ij}^f + \kappa^f \varepsilon_{ij}^f u_{k,j}^f + \bar{\epsilon}(v_i^s - v_i^f) + \left(\frac{\zeta}{T_0} - \frac{b^f}{\rho_0^f}\right) T_{,i} + \frac{a^f}{\rho_0^f} \phi_{,i}^f + \rho_0^f f_i^f,
\]

\[
\rho_0^f j^f \dot{v}_i^f = (\alpha^f + \beta^f) v_{,ij}^f + \gamma^f v_{ij}^f + \kappa^f (\varepsilon_{ij}^f u_{k,j}^f - 2 v_{ij}^f) + \bar{\omega}(v_i^s - v_i^f) + \rho_0^f \ell_i^f,
\]

\[
0 = -C_0 \ddot{T} + (b^s T_0 + \zeta) v_{i}^s - (b^f T_0 + \zeta) v_{i}^f + \frac{K}{T_0} T_{,ii} + \rho_0 h.
\]
To these equations we adjoin the following boundary conditions
\[ u^i_t(x, t) = w^i_t(x, t), \quad v^i_t(x, t) = \phi^i_t(x, t), \quad \psi^j_t(x, t) = \psi^j_t(x, t), \quad T(x, t) = \theta(x, t) \quad \text{pe} \quad \partial B \times I \] (11.10)
and the following initial conditions
\[ u^i_t(x, 0) = g^i_t(x), \quad v^i_t(x, 0) = h^i_t(x), \quad T(x, 0) = r(x) \] (11.11)
where \( \varphi^s, \psi^s, \theta, g^s, h^s, a^s, b^s, r \) and \( \zeta^j \) are given functions.

Let us introduce the following bilinear forms
\[
\mathcal{W}^s_1(\xi_{ij}, \eta_{ij}) = \lambda^s \xi_{kk}\eta_{ii} + \mu^s \xi_{ji}\eta_{ij} + (\mu^s + \kappa^s)\xi_{ij}\eta_{ij},
\]
\[
\mathcal{W}^s_2(\xi_{ij}, \eta_{ij}) = \alpha^s \xi_{kk}\eta_{ii} + \beta^s \xi_{ji}\eta_{ij} + \gamma^s \xi_{ij}\eta_{ij}.
\] (11.12)

The Clausius–Duhem inequality implies that the dissipation potential
\[
\Phi \equiv \mathcal{W}^s_1(a^s_{ij}, a^s_{ij}) + \mathcal{W}^s_2(b^s_{ij}, b^s_{ij}) + \mathcal{W}((a(v^s_i - v^s_i), b(\frac{T_i}{T_0})), (a(v^s_i - v^s_i), b(\frac{T_i}{T_0})))
\] (11.13)
where the bilinear form \( \mathcal{W} \) is given by (2.14), must be semipositive defined in terms of \( a^s_{ij}, b^s_{ij}, v^s_i - v^s_i, T_i/T_0 \) and \( v^s_i - v^s_i \). This is true if and only if the inequalities (1.73) are satisfied and moreover
\[
3\lambda^f + 2\mu^f + \kappa^f \geq 0, \quad 2\mu^f + \kappa^f \geq 0.
\] (11.14)

Obviously, if these inequalities are strictly satisfied, then the dissipation potential is positive defined.

The internal energy density is given by the following quadratic form in terms of \( e^s_{ij}, \gamma^s_{ij} \) and \( T/T_0, \varphi^f \)
\[
\mathcal{E} = U + S - \frac{1}{2} a^f(\varphi^f)^2
\] (11.15)
where
\[
U = \frac{1}{2} \left[ \mathcal{W}^s_1(e^s_{ij}, e^s_{ij}) + \mathcal{W}^s_2(\gamma^s_{ij}, \gamma^s_{ij}) \right], \quad S = \frac{C_0}{2T_0} T^2
\] (11.16)

This energy will be considered pozitiv defined. Thus, we will impose the conditions (1.75) upon the constitutive coefficients and moreover we will assume that
\[
C_0 > 0, \quad a^f < 0.
\] (11.17)

We suppose that \( \rho^s_0 \) and \( j^s \) are positive constants and \( f^s, \ell^s, h \) are continuous functions on \( B \times [0, \infty) \).
We associate to the solution of the problem \((P)\), defined by the equations (11.8), (11.9), the boundary conditions (11.10) and the initial conditions (11.11), the following quantities

\[
\mathcal{K}(t) = \sum_{\sigma=s,f} \mathcal{K}^\sigma(t), \quad \mathcal{K}^\sigma(t) = \frac{1}{2}(\rho_0^\sigma v^\sigma_i v^\sigma_i + \rho_0^\sigma j^\sigma v^\sigma_i v^\sigma_i).
\] (11.18)

Using an appropriate procedure with that used by Chirită and Ciarletta [38], we obtain the following Lemma:

**Lemma 11.1 (Auxiliary estimates)** If the internal density energy and the dissipation potential are positive definite forms, then the following estimates hold

\[
t_{ij}^s t_{ij}^s \leq (1 + \varepsilon_1)\sigma_M^s \mathcal{W}_1^s(e_i^s, e_j^s) + \left(1 + \frac{1}{\varepsilon_1}\right) 3\beta_0^2 T^2, \quad \forall \varepsilon_1 > 0,
\]

\[
s_{ij}^f s_{ij}^f \leq \sigma_M^f \mathcal{W}_1^f (a_{ij}^f, a_{ij}^f),
\]

\[
m_{ij}^s m_{ij}^s \leq \delta_M^s \mathcal{W}_2^s (\gamma_{ij}^s, \gamma_{ij}^s),
\]

\[
m_{ij}^f m_{ij}^f \leq \delta_M^f \mathcal{W}_2^f (b_{ij}^f, b_{ij}^f),
\]

\[
q_i q_i \leq K \mathcal{W}((a(v_i^s - v_i^f), b_{ij}^f T_i^0), (a(v_i^s - v_i^f), b_{ij}^f T_i^0))
\]

where

\[
\sigma_M^s = \max\{2\mu^s + \kappa^s, \kappa^s, 3\lambda^s + 2\mu^s + \kappa^s\},
\]

\[
\delta_M^s = \max\{3\alpha^s + \beta^s + \gamma^s, \gamma^s + \beta^s, \gamma^s - \beta^s\}.
\] (11.20)

**Proof.** Let first remark that the relations (11.8) imply

\[
t_{ij}^s t_{ij}^s \leq -\beta_0 T \delta_{ij} t_{ij}^s + \lambda^s u_{k,i}^s t_{ij}^s + \mu^s (u_{ij}^s + u_{ji}^s) t_{ij}^s + k^s (u_{ji}^s + \varepsilon_{jik} \phi_k^i) t_{ij}^s
\]

\[
\leq \sqrt{3} |\beta_0| |T| (t_{ij}^s t_{ij}^s)^{\frac{1}{2}} + \mathcal{W}_2^s(e_i^s, t_{ij}^s)
\]

\[
\leq \sqrt{3} |\beta_0| |T| (t_{ij}^s t_{ij}^s)^{\frac{1}{2}} + [\mathcal{W}_1^s(e_i^s, e_j^s)]^{\frac{1}{2}} [\mathcal{W}_2^s(t_{ij}^s, t_{ij}^s)]^{\frac{1}{2}}
\] (11.21)

and taking into account the relation (2.16), we have

\[
(t_{ij}^s t_{ij}^s)^{\frac{1}{2}} \leq \sqrt{3} |\beta_0| |T| + [\sigma_M^s \mathcal{W}_1^s(e_i^s, e_j^s)]^{\frac{1}{2}}.
\] (11.22)

Thus, in view of the arithmetic-geometric mean inequality, from (11.22) we deduce the relation (11.19). Similarly, we show the relations (11.19)_{2,3,4}.

The relation

\[
q_i q_i = \mathcal{W}_5 \left(\left(u_i^s - u_i^f, \frac{T_i^0}{T_0}\right), (0, q_i)\right),
\] (11.23)

leads to (11.19)_5. Hence, the proof is complete.

We have to point out that the estimate (5.35) hold true.
11.2 Spatial behavior for bounded and unbounded bodies

In order to study the spatial behavior of the solutions we first introduce the support of the given data. For fixed $t^* > 0$, we consider the support $D$ of the initial and boundary data, of the body and couple forces and of the heat source on the time interval $I = [0, t^*]$ and further, we assume that it is a regular bounded set.

Following the method developed by Chiriţă and Ciarletta [38], $D$ is the set of all $x \in B$ such that

(i) of $x \in B$, then

$g^a(x) \neq 0$ or $h^a(x) \neq 0$ or $a^s(x) \neq 0$ or $b^s(x) \neq 0$ or $r(x) \neq 0$ or $\varsigma(x) \neq 0$ or $F^s(x, t) \neq 0$ or $L^s(x, t) \neq 0$ or $H(x, t) \neq 0$

for some $t \in I$;

(ii) If $x \in \partial B$, then

$w^s(x, t) \neq 0$ or $\vartheta^f(x, t) \neq 0$ or $\rho^s(x, t) \neq 0$ or $\psi^f(x, t) \neq 0$ or $\theta(x, t) \neq 0$ for some $t \in I$.

We introduce the set $D_r$, $r \geq 0$, by

$$D_r = \{ x \in B, D \cap \Sigma(x, r) \neq 0 \},$$

where $\Sigma(x, r)$ is the open ball with radius $r$ and center $x$ and the sets

$$B_r = B \setminus D_r, \quad B(r_1, r_2) = B_{r_2} \setminus B_{r_1}, \quad r_1 \geq r_2.$$  

The set $S_r$ denotes the surface of $\partial B_r$ contained inside $B$ and whose outward unit normal vector is directed to the exterior of $D_r$. In the case of a bounded body, $r$ ranges on $[0, L]$, $L < \infty$, where

$$L = \max\{ \min\{ [(x_i - y_i)(x_i - y_i)]^{1/2} : y \in D \} : x \in \bar{B} \}.$$  

Corresponding to the solution of the initial-boundary value problem $(\mathcal{P})$, we introduce the following time-weighted surface power function

$$Q(r, t) = - \int_0^t \int_{S_r} e^{-\gamma \tau} \left[ \sum_{\sigma=s,f} (t^\sigma_{ji}(\tau)v_i(\tau) + m^\sigma_{ji}(\tau)v_i(\tau)) + \frac{1}{T_0} q_j(\tau)T(\tau) \right] n_j d\alpha d\tau, \quad r \geq 0, t \in I,$$

where $\gamma$ is a positive parameter at our disposal. Further, we introduce, the quantity

$$\tilde{Q}(r, t) = \int_0^t Q(r, \tau) d\tau.$$  

77
Theorem 11.1 Suppose that \( B \) is a bounded regular region and the internal energy density and the dissipation potential are positive defined. Then, \( Q(r,t) \) and \( \tilde{Q}(r,t) \) are two acceptable measures and for each fixed \( t \in I \) we have the following estimates

\[
Q(r,t) \leq Q(0,t)e^{-\frac{r}{\kappa}}, \quad 0 \leq r \leq L
\]

where \( \kappa \) depends on the constitutive coefficients of the mixture, and

\[
\tilde{Q}(r,t) \leq \tilde{Q}(0,t)e^{-\frac{1}{r\tau(t)}}, \quad 0 \leq r \leq L
\]

where \( \tau(t) \) depends on the constitutive coefficients of the mixture and on time \( t \).

Proof. By taking into account the definition for \( S_r, B_r \) and \( B(r_1,r_2) \), the divergence theorem and relations (11.8)–(11.11) we have

\[
Q(r_1,t) - Q(r_2,t) = -\int_0^t \int_{\partial B(r_1,r_2)} e^{-\gamma \tau} \left[ \sum_{\sigma=s,f} (t^\sigma_{ji}(\tau)\dot{u}^\sigma_i(\tau) + m^\sigma_{ji}(\tau)\dot{\phi}^\sigma_i(\tau)) \right] n_j d\tau
\]

\[
+ \frac{1}{T_0} q_j(\tau)T(\tau) \right] n_j d\tau
\]

\[
= -\int_{B(r_1,r_2)} e^{-\gamma \tau} \left( \mathcal{K}(t) + S(t) + \mathcal{E}(t) - \frac{1}{2}a^f[\varphi^f(t)]^2 \right) dv
\]

\[
-\int_0^t \int_{B(r_1,r_2)} e^{-\gamma \tau} \left[ \gamma \left( \mathcal{K}(\tau) + S(\tau) + \mathcal{E}(\tau) - \frac{1}{2}a^f[\varphi^f(\tau)]^2 \right) + \Phi(\tau) \right] dv d\tau,
\]

\[
r_1 \geq r_2 \geq 0, \quad t \in I
\]

and in consequence

\[
\frac{\partial}{\partial r} Q(r,t) = -\int_{S_r} e^{-\gamma t} \left( \mathcal{K}(t) + S(t) + \mathcal{E}(t) - \frac{1}{2}a^f[\varphi^f(t)]^2 \right) dv
\]

\[
-\int_0^t \int_{S_r} e^{-\gamma \tau} \left[ \gamma \left( \mathcal{K}(\tau) + S(\tau) + \mathcal{E}(\tau) - \frac{1}{2}a^f[\varphi^f(\tau)]^2 \right) + \Phi(\tau) \right] dv d\tau,
\]

for \( r \geq 0, t \in I \). Let remark that \( Q(r,t) \) can be written in the following form

\[
Q(r,t) = Q_1(r,t) + Q_2(r,t),
\]

where

\[
Q_1(r,t) = -\int_0^t \int_{S_r} e^{-\gamma \tau} \left[ t^\sigma_{ji}(\tau)\dot{u}^\sigma_i(\tau) + s^f_{ji}(\tau)v^f_i(\tau) + \sum_{\sigma=s,f} m^\sigma_{ji}(\tau)v^\sigma_i(\tau) \right]
\]

\[
+ \frac{1}{T_0} q_j(\tau)T(\tau) \right] n_j d\tau,
\]

\[
Q_2(r,t) = \int_0^t \int_{S_r} e^{-\gamma \tau} (b^f T + a^f \varphi^f)v^f_i n_i d\tau.
\]
In view of the Schwarz’s inequality, the arithmetic-geometric mean inequality and the Lemma 11.1, we deduce

\[
|Q_1(r,t)| \leq \int_0^t \int_{S_r} e^{-\gamma \tau} \left[ \varepsilon_2 K^s(\tau) + \varepsilon_3 K^f(\tau) + \varepsilon_4 S(\tau) \right. \\
+ \frac{1}{2 \varepsilon_2 \rho^s} \hat{c}(1 + \varepsilon_1) \mathcal{E}\left(\varepsilon_i^s, \gamma_i^s\right) + \frac{1}{2 \varepsilon_2 \rho^s} \left(1 + \frac{1}{\varepsilon_1}\right) \beta_0^2 T^2 \\
+ \frac{1}{2 \varepsilon_3 \rho^f} \hat{c}(\mathcal{W}_1^f(a_{ij}, a_{ij}^f) + \mathcal{W}_2^f(b_{ij}, b_{ij}^f)) \\
+ \frac{1}{2 \varepsilon_4 \frac{T_0 C_0}{2}} \mathcal{W}(\{a(b_{ij}^s - v_{ij}^s), b \frac{T_i^d}{T_0}, (a(b_{ij}^s - v_{ij}^s), b \frac{T_i^d}{T_0})\}) \left. \right] d\alpha \tau \tag{11.34}
\]

for all \(\varepsilon_i > 0, i = 1, 2, 3, 4\), where

\[
\hat{c} = \max\{\sigma_i^s, \frac{\delta_i^s}{\beta_i^s}\}, \quad \tilde{c} = \max\{\sigma_i^f, \frac{\delta_i^f}{\beta_i^f}\}. \tag{11.35}
\]

Now, we choose \(\varepsilon_i, i = 1, 2, 3, 4\) to be

\[
\varepsilon_2 = \frac{\hat{c}}{\gamma \rho^s \sigma}(1 + \varepsilon_1), \quad \varepsilon_3 = \frac{\tilde{c}}{2 \varepsilon_2 \rho^f}, \quad \varepsilon_4 = \frac{K}{2 \varepsilon_3 T_0 C_0}, \tag{11.36}
\]

where

\[
\varepsilon_4 = \frac{1}{\gamma} \sqrt{2 \varepsilon_3 \rho f + \frac{\hat{c}}{\rho^s}(1 + \varepsilon_1)}, \tag{11.37}
\]

and \(\varepsilon_1\) is the positive root of the algebraic equation

\[
\varepsilon_1^2 + \left(1 + \frac{\hat{c} \rho^s \gamma}{2 \varepsilon_2 \rho^f} - \frac{3 \beta_0^2 T_0}{C_0 \hat{c}} - \frac{K \rho^s \gamma}{2 \varepsilon_3 T_0 C_0}\right) \varepsilon_1 - \frac{3 \beta_0^2 T_0}{C_0 \hat{c}} \left(1 + \frac{\hat{c} \rho^s \gamma}{2 \varepsilon_2 \rho^f \tilde{c}}\right) = 0. \tag{11.38}
\]

On the other hand, we deduce

\[
Q_2(r,t) \leq \frac{1}{\sqrt{\rho^f}} \left(\sqrt{\frac{T_0}{C_0}} |b^f| + \sqrt{-a^f}\right) \\
\times \int_0^t \int_{S_r} e^{-\gamma \tau} \left[K^f(\tau) + S(\tau) - \frac{1}{2} a^f |\varphi^f(\tau)|^2\right] d\alpha \tau.
\]

Thus, we obtain that \(Q(r,t)\) satisfies the following first-order differential inequality

\[
|Q(r,t)| + \varkappa \frac{\partial Q}{\partial r}(r,t) \leq 0, \quad r \geq 0, t \in I \tag{11.39}
\]

where

\[
\varkappa = \varepsilon + \frac{1}{\gamma} \sqrt{\rho^f} \left(\sqrt{\frac{T_0}{C_0}} |b^f| + \sqrt{-a^f}\right). \tag{11.40}
\]

Because the body is bounded, \(r \in [0, L], L < \infty\) and in view of the definition of \(D\) we have

\[
Q(L, t) = 0, t \in I. \tag{11.41}
\]
But the function $Q(r, t)$ is a decreasing function with respect to $r$ and thus, by a direct integration of relation (11.39), we obtain the estimate (11.29).

In order to establish the estimate (11.30), let us take into account the following inequality

$$
\int_0^t \int_0^s f^2 d\tau ds \leq t \int_0^t f^2 d\tau.
$$

(11.42)

Thus, we will have

$$
|\bar{Q}(r, t)| \leq \left[ t \int_0^t \int_{S_r} e^{-\gamma \tau} \rho^s \left( \dot{u}_i^s \dot{\phi}_i^s + j^s \phi_i^s \phi_i^s \right) d\alpha d\tau \right]^{1/2}
\times \left[ t \int_0^t \int_{S_r} e^{-\gamma \tau} \frac{1}{\rho^s} \left( t_{ji}^s t_{ji}^s + \frac{1}{j^s} m_{ji}^s m_{ji}^s \right) d\alpha d\tau \right]^{1/2}
+ \left[ \sqrt{t} \int_0^t \int_{S_r} e^{-\gamma \tau} C_0 T^2 d\alpha d\tau \right]^{1/2} \left[ \sqrt{t} \int_0^t \int_{S_r} e^{-\gamma \tau} K \rho^s q_i q_i d\alpha d\tau \right]^{1/2}
+ \left[ \sqrt{t} \int_0^t \int_{S_r} e^{-\gamma \tau} \rho^f \left( \dot{u}_i^f \dot{\phi}_i^f + j^f \phi_i^f \phi_i^f \right) d\alpha d\tau \right]^{1/2}
\times \left[ \sqrt{t} \int_0^t \int_{S_r} e^{-\gamma \tau} \frac{1}{\rho^f} \left( s_{ji}^f s_{ji}^f + \frac{1}{j^f} m_{ji}^f m_{ji}^f \right) d\alpha d\tau \right]^{1/2}
+ \left| \int_0^t \int_0^s \int_{S_r} e^{-\gamma \tau} (b^f T + a^f \phi) v_i^f n_i d\alpha d\tau d\tau \right|.
$$

(11.43)

By using the arithmetic-geometric mean inequality and the estimate (11.19) we obtain

$$
|\bar{Q}(r, t)| \leq \sqrt{t} \int_0^t \int_{S_r} e^{-\gamma \tau} \left[ \varepsilon_2^s \sqrt{t} \mathcal{K}^s(\tau) + \varepsilon_3^s \mathcal{K}^f(\tau) + \varepsilon_4^s S(\tau) \right]
\times \left[ \frac{1}{\varepsilon_2^s \rho^s} \left( 1 + \varepsilon_2^s \right) + \frac{1}{\varepsilon_3^s \rho^s} \frac{3}{2} \beta_0^2 t^2 \right] d\alpha d\tau
+ \sqrt{t} \int_0^t \int_{S_r} e^{-\gamma \tau} \left[ \frac{c}{2 \varepsilon_2^s \rho^s} (W_1(a_{ij}^f, a_{ij}^f) + W_2(b_{ij}^f, \bar{u}_{ij}^f)) \right] d\alpha d\tau
+ \frac{K}{2 \varepsilon_4^s T_0 C_0} \mathcal{W}(a_i^s - v_i^f, b_i^f - \bar{u}_i^f, T_i^s - T_0) d\sigma d\tau
+ \frac{1}{\sqrt{\rho_0^f}} \left( \sqrt{\frac{T_0}{C_0} |b|^f + \sqrt{-a^f}} \right)
\times \int_0^t \int_{S_r} e^{-\gamma \tau} \left( \mathcal{K}^f(\tau) + S(\tau) - \frac{1}{2} a^f \varphi^f(\tau)^2 \right) d\alpha d\tau,
$$

(11.44)

for all $\varepsilon_i^s > 0, i = 1, 2, 3, 4$.

Let choose $\varepsilon_1^s$ to be the positive root of the algebraic equation

$$(\varepsilon_1^s)^2 + \left( 1 + \frac{\bar{c} \rho^s}{2 \bar{c} \rho^f t} - \frac{3 \beta_0^2 T_0}{C_0 \bar{c}} - \frac{\rho^s K}{2 \bar{c} T_0 C_0 t} \right) \varepsilon_1^s - \frac{3 \beta_0^2 T_0}{C_0 \bar{c}} \left( 1 + \frac{\bar{c} \rho^s}{2 \rho^f \bar{c} t} \right) = 0$$

80
and
\[ \varepsilon^*_2 = \frac{\hat{c}\sqrt{t}}{\gamma \rho^* s(t)} (1 + \varepsilon^*_1), \quad \varepsilon^*_3 = \frac{\hat{c}}{2s(t)\rho^*}, \quad \varepsilon^*_4 = \frac{K}{2s(t)T_0C_0} \]
where
\[ s(t) = \sqrt{\frac{\hat{c}}{2\rho^* \gamma} + \frac{\hat{c}}{\rho^*} t (1 + \varepsilon^*_1)}. \]

In view of (11.32) and (11.44) we deduce the first-order differential inequality
\[ t \tau(t) \frac{\partial}{\partial r} \tilde{Q}(r,t) + |\tilde{Q}(r,t)| \leq 0, \, r \geq 0, \, t \in [0, T], \quad (11.45) \]
where
\[ \tau(t) = s(t) + \sqrt{\frac{t}{\rho_0^f}} \left( \sqrt{\frac{T_0}{C_0}} |b^f| + \sqrt{-a^f} \right) \quad (11.46) \]
and in consequence the estimate (11.30), and the proof is complete.

Let us now consider an unbounded body, that is we assume that $B$ is an unbounded regular region. We derive a result of Phragmén–Lindelöf type described in the next theorem.

**Theorem 11.2** Suppose that $B$ is an unbounded regular region and the internal energy density and the dissipation potential are positive defined. Then, for each fixed $t \in I$, the following alternative holds

(i) If $Q(r,t) \geq 0$ for all $r \geq 0$, then
\[ Q(r,t) \leq Q(0,t) e^{-\frac{\gamma}{\kappa} r}, \, r \geq 0, \quad (11.47) \]

(ii) If there is a value $r_t \geq 0$ so that $Q(r_t,t) < 0$, then we obtain
\[ -Q(r,t) \geq -Q(r_t,t) e^{\frac{\gamma}{\kappa} (r - r_t)}, \, r \geq r_t. \quad (11.48) \]

**Proof.** For the proof of this theorem we follow the method presented in [38] and the method used in the proof of the previous theorem.

11.3 Further estimate for unbounded bodies

In this subsection we consider an unbounded mixture and we try to improve the estimate (11.47) in the sense described by Horgan, Payne and Wheeler [152] and Quintanilla [229].

There seems to be not possible to obtain such kind of results for the whole class of mixtures and so we are restricted to consider mixtures for which
\[ \mu^f = \lambda^f, \alpha^f = \beta^f. \quad (11.1) \]

To this end, we assume that the support $D$ of the initial and boundary data and the body supplies, defined in the previous section, is enclosed in the half-space $x_3 < 0$. We introduce the
notation $S_z$ for the open cross-section of $B$ for which $x_3 = z, \ x \geq 0$ and whose unit normal vector is $(0,0,1)$. We assume that the unbounded set $B$ is so that $S_z$ is bounded for all finite $z \in [0, \infty)$. We denote by $B_z$ that portion of $B$ in which $x_3 > z$.

We denote by $\mathcal{M}$ the set of solutions for which the following relations hold true

$v^{\sigma}_i, \ u^{s}_{i,j}, \ v^{f}_{i}, \ \phi^{s}_{i,j}, \ \nu^{f}_{i,j}, \ T, T_{i}, \ \varphi \rightarrow 0$ \hspace{1cm} (11.2)

uniformly when $x_3 \rightarrow \infty$ as $O(x_3^{-3})$.

Inspired by the work of Quintanilla [229], we associate to a solution of the initial boundary value problem ($\mathcal{P}$) which is in the class $\mathcal{M}$ the following function

$H(z,t) = - \int_0^t \int_{S_z} \left[ \sum_{\sigma=s,f} (t^{\sigma}_{3i} v^{\sigma}_{i} + m^{\sigma}_{3i} \nu^{\sigma}_{i}) + \frac{1}{T_0} q_3 T \right] \, da \, d\tau.$ \hspace{1cm} (11.3)

The integral (11.3) is an improper integral, but in view of the asymptotic condition (11.2) and the conditions upon the constitutive coefficients we can prove that for any finite time we have

$\lim_{z \rightarrow \infty} H(z,t) = 0.$ \hspace{1cm} (11.4)

Using the boundary conditions, the equations (11.9), the divergence theorem and having in mind the definition of the $S_z$ and $B_z$, we deduce that the function $H(z,t)$ can be written in the form

$H(z,t) = \int_0^t \int_{B_z} \left[ \sum_{\sigma=s,f} (t^{\sigma}_{3i} v^{\sigma}_{i} + m^{\sigma}_{3i} \nu^{\sigma}_{i}) + \frac{1}{T_0} q_3 T \right] \, da \, d\tau$

$= \int_{B_z} \left[ \mathcal{K}(t) + S(t) + U(t) - \frac{1}{2} a^f (\varphi^f)^2 \right] \, dv + \int_0^t \int_{B_z} \Phi(\tau) \, dv \, d\tau.$ \hspace{1cm} (11.5)

Further, we introduce the function

$E(z,t) = \int_z^\infty H(p,t) \, dp$ \hspace{1cm} (11.6)

and note that

$\frac{\partial E}{\partial z} = -H(z,t)$ \hspace{1cm} (11.7)

and

$\frac{\partial^2 E}{\partial z^2} = \int_{S_z} \left[ \mathcal{K}(t) + S(t) + U(t) - \frac{1}{2} a^f (\varphi^f)^2 \right] \, da$

$\quad + \int_0^t \int_{S_z} \Phi(\tau) \, da \, d\tau.$ \hspace{1cm} (11.8)

On the other hand, from the constitutive equations (11.8) and the relations (11.3) and (11.6), we have

$\frac{\partial E}{\partial t} = - \int_{B_z} \left[ - b^T v^{s}_{3} + \lambda^s u^{s}_{k,k} v^{s}_{3} + \mu^s u^{s}_{3,i} v^{s}_{i} + (\mu^s + \kappa^s) u^{s}_{i,3} v^{s}_{i} + \kappa^s \varepsilon_{\alpha\beta\gamma} \phi^{s}_{\alpha} \nu^{s}_{\gamma} \right.$

$\left. + \alpha^s \phi^{s}_{k,k} v^{s}_{3} + \beta^s \phi^{s}_{3,i} v^{s}_{i} + \gamma \phi^{s}_{i,i} \nu^{s}_{i} \right.$

$\left. - \left( \frac{b^f T}{\rho_0} + \frac{a^f}{\rho_0} \varphi^f \right) v^{f}_{3} + s^{f}_{3,i} v^{f}_{i} + m^{f}_{3,i} \nu^{f}_{i} + \frac{1}{T_0} q_3 T \right] \, dv.$
Using the Schwarz inequality and the arithmetic geometric inequality, we can conclude that we have the estimate
\[
\int_{B_s} \left[ - b^f T v_3^f + \lambda^f u_{k,k}^f v_3^f + \mu^f u_{k,i}^f v_i^f + (\mu^f + \kappa^f) u_{i,3}^f v_i^f + \kappa^f \varepsilon_{\alpha,\beta} \phi_{\beta} v_{\alpha}^f + \alpha^f \phi_{k,k}^f v_3^f + \beta^f \phi_{i,i}^f v_i^f + \gamma^f \phi_{i,3}^f v_i^f - \left( \frac{b^f}{\rho_0^f} T + \frac{a^f}{\rho_0^f} \phi^f \right) v_3^f \right] dv
\]
\[
\leq A_1 \int_{B_s} \left[ K(t) + S(t) + U(t) - \frac{1}{2} a^f (\phi^f)^2 \right] dv. \tag{11.10}
\]
where \(A_1\) is a positive computable constant which depends on \(\lambda^f, \mu^f, \kappa^f, \alpha^f, \beta^f, \gamma^f, \rho^f, j^f, C_0, T_0, b^f, b^j, \rho_0^f\) and \(a^f\). The exact expression of this constant and of the constants which will be introduced in this section could be obtained as in previous subsection.

The last three terms from the right hand side of the equality (11.9) need a special attention. As in [229], in view of the boundary conditions, we deduce that
\[
\int_{B_s} s_{3a}^f v_i^f dv = - \int s_{\frac{3}{2}} \left[ (\lambda^f + \mu^f)(v_3^f)^2 + (\mu^f + \kappa^f)v_i^f v_j^f \right] da
\]
\[
+ \int_{B_s} (\lambda^f - \mu^f)v_{\alpha,\alpha} v_3^f dv. \tag{11.11}
\]
and
\[
\int_{B_s} \left( m_{3a}^f \nu_i^f + \frac{1}{T^2} q_3 T \right) dv = \int_{B_s} \left[ \frac{\zeta}{T_0} (v_3^s - v_3^f) T + \kappa^f \varepsilon_{\alpha,\beta} \nu_{\beta} v_{\alpha}^f \right] dv
\]
\[
- \int s_{\frac{3}{2}} \left[ (\alpha^f + \beta^f)(\nu_j^f)^2 + \gamma^f \nu_j^f v_j^f + \frac{K}{T_0^2} T^2 \right] da
\]
\[
+ \int_{B_s} (\alpha^f - \beta^f)\nu_{\alpha,\alpha} v_3^f dv. \tag{11.12}
\]
Using (11.16), (11.18) and the Schwarz inequality, we also have a positive computable constant \(A_2\) which depends on \(\alpha^f, \mu^f, \kappa^f, \alpha^f, \beta^f, \gamma^f, K, C_0, T_0, \rho_0^f\) and \(\nu_j^f\) and a positive computable constant \(A_3\) which depends on \(\kappa^f, \zeta, C_0, T_0, \rho_0^f, \rho_0^f\) and \(\nu_j^f\) such that
\[
\int_{B_s} \left[ (\lambda^f + \mu^f)(v_3^f)^2 + (\mu^f + \kappa^f)v_i^f v_j^f \right]
\]
\[
+ (\alpha^f + \beta^f)(\nu_j^f)^2 + \gamma^f \nu_j^f v_j^f + \frac{K}{T_0^2} T^2 \right] da \tag{11.13}
\]
\[
\leq A_2 \int s_{\frac{3}{2}} \left[ \kappa^f \varepsilon_{\alpha,\beta} \nu_{\alpha} v_{\beta}^f + \frac{\zeta}{T_0} (v_3^s - v_3^f) T \right] dv \leq A_3 \int_{B_s} \frac{1}{2} [K(t) + S(t)] dv. \tag{11.14}
\]
In view of the conditions (11.1) and of the relations (11.9)–(11.14) we can conclude that there is a positive constant \(A_4\) such that
\[
\frac{\partial E}{\partial t} \leq -A_4 \frac{\partial E}{\partial z} + A_2 \frac{\partial^2 E}{\partial z^2}. \tag{11.15}
\]
Following Horgan et al. [152], we use the change of variable

$$ w(z, t) = \exp(b_2 t - b_1 z)E(z, t) $$

(11.16)

with

$$ b_1 = \frac{A_4}{2A_2}, \quad b_2 = b_1^2 A_2. $$

(11.17)

Thus, the inequality (11.15) becomes

$$ \frac{\partial w}{\partial t} \leq A_2 \frac{\partial^2 w}{\partial z^2}. $$

(11.18)

We have that $w(z, t)$ satisfied the relation

$$ A_2 \frac{\partial^2 w}{\partial z^2}(z, t) - \frac{\partial w}{\partial t}(z, t) \geq 0, \quad z \in [0, \infty), t \in I, $$

$$ w(z, 0) = 0, \quad z \in [0, \infty), $$

$$ w(0, t) = \exp\left(b_2 t\right)E(0, t) \geq 0, \quad t \in I, $$

$$ w(z, t) \to 0 \quad \text{uniformly in } t \quad \text{for } z \to \infty. $$

(11.19)

Using the maximum principle for parabolic partial differential equations (Protter and Weinberger [152]), we obtain

$$ w(z, t) \leq J(z, t), \quad z \in [0, \infty), \quad t \in I, $$

(3.2.25)

where $J(z, t)$ is solution of the following problem

$$ A_2 \frac{\partial^2 J}{\partial z^2}(z, t) - \frac{\partial J}{\partial t}(z, t) = 0, \quad z \in [0, \infty), t \in I, $$

$$ J(z, 0) = 0, \quad z \in [0, \infty), $$

$$ J(0, t) = \exp\left(b_2 t\right)E(0, t) \geq 0, \quad t \in I, $$

$$ J(z, t) \to 0 \quad \text{uniformly in } t \quad \text{for } z \to \infty. $$

(11.20)

The solution of the above problem is (c.f. Tikhonov and Samarskii [256, pp. 208])

$$ J(z, t) = \frac{A_2}{2\sqrt{\pi}} \int_0^t \frac{z}{[A_2(t - \tau)]^{3/2}} \exp\left(-\frac{z^2}{4A_2(t - \tau)}\right) \exp\left(b_2 \tau\right)E(0, \tau)d\tau. $$

From (11.16) and the above relation we deduce that

$$ E(z, t) \leq \left(\max_{\tau \in [0, t]} E(0, \tau)\right) \exp\left(\frac{A_4 z}{2A_2}\right)G(z, t), $$

(11.21)

where

$$ G(z, t) = \frac{1}{2\sqrt{A_2 \pi}} \int_0^t z\tau^{-3/2} \exp\left(-\frac{z^2}{4A_2 \tau} - \frac{A_4^2}{4A_2} \tau\right)d\tau. $$

(11.22)
We can easily remark that we can have various estimates for $E(r, t)$ using estimates for $G(r, t)$. Such type of estimates was obtained by Horgan et. al. [152]. Using the estimate considered by Pompei and Scalia [227]

$$G(z, t) \leq \frac{2z(A_2 t/\pi)^{1/2}\exp\left(-\frac{A_2^2 t}{4A_2}ight)}{z^2 - A_4^2 t^2} \exp\left(-\frac{z^2}{4A_2 t}\right)$$

for $z > A_4 t$, we find the estimate

$$E(z, t) \leq \left(\max_{\tau \in [0, t]} E(0, \tau)\right) \frac{1}{z^2 - A_4^2 t^2} 2z \left(\frac{A_2 t}{\pi}\right)^{1/2} \exp\left(-\frac{A_4^2 z}{2A_2} - \frac{z^2}{4A_2 t}\right)$$

for $z > A_4 t$.

The estimate (11.24) proves that, for fixed time, at large distance to the support $D$ of the given data the dominant term is $\exp\left(-\frac{z^2}{4A_2 t}\right)$. Moreover, since $A_2$ depends only on the thermal coefficients and on the viscosities of the fluid, we can conclude that at large distance from the support of the external given data, the spatial decay of processes is influenced only by the thermal effect and by the viscosities of the fluid.
An extended bibliography


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