Deformations of Finsler metrics

by

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Abstract. Let $F^n = (M, F(x, y))$ be a Finsler space and $g_{ij}(x, y)$ its Finsler metric. We consider a deformation of $g_{ij}(x, y)$ of the form

$$^*g_{ij}(x, y) = a(x, y)g_{ij}(x, y) + b(x, y)B_i(x, y)B_j(x, y),$$

with two Finsler scalars $a > 0$, $b \geq 0$ and $B_i(x, y)$ a Finsler co-vector. It follows that $^*g_{ij}$ is a generalised Lagrange metric in Miron’s sense, briefly a GL–metric, see the monograph by R. Miron and M. Anastasiei [8]. The metric $^*g_{ij}$ unifies the Antonelli metrics, the Miron–Tavakol metrics, the Synge metrics (all treated in [8]) as well as the Antonelli–Hrimiuc $\phi$-Lagrange metrics, [2], the Beil metrics, [4], and the vertical part of the Cheeger–Gromoll metric, [10]. We prove some general results on the geometry of the GL-space $(M, ^*g_{ij}(x, y))$. Next, the Levi-Civita connection and the curvature of a Riemannian metric on the tangent manifold $TM$, induced by $g_{ij}$ and $^*g_{ij}$ are determined. These are used for the study of a Riemannian submersion involving the Cheeger–Gromoll metric.

1 Deformations of Finsler metrics

Let $F^n = (M, F)$ be a Finsler space with a smooth i.e. $C^\infty$ manifold $M$ and $F : TM \to R, (x, y) \mapsto F(x, y)$. Here $x = (x^i)$ are coordinates on $M$ and $(x, y) = (x^i, y^i)$ are coordinates on the tangent manifold $TM$ projected on $M$ by $\tau$. The indices $i, j, k, ...$ will run from 1 to $n = \dim M$ and the Einstein convention on summation is implied. The geometrical objects on $TM$ whose local components change like on $M$ i.e. ignoring their dependence on $y$, will be called Finsler objects as in [7] or $d$–objects as in [8].
We set $\partial_i := \frac{\partial}{\partial x^i}, \dot{\partial}_i := \frac{\partial}{\partial y^i}$ and notice that the vertical subspace of $T_u T^* M$, i.e. $V_u T^* M = \text{Ker} \, (D\tau)_u, \ u \in TM$, where $D\tau$ means the differential of $\tau$, is spanned by $(\dot{\partial}_i)$. The $d$–objects can be expressed using $(\dot{\partial}_i)$.

The Finsler metric $g_{ij}(x, y) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2$ will be assumed positive definite.

We have $F^2(x, y) = g_{ij}(x, y)y^i y^j$ and $F^2$ will be called the absolute energy of $F^n$. Assume that $F^n$ is endowed with a $d$-vector field or a Finsler vector field $B = B^i(x, y) \dot{\partial}_i$ and let $B_i(x, y) dx^i$ the Finsler 1-form with $B_i = g_{ik} B^k$. Set $B^2 = B_i B^i$ and consider the following deformation of $g_{ij}(x, y)$:

\begin{equation}
(1.1) \quad *g_{ij}(x, y) = a(x, y)g_{ij}(x, y) + b(x, y)B_i(x, y)B_j(x, y),
\end{equation}

with two Finsler scalars $a > 0, b \geq 0$. The metric $*g_{ij}$ is no longer a Finsler metric but it is a positive definite generalised Lagrange metric in Miron’s sense, briefly a GL–metric, see Ch.X in [8]. It is easy to check that $*g^{jk} = \frac{1}{a} g^{jk} - \frac{b}{a(a + b B^2)} c B^j B^k$ is the inverse of $*g_{ij}$ for $c = \frac{b}{a(a + b B^2)}$.

Various particular forms of $*g_{ij}(x, y)$ were previously considered by some authors. The conformal case i.e. $b = 0, a = \exp(2\sigma(x, y))$ was studied and applied by R. Miron and R.K. Tavakol in General Relativity. The case $a = 1$ and $B_i = y_i$ provides, for a convenient form of $b(x, y)$, a metric which generalises the Synge metric from Relativistic Optics. This case was studied by R. Miron and R. Miron, T. Kawaguchi. For $b = 0, a = \exp(2\sigma(x))$ and $g_{ij}(x, y) = g_{ij}(y)$ one gets the Antonelli metric which was used in Ecology. For the results on all these metrics we refer to the chapters XI and XII in [8] and the references therein. The case $a = b = 1$ and $B_i(x, y) = B_i(x) = \frac{\partial f}{\partial x^i}, f : M \rightarrow \mathbb{R}$ was considered by C. Udrişte in [11] for studying the completeness of a Finsler manifold. The Riemannian version of this case i.e. $g_{ij}(x, y) = g_{ij}(x)$ was intensively used by Th. Aubin in [3]. The case $a = 1$ and $g_{ij}(x, y) = g_{ij}(x)$ with various choices of $b$ and $B_i$ was introduced and studied by R. G. Beil for constructing a new unified field theory, [5].

One says that $*g_{ij}$ is reducible to a Lagrange metric, shortly an $L$–metric if there exists a Lagrangian $L : TM \rightarrow \mathbb{R}$ such that $*g_{ij} = \frac{1}{2} \partial_i \partial_j L$. A necessary and sufficient condition for $*g_{ij}$ be reducible to an $L$–metric is the
symmetry in all indices of the Cartan tensor field $C_{ijk} = \frac{1}{2} \partial_k g_{ij}$ i.e.

(1.2) $\hat{\partial}_k g_{ij} = \hat{\partial}_i g_{kj}$.

Using (1.1) this condition becomes

(1.3) $\dot{a}_k g_{ij} - \dot{a}_j g_{ik} + \dot{b}_k B_i B_j - \dot{b}_j B_i B_k + b(\hat{\partial}_k B_i \cdot B_j - \hat{\partial}_i B_k \cdot B_j + B_i \cdot \hat{\partial}_k B_j - B_k \cdot \hat{\partial}_i B_j) = 0,$ $\dot{a}_k := \hat{\partial}_k a, \dot{b}_k := \hat{\partial}_k b.$

Now we suppose that $a(x, y) = a(F^2)$ and $b(x, y) = b(F^2)$ assuming that the ranges of the real functions $a$ and $b$ from the right hand are included in $Im(F^2)$. It results $\dot{a}_k a = 2a'(F^2)y_i$ because of $\dot{\partial}_i F^2 = 2y_i$. Similarly, $\dot{b}_k b = 2b'(F^2)y_i$. We take $B_i = y_i$. For the GL-metric (1.1) subjected to the above conditions, (1.3) reduces to

(1.4) $(2a - b')(g_{ij} y_k - g_{ik} y_j) = 0.$

Now if the equation $g_{ij} y_k - g_{ik} y_j = 0$ is multiplied by $g^{ij}$ one gets $(n-1)y_k = 0$ which is a contradiction for $n \geq 1$. Thus we have

**Theorem 1.1.** The GL-metric (1.1) with $B_i = y_i, a(x, y) = a(F^2), b(x, y) = b(F^2)$ is an L-metric if and only if $2a = b'$.

As always we may take $a = \phi'$, it comes out that the metric from Theorem 1.1 is essentially the $\phi$-Lagrange metric of Antonelli- Hrimiuc,[2], i.e.

(1.5) $\ast g_{ij}(x, y) = ag_{ij}(x, y) + 2a' y_i y_j$

The Cheeger-Gromoll metric is a Riemannian metric on $TM$ of the form

(1.6) $G_{CG} = g_{ij} dx^i \otimes dx^j + \frac{1}{1 + F^2} (g_{ij}(x) + y_i y_j) dy^i \otimes dy^j,$

for $\delta y^i = dy^i + \gamma^i_{jk} y^j dx^k$, where $\gamma^i_{jk}$ are the Christoffel symbols of $g_{ij}(x)$. This suggests considering the following GL-metric of type (1.1) which generalises the "vertical part" in (1.6):

(1.7) $\ast g_{ij} = \frac{1}{1 + F^2} (g_{ij}(x) + y_i y_j),$

which we call a CGL-metric.

**Corollary 1.1.** The CGL-metric (1.7) is never reducible to a L-metric nor to a Finsler metric.
2 Metrical connection of the GL–space 

\((M, *g_{ij}(x, y))\)

The geometry of \(\ast g_{ij}(x, y)\) is naturally connected with the geometry of \(F^n\). It is our purpose to express the geometrical objects associated to \(\ast g_{ij}(x, y)\) using similar ones for \(F^n\). If \(\gamma^i_{jk}(x, y)\) are the generalised Christoffel symbols for \(g_{ij}(x, y)\) and we put

\[ \gamma^i_{00} := \gamma^i_{jk}y^jy^k, \]

then

\[ \begin{align*}
   \circ N^i_{jk} &= \frac{1}{2} \partial_j \gamma^i_{00}, \\
   \circ C^i_{jk} &= \frac{1}{2} g^{ih}(\partial_j g_{hk} + \partial_k g_{jh} - \partial_h g_{jk}),
\end{align*} \]

for \(\delta_j = \partial_j - \circ N^k_{jk}\circ k\).

This connection is \(h\)–metrical, i.e. \(g_{ij}\circ = 0\) and \(v\)–metrical, i.e. \(g_{ij}\circ = 0\).

Here \(\circ k\) and \(\circ k\) denote the \(h\)– and \(v\)–covariant derivatives with respect to \(CT\). Moreover, two torsions of it vanish. We may consider a similar connection for \(\ast g_{ij}(x, y)\). Indeed, let \(\ast CT = (\circ N^i_{jk}, \ast F^i_{jk}, \ast C^i_{jk})\) be the \(d\)–connection given by

\[ \begin{align*}
   \ast F^i_{jk} &= \frac{1}{2} g^{ih}(\delta_j \ast g_{hk} + \delta_k \ast g_{jh} - \delta_h \ast g_{jk}), \\
   \ast C^i_{jk} &= \frac{1}{2} g^{ih}(\partial_j \ast g_{hk} + \partial_k \ast g_{jh} - \partial_h \ast g_{jk}).
\end{align*} \]

This \(d\)–connection is \(h\)–metrical i.e. \(\ast g_{ij}\circ = 0\) and \(v\)–metrical i.e. \(\ast g_{ij}\circ = 0\) and the torsions \(\ast T^i_{jk} := \ast F^i_{jk} - \ast F^i_{kj} = 0\), \(\ast S^i_{jk} := \ast C^i_{jk} - \ast C^i_{kj} = 0\). Moreover, when \(\circ N^i_{jk}(x, y)\) is fixed, \(\ast CT\) is the unique \(d\)–connection with these properties. It will be called the canonical metrical connection of \(\ast g_{ij}(x, y)\). Using (1.1) in (2.2), after some calculation one gets

**Proposition 2.1.** The metrical connection \(\ast CT\) is given by

\[ \begin{align*}
   \ast F^i_{jk} &= F^i_{jk} + \Phi^i_{jk}, \\
   \ast C^i_{jk} &= C^i_{jk} + \Lambda^i_{jk},
\end{align*} \]
\( \Phi_{jk}^i = \frac{1}{2} g^{ih} [a_j g_{hk} + a_k g_{jk} - a_h g_{jk} + \delta_j (bB_j B_h) + \\
+ \delta_k (bB_j B_h) - \delta_k (bB_j B_k)] - acB^i jh F_{jkh} \) (2.4)

\( \Lambda_{jk}^i = \frac{1}{2} g^{ih} [a_j g_{hk} + \dot{a}_j g_{jk} + \dot{a}_k g_{jk} + \dot{\delta}_j (bB_j B_h) + \\
+ \dot{\delta}_k (bB_j B_h) - \dot{\delta}_h (bB_j B_k)] - acB^i jh C_{jkh} \) (2.5)

with the notations

\[
a_k = \delta_k a, \quad \dot{a}_k = \dot{\delta}_k a, \quad F_{jkh} = \frac{1}{2}(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}),
\]

(2.6)

\[
C_{jkh} = \frac{1}{2}(\dot{\delta}_j g_{kh} + \dot{\delta}_k g_{jh} - \dot{\delta}_h g_{jk}).
\]

Proposition 2.2. The torsions of \( ^*\Gamma \) are as follows:

\( *T_{jk}^i = 0, \quad *R_{jk}^i = \check{R}_{jk}^i := \delta_k \check{N}_{j}^i - \delta_j \check{N}_{k}^i, \quad *S_{jk}^i = 0 \)

*\( P_{jk}^i = P_{jk}^i - \Phi_{jk}^i \) where \( P_{jk}^i = \dot{\delta}_k N_j^i - F_{jkh} \) and *\( C_{jk}^i \) from (2.3).

Proposition 2.3. The curvatures of \( ^*\Gamma \) are as follows:

\( *S_{j}^i k h = S_{j}^i k h + \Lambda_{j}^i k h + (C_{j}^i s h \Lambda_{s}^i k h + C_{j}^i s h \Lambda_{j}^i s h - k/h), \) (2.8)

\( *\Lambda_{j}^i k h = \dot{\delta}_h \Lambda_{j}^i k h + \Lambda_{j}^i s h \Lambda_{s}^i k h - k/h, \) (2.8)'

where \(-k/h\) means the subtraction of the preceding terms with \( k \) replaced by \( h \).

\( *R_{j}^i k h = R_{j}^i k h + \Phi_{j}^i k h + (F_{j}^s s h \Phi_{j}^s k h + \Phi_{j}^s s h F_{j}^s k h - k/h) + \Lambda_{j}^i s h R_{k}^s h, \) (2.9)

\( *\Phi_{j}^i k h = \delta_k \Phi_{j}^i k h + \Phi_{j}^s s h \Phi_{j}^s k h - k/h, \) (2.9)'

\( *P_{j}^i k h = P_{j}^i k h + \Phi_{j}^i k h - \Lambda_{j}^i k h + \Lambda_{j}^i s h P_{k}^s k h + C_{k}^s h \Phi_{j}^s s h \Phi_{j}^s j h + \Phi_{j}^s s h \Lambda_{j}^i s h - \Phi_{j}^s s k \Lambda_{j}^i j h. \) (2.10)
3 On a Riemannian metric on $TM$

Let $TM$ be the tangent manifold to $M$ endowed with the fundamental Finsler function $F$ and the Finsler metric $g_{ij}(x, y)$. Consider the Cartan nonlinear connection $(\tilde{N}^a_j(x, y))$ and then $(\delta_i = \partial_i - \tilde{N}^a_i \partial_a, \partial_a)$ is a local frame on $TM$ adapted to the decomposition of $T_uTM$ into a direct sum of vertical and horizontal subspaces. From now on we shall use two types of indices: $a, b, c, ...$ will indicate vertical components and $i, j, k, ...$ will indicate horizontal ones. All have the same range $\{1, 2, ..., n\}$.

Let be $h_{ab}(x, y) = \delta^i_a \delta^j_b g_{ij}(x, y)$, where $\delta^i_a$ is the Kronecker symbol, and

\begin{equation}
G(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + h_{ab}(x, y)\delta y^a \otimes \delta y^b,
\end{equation}

where $\delta y^a = dy^a + \tilde{N}^a_i dx^i$.

Then $(TM, G(x, y))$ is an oriented Riemannian manifold. The horizontal and vertical distributions are mutually orthogonal with respect to $G$. It is our purpose to study the Riemannian metric $G$. First, we compute the coefficients of the Levi--Civita connection $D$ of $G$ in the frame $(\delta_i, \partial_a)$. We set

\begin{align}
D_{\delta_i} \delta_j &= F_{ij}^k \delta_k + A_{ij}^a \partial_a, \quad D_{\partial_b} \delta_j = C_{ija}^b \delta_i + E_{ij}^b \partial_a, \\
D_{\delta_i} \partial_b &= L_{bij}^a \partial_a + D_{bij} \delta_i, \quad D_{\partial_b} \partial_c = C_{abc}^a \partial_a + B_{abc} \delta_i
\end{align}

Let $\mathfrak{T}$ be the torsion of $D$ i.e. $\mathfrak{T}(X, Y) = D_X Y - D_Y X - [X, Y]$ for $X, Y$ vector fields on $TM$. The condition $D$ is torsion--free is equivalent to

\begin{equation}
\mathfrak{T}(\delta_i, \delta_j) = \mathfrak{T}(\delta_i, \partial_a) = \mathfrak{T}(\partial_a, \partial_b) = 0.
\end{equation}

Using the following equations

\begin{align}
[\delta_i, \delta_j] &= R_{ij}^a \partial_a, \quad [\delta_i, \partial_b] = (\partial_b N^a_i) \partial_a, \quad [\partial_a, \partial_b] = 0
\end{align}

where $R_{ij}^a = \delta_j N_i^a - \delta_i N_j^a$, one finds that (3.3) is equivalent to

\begin{align}
F_{ij}^k &= F_{ji}^k, \quad A_{ij}^a = -A_{ji}^a = -R_{ij}^a \\
D_{ai}^a &= C_{ia}^b, \quad L_{ai}^b = \partial_a N^b_i + E_{ia}^b \\
C_{bc}^a &= C_{cb}^a, \quad B_{bc}^i = B_{cb}^i
\end{align}

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The condition that $D$ is metrical, that is, $XG(X,Y) = G(D_XY,Z) + G(Y,D_XZ)$, written in the frame $(\delta_i, \hat{\partial}_a)$ gives
\[
F^h_{jk} g_{hk} + F^h_{kij} g_{hj} = \delta_i g_{jk}, \quad \tilde{C}^i_{ja} g_{ik} + \tilde{C}^i_{ka} g_{ij} = \hat{\partial}_a g_{jk},
\]
(3.6)\[\begin{align*}
A^i_{jk}, h_{ca} + D^h_{ak} g_{kj} = 0, & & E^a_{ja} h_{cb} + B^c_{ba} g_{kj} = 0, \\
L^a_{ai} h_{cb} + L^a_{ia} h_{ca} = \delta_i h_{ab}, & & C^a_{ba} h_{ec} + C^a_{ca} h_{eb} = \hat{\partial}_a h_{bc}.
\end{align*}\]

The systems (3.5) and (3.6) have the unique solution
\[
F^i_{jk} = \frac{1}{2} g^{jk} (\delta_i g_{hi} + \delta_j g_{hi} - \delta_h g_{ij}), \quad A^a_{jk} = \frac{1}{2} (-R^a_{jk} - h^{ab} \hat{\partial}_b g_{jk}),
\]
(3.7)\[
\tilde{C}^i_{ja} = \frac{1}{2} g^{jk} (\hat{\partial}_j g_{hi} + h_{bc} R^c_{hi}), \quad D^i_{bj} = D^i_{bj},
\]
\[
E^a_{ib} = \frac{1}{2} h^{ac} h_{bc ||i}, \quad L^a_{ba} = \hat{\partial}_b N^a_i + \frac{1}{2} h^{ac} h_{be ||i},
\]
\[
B^k_{ab} = -\frac{1}{2} g^{kji} h_{ab ||i}, \quad C^a_{ba} = \frac{1}{2} h^{ac} (\hat{\partial}_b h_{dc} + \hat{\partial}_b h_{bd} - \hat{\partial}_d h_{bc}).
\]

Here $h_{be ||i}$ denotes the $h$–covariant derivative of $h_{bc}$ with respect to the Berwald connection $\overline{\Gamma} = (N^a_i, \hat{\partial}_b N^a_i, 0)$. Now we shall compute the components of the curvature of $D$ in the same frame. To this aim we shall consider an intermediate linear connection $\nabla$ on $TM$:
\[
\nabla_{\delta_j} \delta_k = F^i_{jk} \delta_i, \quad \nabla_{\hat{\partial}_b} \delta_j = D^i_{bj} \delta_i, \quad \nabla_{\delta_i} \hat{\partial}_b = L^a_{bk} \hat{\partial}_a, \quad \nabla_{\hat{\partial}_c} \hat{\partial}_d = C^a_{bd} \hat{\partial}_a.
\]
(3.8)

This connection is metrical with respect to $G$ i.e. $\nabla_X G = 0$, it preserves the horizontal and vertical distributions and it has three non-vanishing torsions:

The curvature of $\nabla$ has six components in the form (see p. 48 of [8]):
\[
\tilde{R}^i_{jk} g_{ij} = \delta_k F^i_{jk} + F^m_{kj} F^i_{mk} - j/k + D^i_{ah} R^a_{jk},
\]
\[
\tilde{R}^a_{jk} = \delta_k L^a_{bj} + L^a_{bj} L^a_{ck} - j/k + C^a_{be} R^c_{jk},
\]
\[
\tilde{P}^i_{jka} = \hat{\partial}_a F^i_{jk} - D^i_{aj} k + D^i_{bj} P^h_{ka},
\]
\[
P^a_{kc} = \hat{\partial}_c L^a_{bk} - C^a_{be ||k} + C^a_{bd} P^d_{jc},
\]
\[
\tilde{S}^i_{jbc} = \hat{\partial}_c D^i_{bj} + D^i_{bj} D^i_{ck} - b/c,
\]
\[
S^a_{cd} = \hat{\partial}_d C^a_{bc} + C^a_{be} C^a_{cd} - c/d.
\]
(3.9)
Here and in the following $\delta_k$ and $|_a$ will denote $h$– and $v$–covariant derivatives with respect to $\nabla$.

**Remark 3.1** $S^{a}_{cd}$ is nothing but $^*S^{i}_{jh}$. And the other tensors in (3.9) can be expressed with $R^{i}_{jkh}, P^{i}_{jkh}, S^{i}_{jkh}$ or with their $^*$–counterparts. For instance, $\hat{R}^{i}_{hjk} = R^{i}_{hjk} + \frac{1}{2} g^{is} h_{ac} R^{c}_{sh} R^{a}_{jk}$.

Let $K$ be the curvature tensor field of the Levi–Civita connection $D$. We shall denote its components by the same letter $K$ indexed with two types of indices with the understanding that different indices means different components. There will be twelve components of $K$. After calculation one finds

$$K(\hat{\partial}_b, \hat{\partial}_c)\hat{\partial}_d := K^{a}_{dcb} \hat{\partial}_a + K^{i}_{dcb} \delta_i,$$

(3.10)

$$K^{a}_{dcb} = S^{a}_{dcb} + B^{a}_{cd} E^{a}_{db} - B^{a}_{db} E^{a}_{ic},$$

$$K^{i}_{dcb} = \delta^{i}_{cd} - B^{i}_{bd} E^{i}_{ac},$$

$$K^{a}_{abdc} = S^{a}_{abdc} + \frac{1}{2} (B^{a}_{ad} h_{bc} - B^{a}_{ac} h_{bd}),$$

$$K^{i}_{abdc} = S^{i}_{abdc} + \frac{1}{2} B^{i}_{ad} h_{bc} - B^{i}_{ac} h_{bd},$$

(3.11)

$$K^{a}_{jkb} = A^{a}_{jkb} + E^{a}_{jb} D^{a}_{kb},$$

$$K^{i}_{jkb} = P^{i}_{jkb} + A^{i}_{jkb} - E^{i}_{jk} D^{i}_{kb},$$

$$K^{a}_{jk} = \tilde{R}^{a}_{jk} + \frac{1}{2} g^{ab} h_{jk} R^{b}_{ac},$$

$$K^{i}_{jk} = \tilde{P}^{i}_{jk} + \frac{1}{2} B^{i}_{jk} R^{i}_{bc} - R^{i}_{jk} B^{i}_{bc},$$

(3.12)

$$K(\delta_j, \delta_k)\hat{\partial}_b := K^{a}_{jk} \hat{\partial}_a + K^{i}_{jk} \delta_i,$$

(3.13)

$$K^{a}_{jk} = A^{a}_{jk} - E^{a}_{jb} h_{kc} + D^{a}_{kb} h_{jc},$$

$$K^{i}_{jk} = P^{i}_{jk} - E^{i}_{jk} D^{i}_{kb} + A^{i}_{jk} h_{bc},$$

$$K^{a}_{j} = \tilde{R}^{a}_{j} + \frac{1}{2} g^{ab} h_{kj} R^{b}_{ac},$$

$$K^{i}_{j} = \tilde{P}^{i}_{j} + \frac{1}{2} B^{i}_{j} R^{i}_{bc} - R^{i}_{jk} B^{i}_{bc},$$

(3.14)
$K(\delta_j, \delta_k)\delta_h := K_h^{\alpha}k_j \dot{\alpha}_a + K_h^{\alpha}k_j \dot{\alpha}_i,$

\begin{align*}
K_h^{i}k_j &= \tilde{R}_h^{i}k_j + A^b_hk D^i_{kj} - A^b_hk D^i_{bk}, \\
K_h^{a}k_j &= A^a_{hkj} - A^a_{hjk} + R^c_{kj} E_{hc}, \\
K_h^{ikj} &= R_{hikj} + D_{ibj} A^b_{hk} - D_{ibk} A^b_{hj},
\end{align*}

Now easily follows

**Proposition 3.1.** The sectional curvatures of $D$ are as follows:

\begin{align*}
K_{ab} &= [S_{a}^{\alpha\beta} + \frac{1}{2}(B_{aa}h_{bb}h_{ii} - B_{ab}h_{ab}h_{ii})]/(h_{aa}h_{bb} - h_{ab}^2), \\
K_{ja} &= (A_{ajj})^a - E_{ajh}^a h_{aj} D^h_{aj} + E_{ajc} P^c_{ja})/g_{jj} g_{aa}, \\
K_{ji} &= (R_{jij} + D_{ibi} A^b_{jj} - D_{ibj} A^b_{ji})/(g_{ii} g_{jj} - g_{ij}^2).
\end{align*}

In the following we assume that $F^n$ reduces to a Riemannian space i.e. $g_{ij}(x, y) = g_{ij}(x)$. The Cartan nonlinear connection reduces to $\tilde{N}_j^i(x, y) = \gamma_{jk}^i(x)y^k$, where $(\gamma_{jk}^i(x))$ are the Christoffel symbols of the metric $g = (g_{ij}(x))$. We consider the corresponding Riemannian metric $G$ given by (3.1) and we have

**Proposition 3.2.** The mapping $\tau : (TM, G) \to (M, g)$ is a Riemannian submersion.

Indeed, $\tau$ is of maximal rank $n$ and its differential $D\tau$ preserves the lengths of horizontal vectors as it follows from $G(\delta_i, \delta_j) = g_{ij}(x)$.

Let $h$ and $v$ denote the projections of $T_x TM$ onto the subspaces of horizontal and vertical vectors, respectively. Following B.O’Neil, [9], the fundamental tensor fields of the Riemannian submersion $\tau$ are as follows:

\begin{align*}
S(X, Y) &= hD_{a}X + vD_{a}hX, \\
N(X, Y) &= vD_{h}X + hD_{h}vY, \quad X, Y \in \mathcal{X}(TM).
\end{align*}

In the frame $(\delta_i, \dot{\alpha}_a)$ we have

\begin{align*}
S(\delta_i, \dot{\delta}_j) &= 0, S(\delta_i, \dot{\alpha}_a) = 0, S(\dot{\alpha}_a, \delta_i) = E^j_{ia} \delta_j, S(\dot{\alpha}_a, \dot{\alpha}_b) = B^i_{ab} \delta_i. \\
N(\delta_i, \dot{\delta}_j) &= \frac{1}{2} R_{ij}^a \dot{\alpha}_a, N(\delta_i, \dot{\alpha}_a) = D^i_{ai} \delta_j, N(\dot{\alpha}_a, \delta_i) = 0, N(\dot{\alpha}_a, \dot{\alpha}_b) = 0
\end{align*}

By (3.19) and (3.7) it follows
Proposition 3.3. The Riemannian submersion \( \tau : (TM, G) \to (M, g) \) is totally geodesics, i.e. \( S = 0 \) if and only if

\[
* g_{ij} \parallel k = 0,
\]

where \( \parallel k \) denotes the \( h \)-covariant derivative with respect to the Berwald connection \( (\gamma^i_{jk}(x)y^k, \gamma^i_{jk}(x), 0) \).

Proposition 3.4. The tensor field \( N \) vanishes if and only if the Riemannian metric \( g \) is flat.

4 Deformations of Riemannian metrics

The geometrical objects associated to \( * g_{ij}(x,y) \) are generally complicated. Some simplifications appear for particular choices of \( a, b \) and \( B_i \). We studied in a previous paper, [1], the case \( a = 1 \) and a concurrent d-vector field \( B^i(x,y) \) while M. Kitayama studied the case \( a = 1 \) and a parallel d-vector field \( B^i(x,y) \), [6]. Here we selected for a detailed analysis the following deformation of a Riemannian metric \( g = (g_{ij}(x)) \):

\[
\begin{align*}
* g_{ij}(x,y) &= a(F^2)g_{ij}(x) + b(F^2)y_iy_j, \\
G(x,y) &= g_{ij}(x,y)dx^i \otimes dx^j + (a(F^2)g_{ij}(x) + b(F^2)y_iy_j)\delta x^a \otimes \delta x^b,
\end{align*}
\]

where \( F^2(x,y) = g_{ij}(x)y^iy^j, y_i = g_{ij}(x)y^j \).

Accordingly, we consider the Riemannian submersion \( \tau : (TM, G) \to (M, g) \), where

\[
\begin{align*}
\Gamma &= (\gamma^i_{jk}(x)y^i, \gamma^i_{jk}(x), 0). \\
\n\end{align*}
\]

The Cartan connection for \((M, g_{ij}(x))\) reduces to 

\[
\begin{align*}
\n\end{align*}
\]

The \( GL \)-metric (4.1) contains as a particular case the \( \phi \)-Lagrange metric associated to a Riemannian space while \( G \) generalises the Cheeger–Gromoll metric studied by Sekizawa [10]. The Cartan connection for \((M, g_{ij}(x))\) reduces to 

\[
\begin{align*}
\n\end{align*}
\]

The \( v \)-covariant derivative \( \nabla^v \parallel k \) coincides with the partial derivative with respect to \( (y^k) \). The \( h \)-covariant derivative \( \nabla^h \parallel k \) reduces to the usual covariant derivative for the objects which do not depend on \( (y^i) \) and coincides with \( \parallel k \) for the others.

We notice for the later use the following formulae

\[
\begin{align*}
\delta_k F^2 = 0, y^i_{\parallel k} = 0, y^i_{\nabla^v k} = 0, y^i_{\nabla^h k} = \delta^i_k, y^i_{\nabla^h k} = g_{ik}(x)
\end{align*}
\]
\[ \delta_k a = 0, \quad \delta_k b = 0 \]
\[ \dot{\delta}_k a = 2a' y_k, \quad \dot{\delta}_k b = 2b' y_k. \]

By a direct calculation one proves

**Proposition 4.1.** The d–connection \( C \) of the GL–metric (4.1) is given by

\[ *F_{jk}^i = \gamma_{jk}^i(x) \text{i.e.} \quad \Phi_{jk}^i = 0 \]
\[ *C_{jk}^i(x, y) = \Lambda_{jk}^i(x, y) = \frac{a'}{a} (\delta_k^j y_j + \delta_j^k y_k) + \frac{b - a'}{a + bF^2} y_j y_k + \frac{ab' - 2a'b}{a(a + bF^2)} y^i y_j y_k. \]

From (4.3) and (4.4) it results

\[ *g_{ij}^k = 0, \quad *g_{ij}^k|_k = 2a' g_{ij} y_k + b(g_{ik} y_j + g_{jk} y_i) + 2b' y_i y_j y_k. \]

Thus \( *g_{ij} \) is \( h \)–metrical and not \( v \)–metrical with respect to \( C \). The torsions of \( *C \) of the GL–metric (4.1) are vanishing excepting \( *R_{jk}^i = \gamma_{jk}^i(x)y^h \) and \( *C_{jk}^i \) from (4.5). As for its curvatures we find

\[ *R_{jk}^i = r_{jk}^i(x) + \Lambda_{jk}^i R_{kh}^i, \]
\[ *P_{jk}^i = 0 \text{ because of } \Lambda_{jk}^i \neq 0, \]
\[ *S_{jk}^i = \Lambda_{jk}^i \text{ from (2.8)'}, \]

where \( r_{jk}^i \) is the curvature tensor of \( (g_{ij}(x)) \).

Using \( y_s R_{kh}^s = y_s r_{kh}^s p^p y^p = r_{ikh} y^i y^k = 0 \), one gets

\[ *R_{jk}^i = r_{jk}^i(x) + \frac{a'}{a} y_j R_{kh}^i + \frac{b - a'}{a + bF^2} y_j R_{jkh}, \]
\[ *R_{0}^i = \left( 1 + \frac{a' F^2}{a} \right) R_{kh}^i. \]
where “0” denotes the contraction by $(y^j)$.

Now we consider the Riemannian metric $G$ given by (4.2). The Levi–Civita connection of it has the local coefficients

$$
F^k_{ij} = \gamma^k_{ij}(x), \quad A^a_{jk} = -\frac{1}{2}r^a_{0\ jk},
$$

(4.10)

$$
D^i_{bj} = \frac{a}{2}r^i_{b\ j0} = \tilde{C}^i_{jb},
$$

$$
E^a_{ib} = 0 = B^k_{ab}, \quad L^a_{bi} = \gamma^a_{bi}(x), \quad C^a_{bc} = \Lambda^a_{bc}.
$$

The curvature of $\nabla$ from (3.9) reduces to

$$
\tilde{R}^i_{h\ jk} = r^i_{h\ jk}(x) + \frac{a}{2}r^i_{h\ a0} \cdot r^a_{0\ jk},
$$

$$
\tilde{R}^a_{b\ jk} = *R^a_{b\ jk},
$$

$$
\tilde{P}^i_{j\ ka} = -\frac{a}{2}r^i_{j\ a0\ k},
$$

(4.11)

$$
P^a_{b\ kc} = 0 \text{ because of } \Lambda^a_{bc|k} = 0,
$$

$$
\tilde{S}^i_{j\ bc} = ar^i_{j\ bc} + \left(a'y_c r^i_{j\ b0} + \frac{a^2}{4}r^h_{j\ b0}a^h_{k0} - b/c\right),
$$

$$
S^a_{b\ cd} = \Lambda^a_{bcd}.
$$
The curvature of the Levi–Civita connection $D$ are given by

$$
K^a_{bc} = \Lambda^a_{bc}, \quad K^i_{cb} = 0,
$$

$$
K^i_{bc} = \tilde{S}^i_{bc}, \quad K^a_{bc} = 0,
$$

$$
K^a_{jb} = 0, \quad K^i_{jb} = \frac{a}{2} r^i_{jb} - \frac{a'}{2} y b r^i_{0 j} - \frac{a'}{2} y c r^i_{0 j} + \frac{a^2}{4} s^i_{s b} r^s_{j c},
$$

$$
K^a_{jkc} = \tilde{S}^a_{jkc}, \quad K^i_{jkc} = 0,
$$

$$
K^a_{kjc} = 0, \quad K^i_{kjc} = \frac{a}{2} r^a_{jkc} - \frac{a'}{2} y b r^a_{0 kc} - \frac{b - a'}{2(a + b F^2)} y^a r^b_{0 jk},
$$

$$
K^a_{ikb} = - \frac{a}{2} r^a_{ikb} + \frac{a'}{4} r^a_{0 ik} - \frac{a'}{2} y b r^a_{0 jk} - \frac{b - a'}{2(a + b F^2)} y^a r^b_{0 jk},
$$

$$
K^i_{ikb} = \frac{a}{2} (r^i_{0 ib} - r^i_{0 b}; k),
$$

$$
K^a_{hjk} = r^a_{hjk} + \frac{a}{2} r^a_{0 hjk} - \frac{a'}{4} r^0_{hjk} r^i_{a 0},
$$

$$
K^i_{hjk} = \frac{1}{2} (r^0_{hjk} - r^0 a h k i).
$$

An inspection of (4.11) and (4.12) gives

**Theorem 4.1.** If $(M, g)$ is flat, then $(TM, G)$ is flat if and only if $\Lambda^i_{jkh} = 0$.

This theorem shows that $G$ is less "rigid" than the Sasaki metric of $(g_{ij}(x))$ which is locally flat if and only if $(g_{ij}(x))$ is locally flat.

Now if we fix $x = x_0$, then $g_{ij}(x_0, y)$ is a Riemannian metric in the fibre $T_{x_0}M$ and $\Lambda^i_{jkh}$ is just its curvature tensor field. Thus we may reformulate Theorem 4.1 in the form

**Theorem 4.1’.** If $(M, g)$ is flat, then $(TM, G)$ is flat if and only if $(T_{x_0}(M), g_{ij}(x_0, y))$ is a flat Riemannian manifold for every $x_0 \in M$.

For the conformal case i.e. $b = 0$ one finds

$$
\Lambda^i_{jk} = \frac{a'}{a} (\delta^i_k y_j - \delta^i_j y_k - y^i g_{jk})
$$

(4.13)
\[ \Lambda^{\dot{j}}_{k^h} = \left[ 2 \left( \frac{a'}{a} \right)' - \frac{a'^2}{a} \right] (\delta^i_k y_i y_j + y^i y_k g_{j} - h/k) + \frac{a'^2}{a} F^2 (\delta^i_k g_{j} - \delta^i_k g_{j}) \] (4.14)

It follows

**Proposition 4.2.** \( \Lambda^{\dot{j}}_{k^h} = 0 \iff a = \text{constant} \).

From Theorem 3.3 and (4.6) one deduces

**Proposition 4.3.** The Riemannian submersion \( \tau : (TM, G) \rightarrow (M, g) \) with \( G \) given by (4.2) is totally geodesics.

The other consequences of the previous formulae will be presented elsewhere.

**References**


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