On the geometry of higher order Lagrange spaces.

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Abstract

A Lagrange space of order $k \geq 1$ is the space of accelerations of order $k$ endowed with a regular Lagrangian. For these spaces we discuss: certain natural geometrical structures, variational problem associated to a given regular Lagrangian and the induced semispray, nonlinear connection, metrical connections. A special attention is paid to the prolongations of the Riemannian and Finslerian structures. In the end we sketch the geometry of time dependent Lagrangian. The geometry, which we have developed, is directed to Mechanicists and Physicists. The paper is a brief survey of our results in the higher order geometry. For details we refer to the monograph [3].

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Introduction

In the last twenty years the problem of the geometrisation of higher order Lagrangians was intensively studied for its applications in Mechanics, Theoretical Physics, Variational Calculus, etc. These Lagrangians are defined on k-jets spaces. The theory had as consequence the appearance of the notion of higher order Lagrange space introduced by R. Miron in [3]. The geometry of these spaces derives from the principles of the Higher Order Mechanics. The integral of action for a Lagrangian leads to a canonical k-semispray from which one constructs the whole geometry of the Lagrange space $L^{(k)n}$. 
1 The k-osculator bundle of a manifold.

In this section we consider the bundle of jets of order k for the maps from $\mathcal{R}$ to a manifold $M$ usually denoted by $J^k(M)$ or $T^kM$. In order to stress that we encounter only this jet bundle and for some historical reasons we call it the k-osculator bundle and denote it by $(\text{Osc}^kM, \pi^k, M)$.

For a local chart $(U, \varphi = (x^i))$ in $p \in M$ its lifted local chart in $u \in (\pi^k)^{-1}(p)$ will be denoted by $((\pi^k)^{-1}(U), \Phi = (x^i, y^{(1)i}, ..., y^{(k)i}))$.

For each $u = (x, y^{(1)i}, ..., y^{(k)i}) \in E := \text{Osc}^kM$, the natural basis of the tangent space $T_uE$ is $\{\frac{\partial}{\partial x^i}|_u, \frac{\partial}{\partial y^{(1)i}}|_u, ..., \frac{\partial}{\partial y^{(k)i}}|_u\}$. The summation over repeated indices will be implied.

We have k-canonical surjective submersion $\pi^k : \text{Osc}^kM \to M$ and $\pi^k_\alpha : \text{Osc}^kM \to \text{Osc}^\alpha M$, $\forall \alpha \in \{1, ..., k-1\}$ which are locally expressed by $\pi^k_\alpha : (x, y^{(1)i}, ..., y^{(k)i}) \mapsto (x)$ and $\pi^k_\alpha : (x, y^{(1)i}, ..., y^{(k)i}) \mapsto (x, y^{(1)i}, ..., y^{(\alpha)i})$. For each of them determines a vertical distribution $V_{\alpha+1}E = \text{Ker}(\pi^k_\alpha)^*$, where $(\pi^k_\alpha)^*$ is the tangent map associated to $\pi^k_\alpha$, $\alpha \in \{0, 1, ..., k-1\}$. The tensor field: $J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \cdots + \frac{\partial}{\partial y^{(k)i}} \otimes dy^{(k-1)i}$ is called the $k$-almost tangent structure on $E$. It has the properties: 1. $J^{k+1} = 0$, 2. $\text{Im}f^\alpha = \text{Ker}J^{k-\alpha+1} = V_\alpha E$, 3. rank $J^\alpha = (k-\alpha+1)n$, $\forall \alpha \in \{1, 2, ..., k\}$.

The vector fields $\hat{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}}, \hat{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(k-1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}$, ... $k^\alpha = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(2)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k)i}}$ are called the Liouville vector fields and they are globally defined on $E$.

A vector field $S \in \chi(E)$ is called a semispray or a k-semispray on $E$ if $JS = \hat{\Gamma}$. The local expression of a semispray is:

\begin{equation}
(1.1) \quad S = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} = (k+1)G^i\frac{\partial}{\partial y^{(k)i}},
\end{equation}

where the functions $G^i$ are defined on every domain of local charts.

We consider also the operator:

\begin{equation}
(1.2) \quad \Gamma = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}.
\end{equation}
2 Variational problem for the higher order Lagrangians

As for Lagrangians of order one it can be considered the integral action for a Lagrangian of order \( k > 1 \). In this section we show how the variational problem associated to it leads to the Euler-Lagrange equations and to the Synge equations, as well. From the latter a semispray of order \( k \) is derived.

A Lagrangian of order \( k \), \((k \in \mathbb{N}^*)\), is a mapping \( L : E := \text{Osc}^k M \rightarrow \mathbb{R} \). \( L \) is called differentiable if it is of class \( C^\infty \) on \( \tilde{E} := E \setminus \{0\} \), \((0 \) denotes the null section of the k-osculator bundle) and continuous on the null section.

The Hessian matrix of a differentiable Lagrangian \( L \), with respect to \( y^{(k)}_i \), on \( \tilde{E} \) has the elements

\[
g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)}_i \partial y^{(k)}_j}.
\]

One can see that \( g_{ij}(x, y^{(1)}, \ldots, y^{(k)}) \) is a symmetric \( d \)-tensor field. In general, a geometric object on \( \text{Osc}^k M \) which behaves like a geometric object on \( M \) will be called a \( d \)-geometric object.

If

\[
\text{rank} \| g_{ij}(x, y^{(1)}, \ldots, y^{(k)}) \| = n \text{ on } \tilde{E}
\]

we say that \( L(x, y^{(1)}, \ldots, y^{(k)}) \) is a regular Lagrangian, otherwise we say that \( L \) is degenerate.

For the beginning we consider the higher order differentiable Lagrangians without the regularity condition \((2.2)\).

The Lie derivatives of a differentiable Lagrangian \( L(x, y^{(1)}, \ldots, y^{(k)}) \) with respect to the Liouville vector fields \( \Gamma^1, \ldots, \Gamma^k \) determine the scalars

\[
I^1(L) = L_{\Gamma^1} L, \ldots, I^k(L) = L_{\Gamma^k} L.
\]

These are differentiable functions on \( \tilde{E} \), called the main invariants of the Lagrangian \( L \) because of their importance in this theory.

Let us consider a smooth parameterised curve \( c : [0, 1] \rightarrow M \) represented in the domain \( U \) of local chart by \( x^i = x^i(t), t \in [0, 1] \). The integral of action for \( L(x, y^{(1)}, \ldots, y^{(k)}) \) is

\[
I(c) = \int_0^1 L(x(t), \frac{dx(t)}{dt}, \ldots, \frac{d^k x(t)}{dt^k}) dt.
\]
On the open set $U$ we consider also the curves

$$(2.5)\quad c_{\varepsilon} : t \in [0, 1] \mapsto (x^i(t) + \varepsilon V^i(t)) \in M,$$

where $\varepsilon$ is a real number, sufficiently small in absolute value so that $\text{Im} \ c_{\varepsilon} \subset U$, $V^i(t) = V^i(x(t))$ being a regular vector field on $U$, restricted to the curve $c$. We assume that all curves $c_{\varepsilon}$ have the same endpoints $c(0)$ and $c(1)$ with the curve $c$ and their derivatives of order $1, \ldots, k - 1$ coincide at the points $c(0)$ and $c(1)$.

**Theorem 2.1** In order that the integral of action $I(c)$ be an extremal value for the functional $I(c_{\varepsilon})$ it is necessary that the following Euler–Lagrange equations hold:

$$(2.6)\quad \dot{E}_i (L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^{(1)}i} + \cdots + (-1)^k \frac{1}{k!} \frac{d^k}{dt^k} \frac{\partial L}{\partial y^{(k)}i} = 0,$$

where $\dot{E}_i (L) = \frac{dx^i}{dt}, \ldots, y^{(k)}i = \frac{1}{k!} \frac{d^k}{dt^k} x^i$.

The curves $c : [0, 1] \to M$, solutions of equations (2.6), are called extremal curves of the integral action $I(c)$. The equality (2.6) implies the following result:

**Theorem 2.2** $\dot{E}_i (L)$ is a $d$-covector field.

Now let us consider a nondegenerate Lagrangian $L$ with the fundamental tensor field $g_{ij}$. We consider the $d$-covector field $\dot{E}_i (L)$ of the following form

$$(2.7)\quad \frac{1}{(k - 1)!} \left\{ \frac{\partial L}{\partial y^{(k - 1)}i} - \Gamma \left( \frac{\partial L}{\partial y^{(k)}i} \right) - \frac{2}{k!} g_{ij} \frac{d^{k+1}x^i}{dt^{k+1}} \right\},$$

where $\Gamma$ is the operator (1.2). It follows

**Theorem 2.3** The system of differential equations (Synge equations)

$$(2.8)\quad g^{ij} \frac{1}{k} \dot{E}_j (L) = 0,$$

determines a $k$-semispray $S$ with the coefficients

$$(2.9)\quad (k + 1) G^i = \frac{1}{2} g^{ij} \left\{ \Gamma \left( \frac{\partial L}{\partial y^{(k)}j} \right) - \frac{\partial L}{\partial y^{(k - 1)}j} \right\}.$$
3 Higher order Lagrange spaces

In the book [3], R.Miron defines the Lagrange spaces of order \( k \) as follows: A Lagrange space of order \( k \) is a pair \( L^{(k)n} = (M, L) \), where \( M \) is a real \( n \)-dimensional manifold, \( L : \text{Osc}^{k}M \rightarrow \mathbb{R} \) is a differentiable Lagrangian for which the fundamental tensor given by (2.1) satisfies (2.2) and the quadratic form \( \psi = g_{ij} \xi^i \xi^j \) has the constant signature on \( \tilde{E} \).

We prove:

**Theorem 3.1** ([2], [3]) If the base manifold \( M \) is paracompact then there exist the Lagrange spaces of order \( k \), \( L^{(k)n} = (M, L) \), for which the fundamental tensor \( g_{ij} \) is positively defined.

**Proof.** \( M \) being a paracompact manifold, there exists at least a Riemannian structure \( \gamma_{ij} \) on \( M \). Denote by \( D \) the Levi-Civita connection of \( (M, \gamma) \) and by \( \gamma_{ij}^{lk} = \gamma_{kj}^{li} \) the local coefficients of \( D \). By a straightforward calculation one proves that:

\[
\begin{align*}
    z^{(1)m} &= y^{(1)m}, \\
    z^{(2)m} &= \frac{1}{2} [Tz^{(1)m} + \gamma_{ij}^{m} z^{(1)i} z^{(1)j}], \\
    z^{(k)m} &= \frac{1}{k} [Tz^{(k-1)m} + \gamma_{ij}^{m} z^{(1)j} z^{(k-1)i}],
\end{align*}
\]

(3.1)

are \( d \)-vector fields.

The Lagrangian

\[
L(x, y^{(1)}, ..., y^{(k)}) = \gamma_{ij}(x) z^{(k)i} z^{(k)j}
\]

is defined on \( \tilde{E} \), is a differentiable Lagrangian, and has the fundamental tensor \( g_{ij}(x, y^{(1)}, ..., y^{(k)}) = \gamma_{ij}(x) \). Thus, the pair \( (M, L) \) is a Lagrange space of order \( k \). \( \text{q.e.d.} \)

In a Lagrange space \( L^{(k)n} = (M, L) \) there exists a \( k \)-semispray on \( E \), with the coefficients given by (2.9) which depends only on the fundamental function \( L \). A nonlinear connection on the manifold \( E \) is a distribution on \( E \), \( N : u \in E \rightarrow N(u) \subset T_u E \) which is supplementary to the vertical distribution \( V_1 : u \in E \rightarrow V_1(u) \subset T_u E \) i.e. we have

\[
T_u E = N(u) \oplus V_1(u), \; \forall u \in E.
\]

For a nonlinear connection \( N \) and for every \( u \in E \) the mapp \((\pi^k)_{*u}|_{N(u)} : N(u) \rightarrow T^{\ast}_{\pi^k(u)} M\) is a linear isomorphism. Its inverse map will be denoted
by \( l_h \) and is called the horizontal lift associated \( N \). Set \( \frac{\delta}{\delta x^i} = l_h(\frac{\partial}{\partial x^i}) \). The linearly independent vector fields \( \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n} \) can be uniquely written in the form:

\[
(3.2) \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_{(1)} \frac{\partial}{\partial y^{(1)}j} - \cdots - N^j_{(k)} \frac{\partial}{\partial y^{(k)}j}.
\]

The functions \( N^j_{(\alpha)}(x, y^{(1)}, \ldots, y^{(k)}) \), \( (\alpha = 1, \ldots, k) \) are called the coefficients of the nonlinear connection \( N \) and \( (\frac{\delta}{\delta x^i}) \), \( (i = 1, \ldots, n) \) is called the adapted basis to the horizontal distribution \( N \). Let be \( N_\alpha = J^\alpha(N) \) and \( \frac{\delta}{\delta y^{(\alpha)i}} = J^\alpha(\frac{\delta}{\delta x^i}), \alpha = 0, \ldots, k - 1 \). Then the following direct decomposition of linear spaces holds:

\[
(3.3) \quad T_u(E) = N_0(u) \oplus N_1(u) \oplus \cdots \oplus N_{k-1}(u) \oplus V_k(u), \; \forall u \in E.
\]

The basis adapted to this decomposition is

\[
(3.4) \quad \{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)}i}, \ldots, \frac{\delta}{\delta y^{(k)}i} \}.
\]

The dual basis of the adapted basis (3.4) is given by:

\[
(3.5) \quad \delta x^i = dx^i, \delta y^{(1)}i = dy^{(1)}i + M^i_j dx^j, \ldots,
\]

\[
\delta y^{(k)}i = dy^{(k)}i + M^i_j \overset{(1)}{dy}^{(k-1)}j + \cdots + M^i_j \overset{(k-1)}{dy}^{(1)}j + M^i_j dx^j,
\]

where

\[
M^i_j = \overset{(1)}{N}^i_j, M^i_j = \overset{(2)}{N}^i_j + \overset{(1)}{N}^m_j M^m_j, \ldots,
\]

\[
M^i_j = \overset{(k-1)}{N}^i_j + \overset{(k-2)}{N}^i_j M^m_j + \cdots + \overset{(1)}{N}^m_j M^m_j.
\]

The set of functions \( (M^i_j)_{\alpha=1}^{k} \) are called the dual coefficients of the nonlinear connection \( N \).

**Theorem 3.2** (R.Miron, [3]) *In a Lagrange space \( L^{(k)n} = (M, L) \) there*
exists a canonical nonlinear connection which has the dual coefficients

$$M^i_j = \frac{\partial G^i}{\partial y^{(k)j}},$$

(3.7)

$$M^i_j = \frac{1}{\alpha} \left\{ S M^i_j + M^i_m M^m_j \right\}, \quad (\alpha = 2, \ldots, k).$$

Theorem 3.3 (I. Bucataru, [2]) Let $S$ be a $k$-semispray with $G^i$ as coefficients. The system of functions:

$$(3.8) \quad M^i_j = \frac{\partial G^i}{\partial y^{(k)j}}, \quad M^i_j = \frac{\partial G^i}{\partial y^{(k-1)j}}, \ldots, \quad M^i_j = \frac{\partial G^i}{\partial y^{(1)j}},$$

are the dual coefficients of a nonlinear connection $N$ on the $k$-osculator bundle.

For a nonlinear connection $N$, an $N$-linear connection is a linear connection $D$ on $E$ with the properties

1. $D$ preserves by parallelism the horizontal distribution $N$.
2. The $k$-tangent structure $J$ is absolutely parallel with respect to $D$.

An $N$-linear connection $D$ on $E$ can be represented in the adapted basis (3.4), in the form

$$D_{\delta} \frac{\delta}{\delta x^i} = L^m_{ij} \frac{\delta}{\delta x^m}, \quad D_{\delta} \frac{\delta}{\delta y^i} = L^m_{ij} \frac{\delta}{\delta y^m}, \quad \alpha = 1, \ldots, k$$

(3.9)

$$D_{\delta} \frac{\delta}{\delta y^i} = C^m_{ij} \frac{\delta}{\delta x^m}, \quad D_{\delta} \frac{\delta}{\delta y^i} = C^m_{ij} \frac{\delta}{\delta y^m},$$

$$(\alpha, \beta = 1, \ldots, k).$$

Theorem 3.4 The following properties hold:

1° There exists a unique $N$-linear connection $D$ on $\tilde{E}$ verifying the axioms:

$$(3.10) \quad g_{ij|h} = 0, \quad g_{ij \mid h} = 0, \quad (\alpha = 1, \ldots, k)$$

7
\[ T^i_{(0)} jh = L^i_{jh} - L^i_{bj} = 0, \]
\[ S^i_{(\alpha)} jh - C^i_{(\alpha)} hj = 0, \quad (\alpha = 1, ..., k). \]

\[ S^i_{(\alpha)} jh = C^i_{(\alpha)} hj - \frac{\partial g_{ij}}{\partial x^\alpha} = 0, \quad (\alpha = 1, ..., k). \]

\[ \Gamma^m_{ij} = \frac{1}{2} g^{ms} \left( \frac{\partial g_{is}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s} \right), \]
\[ C^m_{ij} = \frac{1}{2} g^{ms} \left( \frac{\partial g_{is}}{\partial y^{(\alpha)} j} + \frac{\partial g_{sj}}{\partial y^{(\alpha)} i} - \frac{\partial g_{ij}}{\partial y^{(\alpha)} s} \right) \quad (\alpha = 1, ..., k). \]

\[ \Gamma = (L^i_{jh}, C^i_{(1)} jh, ..., C^i_{(k)} jh) \] of this connection are given by the generalised Christoffel symbols:

\[ \text{2°} \] This connection depends only on the fundamental function \( L(x, y^{(1)}, ..., y^{(k)}) \) of the space \( L^{(k)n} \).

\[ \text{3°} \] The prolongation of the Riemannian and Finslerian structures to the higher order jets bundle

In this section we shall give a solution for the difficult problem of the prolongation to the manifold \( \text{Osc}^k M \) of the Riemannian and Finslerian structures, defined on the base manifold \( M \), \([3]\).

Let \( \mathcal{R}^n = (M, g) \) be a Riemannian space, \( g \) being a Riemannian metric defined on \( M \), having the local coordinates \( g_{ij}(x), \quad x \in U \subset M. \) We showed in the proof of the Theorem 3.1 that the Riemann structure \( g \) determines a regular Lagrangian \( L \). Let \( S \) be the canonical semispray with the local coefficients \( G^i \) given by \( (2.19) \). From Theorem 3.2 or 3.3 we obtain the dual coefficients of a nonlinear connection \( N \) associated to the Lagrange space \( L^{(k)n} = (M, L) \). Now, we can use the canonical nonlinear connection \( N \) with the dual coefficients \( (M^i_{(1) j}, ..., M^i_{(k) j}) \) and adapted cobasis \( (dx^i, \delta y^{(1)i}, ..., \delta y^{(k)i}) \) given by \( (3.5) \).

**Theorem 4.1** The pair \( \text{Prol}^k \mathcal{R}^n = (\text{Osc}^k M, G) \), where

\[ G = g_{ij}(x) dx^i \otimes dx^j + g_{ij}(x) \delta y^{(1)i} \otimes \delta y^{(1)j} + \cdots + g_{ij}(x) \delta y^{(k)i} \otimes \delta y^{(k)j}, \]
is a Riemannian space of dimension \((k + 1)n\), whose metric structure \(G\) depends only on the structure \(g_{ij}(x)\) of the apriori given Riemann space \(\mathcal{R}^n = (M, g)\).

The existence of the Riemannian space \(\text{Pro}^k \mathcal{R}^n = (\widetilde{\text{Osc}}^k M, G)\) solves the posed problem. This space is called the prolongation of order \(k\) of the space \(\mathcal{R}^n = (M, g)\). Also, we say that \(G\) is Sasaki \(N\)-lift of the Riemannian structure \(g\). The prolongation of Finsler structures is obtained on a similar way.

5 Time dependent Lagrangians

The case when a Lagrangian of order \(k > 1\) explicitly depends on time was considered by M. Anastasiei in [1]. In this section the notion of time-dependent \(k\)-spray is introduced and characterised. Then it is shown that any time-dependent \(k\)-spray induces a nonlinear connection and any time-dependent Lagrangian of order \(k\) determines a \(k\)-spray via the Euler-Lagrange equations.

The explicit appearance of time is modelled by considering the manifold \(E = \mathbb{R} \times \text{Osc}^k M\) projected over \(\mathbb{R} \times M\) by \(\pi(t, u) = (t, x), \ x = \pi^k(u), \ u \in \text{Osc}^k M\). The local coordinates on \(E\) are those on \(\text{Osc}^k M\) together with a new one \(t \in \mathbb{R}\) with the meaning of absolute time.

Let \(\pi^k_h : \mathbb{R} \times \text{Osc}^k M \to \mathbb{R} \times \text{Osc}^h M, \ h < k, h, k \in \mathbb{N}\), be given by \((t, x, y^{(1)}, ..., y^{(k)}) \to (t, x, y^{(1)}, ..., y^{(h)})\), \(\pi^k_0 := \pi^k\) and \(V_1 = \ker(\pi^k), V_2 = \ker(\pi^k_1), ..., V_k = \ker(\pi^k_{k-1}), \) where \((\pi^k_h)_*\) means the differential (tangent map) of the mapping \(\pi^k_h\).

Now, consider the linear operators \(J, \tilde{J} : T_u E \to T_u E\) defined with respect to the natural basis as follows:

\[
J(\frac{\partial}{\partial t}) = 0, J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^{(1)} i}, ..., J(\frac{\partial}{\partial y^{(k-1)} i}) = \frac{\partial}{\partial y^{(k)} i}, J(\frac{\partial}{\partial y^{(k)} i}) = 0 \tag{5.1}
\]

\[
\tilde{J}(\frac{\partial}{\partial t}) = -\frac{k}{\Gamma}, \tilde{J}(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^{(1)} i}, ..., \tilde{J}(\frac{\partial}{\partial y^{(k-1)} i}) = \frac{\partial}{\partial y^{(k)} i}, \tilde{J}(\frac{\partial}{\partial y^{(k)} i}) = 0.
\]

A direct calculation gives

**Proposition 5.1**

a) \(J^k(\Gamma) = \Gamma, ..., J^2(\Gamma) = \frac{1}{\Gamma}, J(\Gamma) = 0.\)

b) \(J \circ J \circ \cdots \circ J = 0.\) The same holds for \(\tilde{J}\).
c) \( J \) is an integrable \( k \)-tangent structure.

We notice that \( \tilde{J} \) is not integrable as \( k \)-tangent structure.

A time-dependent vector field on \( \text{Osc}^k M \) is a smooth mapping \( X^\circ : \mathbb{R} \times \text{Osc}^k M \to T(\text{Osc}^k M), (t, u) \to X^\circ(t, u) \in T_u(\text{Osc}^k M), u \in \text{Osc}^k M \). It induces a vector field on \( \mathbb{R} \times \text{Osc}^k M \) by setting \( X(t, u) = \frac{\partial}{\partial t} + X^\circ(t, u) \).

**Definition 5.1** A time–dependent \( k \)-semispray is a vector field \( S = \frac{\partial}{\partial t} + \tilde{S} \), where \( \tilde{S} \) is a time–dependent vector field on \( \text{Osc}^k M \) verifying \( J \tilde{S} = \Gamma \).

It is not difficult to see that \( J \tilde{S} = \Gamma \) implies \( \tilde{S} = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - \frac{ky}{k} y^{(k)i} dt \), where the form of the last term was chosen for the sake of convenience. Thus we get

**Proposition 5.2** A time-dependent \( k \)-semispray is of the form

\[
S = \frac{\partial}{\partial t} + y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i(t, x, y^{(1)}, \ldots, y^{(k)}) \frac{\partial}{\partial y^{(k)i}},
\]

where \( y^{(k)i} = \frac{\partial}{\partial y^{(k)i}} \), \( G^i = \frac{\partial}{\partial y^{(k)i}} \), and \( \frac{\partial}{\partial y^{(k)i}} \) is the last term was chosen for the sake of convenience.

**Proposition 5.3** A vector field \( S \) on \( E = \mathbb{R} \times \text{Osc}^k M \) is a time-dependent \( k \)-semispray if and only if \( JS = \Gamma, \tilde{J}(S) = 0 \).

Let \( \Psi^{(1)i} = dx^i - y^{(1)i} dt, \Psi^{(2)i} = dy^{(1)i} - 2y^{(2)i} dt, \ldots, \Psi^{(k)i} = dy^{(k-1)i} - ky^{(k)i} dt \) be 1–forms on \( E \).

The Propositions 5.2 and 5.3 yield

**Proposition 5.4** A vector field \( S \) on \( \mathbb{R} \times \text{Osc}^k M \) is a time-dependent \( k \)-semispray if and only if

\[
dt(S) = 1, \Psi^{(1)i}(S) = 0, \ldots, \Psi^{(k)i}(S) = 0.
\]

Let \( c : t \to x^i(t) \) be a curve on \( M \) and \( \tilde{c}(t) = (t, x^i(t), \frac{dx^i}{dt}, \frac{1}{2} \frac{d^2x^i}{dt^2}, \ldots, \frac{1}{k!} \frac{d^kx^i}{dt^k}) \) its prolongation to \( \mathbb{R} \times \text{Osc}^k M \). We have
Proposition 5.5 The curve $\tilde{c}$ is an integral curve of a time–dependent $k$–semispray $S$ i.e. $\frac{d\tilde{c}}{dt} = S(\tilde{c})$ if and only if the functions $t \to x^i(t)$ are solutions of the system of differential equations

$$\frac{1}{(k+1)!} \frac{d^{k+1}x^i}{dt^{k+1}} + G^i(t, \frac{dx^i}{dt}, \frac{1}{2!} \frac{d^2x^i}{dt^2}, \ldots, \frac{1}{k!} \frac{d^kx^i}{dt^k}) = 0.$$ 

It is known that a time independent $k$-semispray induces a nonlinear connection, [3]. This also happens for time dependent $k$-semisprays.

A regular time-dependent Lagrangian is a smooth function $L : \mathbb{R} \times \text{Osc}^k M \to \mathbb{R}, (t, x, y^{(1)}, ..., y^{(k)}) \to L(t, x, y^{(1)}, ..., y^{(k)})$ with the property that the matrix

$$(g_{ij}(t, x, y^{(1)}, ..., y^{(k)})) := \left(\frac{1}{2} \frac{\partial}{\partial y^{(k)}_j} \frac{\partial}{\partial y^{(k)}_i} L\right)$$

has rank $n$. The solutions of the variational problem $\delta \int_{t_0}^{t_1} L \, dt = 0$ are given by the well-known Euler-Lagrange equations

$$(5.4) \quad \mathcal{E}_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial y^{(1)}_i}\right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial y^{(2)}_i}\right) - \cdots (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial y^{(k)}_i}\right) = 0,$$

$$(5.4) \quad y^{(1)}_i = \frac{dx^i}{dt}, ..., y^{(k)}_i = \frac{d^kx^i}{dt^k}.$$ 

Theorem 5.1 A regular time dependent Lagrangian determines a $k$-semispray.

Corollary 5.1 Every regular time–dependent Lagrangian determines a nonlinear connection.

References


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