Finite groups with a certain number of values of the Chermak–Delgado measure

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In this paper, we study the finite groups whose Chermak–Delgado measure has exactly \( k \) values. We will focus especially on the case \( k = 2 \). These groups determine an interesting class of \( p \)-groups containing cyclic groups of prime order and extraspecial \( p \)-groups.

Keywords: Chermak–Delgado measure; Chermak–Delgado lattice; subgroup lattice; generalized quaternion 2-group; extraspecial \( p \)-group; outer abelian \( p \)-group; \( p \)-group of maximal class.

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1. Introduction

Throughout this paper, let \( G \) be a finite group and \( L(G) \) be the subgroup lattice of \( G \). Denote by

\[
m_G(H) = |H||C_G(H)|
\]

the Chermak–Delgado measure of a subgroup \( H \) of \( G \) and let

\[
m^*(G) = \max\{m_G(H) \mid H \leq G\} \quad \text{and} \quad \mathcal{CD}(G) = \{H \leq G \mid m_G(H) = m^*(G)\}.
\]

Then the set \( \mathcal{CD}(G) \) forms a modular, self-dual sublattice of \( L(G) \), which is called the Chermak–Delgado lattice of \( G \). It was first introduced by Chermak and Delgado [7], and revisited by Isaacs [9]. In the last years, there has been a growing interest in understanding this lattice (see e.g. [8–13, 15, 18, 20]). We recall several important properties of the Chermak–Delgado measure that will be used in our paper:

- if \( H \leq G \), then \( m_G(H) \leq m_G(C_G(H)) \), and if the measures are equal then \( C_G(C_G(H)) = H \);
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- if \( H \in \text{CD}(G) \), then \( C_G(H) \in \text{CD}(G) \) and \( C_G(C_G(H)) = H \);
- the minimum subgroup \( M(G) \) of \( \text{CD}(G) \) (called the Chermak–Delgado subgroup of \( G \)) is characteristic, abelian, and contains \( Z(G) \).

We remark that the Chermak–Delgado measure associated to a finite group \( G \) can be seen as a function

\[
m_G : L(G) \to \mathbb{N}^*, \quad H \mapsto m_G(H), \quad \forall H \in L(G).
\]

The starting point for our discussion is given by [16, Corollary 3], which states that there is no finite nontrivial group \( G \) such that \( \text{CD}(G) = L(G) \). In other words, \( m_G \) has at least two distinct values for every finite nontrivial group \( G \). This leads to the following natural question:

Given an integer \( k \geq 2 \), which are the finite groups \( G \) whose Chermak–Delgado measure \( m_G \) has exactly \( k \) values?

The paper is organized as follows. In Sec. 2, we present some general results on the above groups. Section 3 deals with the case \( k = 2 \). In the final section, some further research directions are indicated.

We recall several basic definitions:

- a generalized quaternion 2-group is a group of order \( 2^n \), \( n \geq 3 \), defined by the presentation
  \[
  Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, b^{-1}ab = a^{-1} \rangle;
  \]
- a finite \( p \)-group \( G \) is said to be extraspecial if \( Z(G) = G' = \Phi(G) \) has order \( p \);
- a finite \( p \)-group \( G \) is said to be outer abelian if \( G \) is non-abelian, but every proper quotient group of \( G \) is abelian;
- a finite \( p \)-group \( G \) of order \( p^n \) is said to be of maximal class if the nilpotence class of \( G \) is \( n - 1 \).

The following well-known facts on \( p \)-groups will be useful to us. The first two appear in [14, Eqs. (4.26) and (4.4)], the third in [19, Corollary 10], while the fourth in [11, Proposition 1.8].

- Any group of order \( p^4 \) contains an abelian subgroup of order \( p^3 \).
- A finite \( p \)-group \( G \) has a unique subgroup of order \( p \) if and only if either it is cyclic or \( p = 2 \) and \( G \cong Q_{2^n} \) for some \( n \geq 3 \).
- A finite \( p \)-group \( G \) is outer abelian if and only if \( |G'| = p \) and \( Z(G) \) is cyclic, and \( G \) is one of the following non isomorphic groups:
  1. \( M(n, 1) = \langle a, b \mid a^{p^n} = b^p = 1, a^b = a^{1+p^{n-1}} \rangle, n \geq 3 \);
  2. an extraspecial \( p \)-group;
  3. \( G = E \ast A \), where \( E \) is an extraspecial \( p \)-group and \( A \cong M(n, 1), n \geq 3 \);
  4. \( G = E \ast A \), where \( E \) is an extraspecial \( p \)-group and \( A \cong C_{p^t}, t \geq 2 \).
- A finite \( p \)-group \( G \) is of maximal class if and only if it has a subgroup \( A \) of order \( p^2 \) such that \( C_G(A) = A \).
2. Some General Results

First of all, we indicate a lower bound for the number of values of the Chermak–Delgado measure associated to a finite group.

Theorem 2.1. Let $G$ be a finite group. For each prime $p$ dividing the order of $G$ and $P \in \text{Syl}_p(G)$, let $|Z(P)| = p^{n_p}$. Then

$$|\text{Im}(m_G)| \geq 1 + \sum_p n_p. \quad (1)$$

Proof. Let $p$ be a prime dividing the order of $G$ and $P \in \text{Syl}_p(G)$ with $|P| = p^{m_p}$ and $|Z(P)| = p^{n_p}$. For each such $p$ and each $i$, $1 \leq i \leq n_p$, there is a subgroup $H_{p,i} \leq Z(P)$ with $|H_{p,i}| = p^i$. Since $P \subseteq C_G(H_{p,i})$, it follows that the exponent of $p$ in $m_G(H_{p,i})$ is $i + m_p$, and so $m_G(H_{p,i}) \neq m_G(H_{p,j})$ for $i \neq j$. We also observe that we have $m_G(H_{p,i}) \neq m_G(H_{p,j})$ for $p_1 \neq p_2$. Then $\text{Im}(m_G)$ has at least $1 + \sum_p n_p$ distinct elements, namely $m_G(1)$ and $m_G(H_{p,i})$, where $p$ runs over all prime divisors of $|G|$ and $i = 1, 2, \ldots, n_p$. This completes the proof. \qed

Note that for a $p$-group $G$ we have equality in $(1)$ if and only if $G \in \text{CD}(G)$. This happens for large classes of $p$-groups, such as for all abelian $p$-groups. Assume now that $|G| = p_1^{m_{p_1}} p_2^{m_{p_2}} \cdots p_k^{m_{p_k}}$ with $k \geq 2$ and that equality occurs in Eq. (1). Then either $m_G(G) = m_G(1)$ or there are $i \in \{1, \ldots, k\}$ and $1 \leq j \leq n_{p_i}$ such that $m_G(G) = m_G(H_{p,i,j})$. These conditions easily lead to $Z(G) = 1$ or $Z(G) = H_{p,i,j}$. We observe that none of them assure the equality in Eq. (1), as show the examples $G = S_4$ and $G = D_{12}$, respectively. We are also able to give several examples of non-$p$-groups where equality occurs in Eq. (1), such as $A_4$ and all non-abelian groups of order $pq$ with $p, q$ distinct primes.

The following theorem gives a precise formula of computing $|\text{Im}(m_G)|$ for finite abelian groups $G$.

Theorem 2.2. Let $G$ be a finite abelian group of order $n$. Then

$$|\text{Im}(m_G)| = \tau(n),$$

where $\tau(n)$ denotes the number of divisors of $n$.

Proof. Let $H$ be a subgroup of order $d$ of $G$. Then

$$m_G(H) = |H||C_G(H)| = |H||G| = dn$$

and $d \mid n$. Conversely, let $d$ be a divisor of $n$. Since $G$ is abelian, we infer that there is a subgroup $H \leq G$ such that $|H| = d$, and so $m_G(H) = dn$. Thus,

$$\text{Im}(m_G) = \{dn : d \mid n\},$$

which implies that

$$|\text{Im}(m_G)| = \tau(n),$$

as desired. \qed
We remark that Theorem 2.2 can be used to determine all finite abelian groups $G$ whose Chermak–Delgado measure has a small number of values.

**Corollary 2.3.** Let $G$ be a finite abelian group with $|\text{Im}(m_G)| = k$.

(a) If $k = 2$, then $G \cong C_p$ for some prime $p$.

(b) If $k = 3$, then either $G \cong C_{p^2}$ or $G \cong C_p \times C_p$ for some prime $p$.

3. The Case $k = 2$

In this section, we study the class $C$ of finite groups $G$ whose Chermak–Delgado measure $m_G$ has exactly two values. Note that the abelian groups in $C$ have been determined in the above corollary. We easily infer that $C$ is not closed under subgroups, homomorphic images, direct products or extensions. Also, by Theorem 2.1 one obtains that.

**Proposition 3.1.** All groups in $C$ are $p$-groups with center of order $p$.

Since our study can be reduced to $p$-groups and it is completely finished for abelian groups, in what follows, we will suppose that $G$ is a non-abelian $p$-group of order $p^n$ ($n \geq 3$) belonging to $C$. Then:

(a) $G \in \mathcal{C}(G)$;

(b) $\text{Im}(m_G) = \{p^n, p^{n+1}\}$, and consequently $m^*(G) = p^{n+1}$;

(c) $Z(G)$ is the unique minimal normal subgroup of $G$, and consequently $Z(G) \subseteq G' \subseteq \Phi(G)$;

(d) $HZ(G) \in \mathcal{C}(G)$, $\forall H \leq G$ satisfying $Z(G) \not\subseteq H$.

If $Z(G) \nsubseteq H$, then $H \not\in \mathcal{C}(G)$. Since $C_G(H) = C_G(HZ(G))$, it follows that $m_G(H) \neq m_G(HZ(G))$, and consequently $HZ(G) \in \mathcal{C}(G)$.

There are many examples of finite non-abelian $p$-groups $G$ such that $\mathcal{C}(G) = \{Z(G), G\}$ (see e.g. [5, Corollary 2.2 and Proposition 2.3]). Using Corollary 2.3(a), and the above item (d), we are able to prove that the intersection between this class of groups and $C$ is empty.

**Corollary 3.2.** $C$ does not contain non-abelian $p$-groups $G$ with $\mathcal{C}(G) = \{Z(G), G\}$.

**Proof.** Assume that $C$ contains a non-abelian $p$-group $G$ satisfying $\mathcal{C}(G) = \{Z(G), G\}$.

If $G$ possesses a minimal subgroup $H \neq Z(G)$, then $HZ(G) \in \mathcal{C}(G)$ by (d). On the other hand, we obviously have $HZ(G) \neq Z(G)$, and since $\mathcal{C}(G) = \{Z(G), G\}$, we get $HZ(G) = G$. Then $|G| = p^2$, implying that $G$ is abelian, a contradiction.

If $Z(G)$ is the unique subgroup of order $p$ in $G$, then $G$ is a generalized quaternion 2-group, i.e. $p = 2$ and

$$G \cong Q_{2n} = \langle a, b \mid a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, b^{-1} ab = a^{-1} \rangle$$

for some $n \geq 3$.  

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It results that $G$ has a cyclic maximal subgroup $H \cong \langle a \rangle$. So,

$$m_G(H) = 2^{2n-2} \leq 2^{n+1} = m^*(G),$$

which means $n \leq 3$. Since $G$ is non-abelian, we get $n = 3$, that is $G \cong Q_8$. Then $\mathcal{CD}(G)$ is a quasi-antichain of width 3, contradicting the hypothesis.

Next, we will focus on giving examples of non-abelian $p$-groups in $C$.

**Theorem 3.3.** All extraspecial $p$-groups are contained in $C$.

**Proof.** Let $G$ be an extraspecial $p$-group. It is well known that $\mathcal{CD}(G)$ consists of all subgroups $H$ of $G$ containing $Z(G)$ (see e.g. [8] Example 2.8 or [17] Theorem 4.3.4). Consequently, all these subgroups have the same Chermak–Delgado measure. On the other hand, by [2, Lemma 2.6] any subgroup $H$ of $G$ with $Z(G) \not\subseteq H$ satisfies $m_G(H) = |G|$. Thus the function $m_G$ has exactly two values, as desired.

Using GAP, we are also able to give an example of a nonextraspecial non-abelian $p$-group in $C$, namely SmallGroup(32,8):

$$G = \langle a, b, c | a^4 = 1, b^4 = a^2, c^2 = bab^{-1} = a^{-1}, ac = ca, cbc^{-1} = a^{-1}b^3 \rangle.$$ 

Note that the nilpotence class of $G$ is 3. Also, $\mathcal{CD}(G)$ is described in [17, Lemma 4.5.16 and Corollaries 4.5.20 and 4.5.21].

We observe that all non-abelian groups of order $p^3$ belong to $C$ because they are extraspecial. The same thing cannot be said about non-abelian groups of order $p^4$: such a group $G$ has an abelian subgroup $A$ of order $p^3$, and so

$$m^*(G) \geq m_G(A) = p^6 > p^5,$$

implying that $G$ is not contained in $C$. This argument can be extended in the following way.

**Proposition 3.4.** If a non-abelian group of order $p^n$ contains an abelian subgroup of order $\geq p^{\left\lceil \frac{n-1}{2} \right\rceil}$, then it does not belong to $C$.

From Proposition 3.4 we easily infer that certain groups does not belong to $C$. For example, no group of order 64 is contained in $C$ because such a group possesses an abelian subgroup of order 16. Another interesting application of Proposition 3.4 is the following.

**Theorem 3.5.** Let $G$ be a finite $p$-group of nilpotence class 2 contained in $C$. Then $G$ is extraspecial.

**Proof.** Since the nilpotence class of $G$ is 2, we have that $G/Z(G)$ is abelian and so $G' \subseteq Z(G)$. By Proposition 3.1 we get $G' = Z(G)$, which implies that $G$ is an outer abelian $p$-group. Then $G$ belongs to one of the four classes of groups (1)–(4) described in Sec. 1.
We observe that $M(n,1)$ has a cyclic subgroup of order $p^n$, namely $\langle a \rangle$, and $n \geq \left\lceil \frac{m+4}{2} \right\rceil$ for $n \geq 3$. Thus, it cannot be contained in $C$ by Proposition 3.4. Also, it is easy to see that a central product $E * A$, where $E$ is an extraspecial $p$-group of order $p^{2m+1}$ and $A \cong M(n,1)$, $n \geq 3$, always has an abelian subgroup of order $p^n + n$. Since $m + n \geq \left\lceil \frac{2m+n+4}{2} \right\rceil$ for $n \geq 3$, by Proposition 3.4, we infer that $E * A$ does not belong to $C$. Similarly, a central product $E * A$, where $E$ is an extraspecial $p$-group of order $p^{2m+1}$ and $A \cong C_{p^t}$ with $t \geq 2$, always has an abelian subgroup of order $p^{m+t}$. If $t \geq 3$ then $m + t \geq \left\lceil \frac{2m+n+4}{2} \right\rceil$, implying that $E * A$ is not contained in $C$. If $t = 2$, it suffices to observe that the center of $E * A$ is of order $p^2$, and consequently $E * A$ is not contained in $C$ by Proposition 3.1. These shows that the unique possibility is that $G$ be an extraspecial $p$-group, as desired.

Our last result shows that the non-abelian groups of order $p^3$ are in fact the unique $p$-groups of maximal class in $C$.

**Theorem 3.6.** Let $G$ be a finite $p$-group of maximal class contained in $C$. Then $G$ is non-abelian of order $p^3$.

**Proof.** Obviously, $G$ is non-abelian. Let $|G| = p^n$. We know that $G$ possesses a subgroup $A$ of order $p^2$ such that $C_G(A) = A$. It follows that $m_G(A) = p^4$, and therefore, we have either $n = 3$ or $n = 4$. Since the case $n = 4$ is impossible, we get $n = 3$, as desired.

4. Further Research

We end our paper by indicating three natural open problems concerning the above results.

**Problem 1.** Which are the pairs $(p, n)$, where $p$ is a prime and $n$ is a positive integer, such that $C$ contains groups of order $p^n$?

Note that all pairs $(p, n)$ with $n$ odd satisfy this property by Corollary 2.3(a), and Theorem 5.3.

**Problem 2.** Study the class $C'$ of finite groups $G$ whose Chermak–Delgado measure $m_G$ has exactly three values.

We know that $C'$ contains all abelian $p$-groups of order $p^2$ by Corollary 2.3(b); it also contains several classes of non-abelian groups, such as the non-abelian groups of order $pq$ ($p, q$ primes), and in particular the dihedral groups $D_{2p}$ with $p$ an odd prime.

**Problem 3.** Given a finite group $G$, we consider the natural action of Aut($G$) on $L(G)$ and denote by $k$ the number of distinct orbits. Since every two subgroups in the same orbit have the same Chermak–Delgado measure, we have

$$|\text{Im}(m_G)| \leq k. \quad (2)$$

Determine the finite groups $G$ for which equality occurs in (2).
Clearly, cyclic groups satisfy this property. Note that there are also examples of noncyclic groups satisfying it, such as elementary abelian $p$-groups.

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