A note on $U$-decomposable groups

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Abstract

The starting point for our discussion is given by the paper [5], where there is introduced the class of $U$-decomposable groups. Some general results related to these groups are also presented in [5]. The aim of the present paper is to complete these results by obtaining a characterization of $U$-decomposable abelian $p$-groups.

Keywords: subgroups, $U$-decomposable groups, elementary abelian $p$-groups.

1 Preliminaries

We say that a group $(G, \cdot, e)$ is $U$-decomposable, if there exists a finite family of proper subgroups $(H_i)_{i=1}^n$ of $G$ (called a $U$-decomposition of $G$) such that:

i) $G = \bigcup_{i=1}^n H_i$;

ii) $H_i \cap H_j = \{e\}$, for any $i, j = 1, n$ with $i \neq j$.

Otherwise we say that $G$ is $U$-indecomposable.

We proved in Corollary 2, [5], that a finite nilpotent group is $U$-decomposable, if and only if it is a $U$-decomposable $p$-group, therefore it is essential to study the decomposability of $p$-groups. With respect to these groups
we obtained that if \( G \) is a \( U \)-decomposable \( p \)-group and \((H_i)_{i=1}^m\) is a \( U \)-decomposition of \( G \) having the property that there exists \( i_0 \in \{1, 2, ..., n\} \) such that \( \Phi(G) \subseteq H_{i_0} \), then all nontrivial elements in \( G \) are of order \( p \) or \( Z(G) \) is an elementary abelian \( p \)-group (see Proposition 4, [5]).

2 Main results

**Proposition 1.** Let \( G \) be a noncyclic abelian \( p \)-group. Then the following conditions are equivalent:

a) \( G \) is \( U \)-decomposable.

b) \( G \) is an elementary abelian \( p \)-group.

**Proof.** b) \( \implies \) a) Obvious (see Example 1, [5]).

a) \( \implies \) b) From the fundamental theorem on finitely generated abelian groups, there exist (uniquely determined by \( G \)) the natural numbers \( k, \alpha_1, \alpha_2, ..., \alpha_k \) such that \( k \geq 2, 1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \) and \( G \cong \bigoplus_{i=1}^{k} \mathbb{Z}_{p^{\alpha_i}} \).

For each \( i \in \{1, 2, ..., k\} \), let \( \sigma_i : \mathbb{Z}_{p^{\alpha_i}} \rightarrow G \) be the natural map and \( G_i \) be the image of \( \sigma_i \). Then \( G \) is the direct sum of its subgroups \( G_i, i = 1, \ldots, k \). We prove our statement by induction on \( k \).

Suppose that \( k = 2 \) and let \((H_i)_{i=1}^2\) be a \( U \)-decomposition of \( G \). Since \( G_1, G_2 \) are cyclic (and so \( U \)-indecomposable), there exist \( i_1, i_2 \in \{1, 2, ..., n\} \) such that \( i_1 \neq i_2 \) and \( G_1 \subseteq H_{i_1}, G_2 \subseteq H_{i_2} \). Let \( x \in H_{i_1} \setminus G_1 \). Then, writing \( x = x_1 + x_2 \), where \( x_1 \in G_1 \) and \( x_2 \in G_2 \), it obtains \( x - x_1 = x_2 \in H_{i_1} \cap H_{i_2} \), therefore \( x = x_1 \in G_1 \); contradiction. Thus we have \( G_1 = H_{i_1} \) and, in a similar manner, \( G_2 = H_{i_2} \). Moreover, \( H_i \cap G_1 = H_i \cap G_2 = \{e\} \), for any \( i \in \{1, 2, ..., n\} \setminus \{i_1, i_2\} \) (where by \( e \) we denote the identity of \( G \)). First we show that \( \alpha_1 = \alpha_2 \). Indeed, if we assume \( \alpha_1 < \alpha_2 \), then, choosing \( x_0 \) a generator of the cyclic group \( G_i, i = 1, 2 \), it obtains that the element \( x_0 = x_{10} + x_{20} \) is contained in a subgroup \( H_i \) with \( i \neq i_1 \) and \( i \neq i_2 \). It results \( p^{\alpha_1}x_0 \in H_i \cap G_2 \), therefore \( p^{\alpha_1}x_0 = e \); contradiction. Next, we prove that \( \alpha_1 = \alpha_2 = 1 \). If we have \( \alpha_1 = \alpha_2 \geq 2 \), then, considering \( x_{11} \) an element of order \( p^{\alpha_1-1} \) in \( G \), it obtains that there exists \( i \in \{1, 2, ..., n\} \setminus \{i_1, i_2\} \) such that \( x_1 = x_{11} + x_{20} \in H_i \). It follows \( p^{\alpha_1-1}x_1 \in H_i \cap G_2 \) and so \( p^{\alpha_1-1}x_1 = e \); contradiction. Hence \( \alpha_1 = \alpha_2 = 1 \).
Let us suppose the statement to hold for every abelian $p$-group having $k - 1$ invariant factors and let $G$ be an abelian $p$-group with $k$ invariant factors. Denoting $G' = \bigoplus_{i=1}^{k-1} G_i$, we have $G = G' \bigoplus G_k$. If $(H_i)_{i=1}^{n}$ is a $U$-decomposition of $G$, then there exists $i'_2 \in \{1, 2, \ldots, n\}$ such that $G_k \subseteq H_{i'_2}$.

**Case 1.** $G'$ is $U$-indecomposable.

In this situation, we can choose an index $i'_1 \in \{1, 2, \ldots, n\} \setminus \{i'_2\}$ with $G' \subseteq H_{i'_1}$. By a similar reasoning as above, it obtains $G' = H_{i'_1}$, $G_k = H_{i'_2}$ and $H_{i'_1} \cap G' = H_{i'_2} \cap G_k = \{e\}$, for any $i'' \in \{1, 2, \ldots, n\} \setminus \{i'_1, i'_2\}$. Now, following the same steps as in the case $k = 2$, it results $\alpha_{k-1} = \alpha_k = 1$. Hence $\alpha_i = 1$, for any $i = 1, k$.

**Case 2.** $G'$ is $U$-decomposable.

In this situation, from the inductive hypothesis, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 1$. It results that the Frattini subgroup $\Phi(G)$ of $G$ is cyclic of order $p^{\alpha_k - 1}$ and $\Phi(G) \subseteq G_k$. Hence $G$ is an elementary abelian $p$-group.

From the previous proposition and Corollary 2, [5], it obtains the following result, which gives us a characterization of $U$-decomposable finite abelian groups.

**Corollary.** A finite abelian group is $U$-decomposable, if and only if it is an elementary abelian $p$-group.

**References**


