A PROPERTY OF THE FUNCTORS Tor AND Ext

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Let $R$ be the class of domains $R$ satisfying the property:

* for every injective $R$-module $E$ and for every multiplicatively closed subset $S$ of $R$ with $1 \in S$, $E/t_S(E)$ is an injective $R$-module,

where $t_S(E) = \{e \in E \mid (\exists) s \in S$ such that $se = 0\}$. We remark that $\mathcal{R}$ includes the class of the Dedekind rings.

Let $R$ be an element of $\mathcal{R}$, $S$ be a multiplicatively closed subset of $R$ such that $1 \in S$, $S^{-1}R$ be the ring of quotients associated to $R$ and $S$ and $A, B$ be two $R$-modules. The main result of the paper states that, if $t_S(B') = 0$ and if for each $R$-monomorphism $f : B \to B'$ with $t_S(B') = 0$ from $sb' \in f(B)$, $s \in S$, $b' \in B'$ it results $b' \in f(B)$, then we have:

$$S^{-1}\text{Ext}^n_R(A, B) \cong \text{Ext}^n_{S^{-1}R}(S^{-1}A, S^{-1}B).$$

A similar property holds for the functor Tor, i.e.:

$$S^{-1}\text{Tor}_n^R(A, B) \cong \text{Tor}_n^{S^{-1}R}(S^{-1}A, S^{-1}B),$$

in the general case ($R$ is a domain and $A, B$ are two $R$-modules).

1. Preliminaries

Let $R$ be a domain, $S$ be a multiplicatively closed subset of $R$ with $1 \in S$, $S^{-1}R$ be the ring of quotients associated to $R$ and $S$, and $\mathcal{M}_R$, $\mathcal{M}_{S^{-1}R}$ be the categories of $R$-modules, respectively of $S^{-1}R$-modules.

We remind the necessary notions and results (see also [1] and [2]):

We have a functor $S^{-1} : \mathcal{M}_R \to \mathcal{M}_{S^{-1}R}$ given by:

(i) for an $R$-module $A$, $S^{-1}A$ is the module of quotients associated to $A$ and $S$;

(ii) for an $R$-morphism $f : A \to B$, $S^{-1}f : S^{-1}A \to S^{-1}B$ is the $S^{-1}B$-morphism defined by:

$$\frac{a}{s} \mapsto \frac{f(a)}{s} \quad (\forall) a \in A, \ (\forall) s \in S.$$
The functor $S^{-1}$ has the following properties:

(a) it is an exact functor;
(b) for an $R$-module $A$, we have $S^{-1}A \cong S^{-1}R \otimes_R A$;
(c) for an $R$-module $A$, and for an $R$-submodule $A'$ of $A$ we have

(1) $S^{-1}(A/A') \cong S^{-1}A/S^{-1}A'$;
(d) if $A$ and $B$ are two $R$-modules then

$S^{-1}(A \otimes_R B) \cong S^{-1}A \otimes_{S^{-1}R} S^{-1}B$;
(e) if $f : A \to A'$ and $g : B \to B'$ are two $R$-morphisms then:

$$\text{Ker } S^{-1}(f \otimes g) \cong \text{Ker } (S^{-1}f \otimes S^{-1}g)$$

$$\text{Im } S^{-1}(f \otimes g) \cong \text{Im } (S^{-1}f \otimes S^{-1}g)$$

Let $A$ be an $R$-module. We say that $A$ is $S$-divisible if for any $a \in A$, $s \in S$ there exists $a' \in A$ such that $a = sa'$. $A$ is called divisible if it is an $R^*$-divisible module. $t(A) = \{a \in A \mid (\exists) r \in R^*, ra = 0\}$ is called the torsion of $A$. We have:

(i) $t(A) = R-$submodule of $A$;
(ii) $t(A/t(A)) = 0$.

We have also:

(2) if $A$ is a torsion-free $R$-module then $A$ is divisible if and only if $A$ is injective;
(3) if $K$ is the quotient field of $R$ and $A$ is an $R$-module then $A$ is torsion $(t(A) = A)$ if and only if $K \otimes_R A = 0$.

A ring $R$ is called a Dedekind ring if each ideal of $R$ is a projective $R$-module.

About the Dedekind rings we have the following equivalent statements:

(i) a domain $R$ is a Dedekind ring;
(ii) every $R$-submodule of a projective $R$-module is projective;
(iii) every quotient of an injective $R$-module is injective.

If $A$ and $B$ are $R$-modules then:

(a) $\text{Tor}_n^R(A, B) = \frac{\text{Ker } 1_A \otimes d_n}{\text{Im } 1_A \otimes d_{n+1}}$, $(\forall)n \in \mathbb{Z}$, where

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to B \to 0$$

is a projective resolution of $B$;
(b) \( \Ext^n_R(A, B) = \frac{\Ker \Hom(A, d_n)}{\Im \Hom(A, d_{n-1})} \), \((\forall)n \in \mathbb{Z}\), where
\[
0 \rightarrow B \rightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \rightarrow \cdots
\]
is an injective resolution of \( B \).

We remark that \( \Tor^n_R(A, B) \) and \( \Ext^n_R(A, B) \) are \( R \)-modules for any \( R \)-modules \( A, B \) and for any \( n \in \mathbb{Z} \).

2. Main results

Let \( R \) be a domain, \( S \) be a multiplicatively closed subset of \( R \) such that \( 1 \in S \), \( S^{-1}R \) be the ring of quotients associated to \( R \) and \( S \) and \( K \) be the quotient field of \( R \). Let \( B \) be an \( R \)-module.

Definition 1. \( t_S(B) = \{ b \in B \mid (\exists)s \in S \text{ such that } sb = 0 \} \) is called the \( S \)-torsion of \( B \).

Proposition 1.
(i) \( t_S(B) \) is an \( R \)-submodule of \( B \).
(ii) \( t_S(B/t_S(B)) = 0 \).

Proof. (i) Let \( b_1, b_2 \in t_S(B) \); then \((\exists)s_1, s_2 \in S \text{ such that } s_1b_1 = s_2b_2 = 0\). It follows that \((\exists)s = s_1s_2 = s_2s_1 \in S \text{ such that } s(b_1 - b_2) = 0\); then \( b_1 - b_2 \in t_S(B) \). Let \( r \in R \) and \( b \in t_S(B) \); then \((\exists)s \in S \text{ such that } sb = 0\). We have \( s(rb) = (sr)b = (rs)b = r(sb) = r \cdot 0 = 0 \), which gives \( rb \in t_S(B) \).

(ii) Let \( b + t_S(B) \in t_S(B/t_S(B)) \); then \((\exists)s \in S \text{ such that } s(b + t_S(B)) = 0\). Therefore \( sb \in t_S(B) \) and \((\exists)s' \in S \text{ such that } s'(sb) = 0\). It follows that \( (\exists)s'' = s's \in S \text{ such that } s''b = 0 \), i.e. \( b \in t_S(B) \), so \( b + t_S(B) = 0 \).

Definition 2.
(a) We say that \( B \) is \( S \)-saturated if for any \( R \)-monomorphism \( f : B \rightarrow B' \) with \( t_S(B') = 0 \), from \( sb' \in f(B) \), \( s \in S \), \( b' \in B' \) it results \( b' \in f(B) \).
(b) We say that \( B \) is saturated if it is \( R \)-saturated.

Example. Let \( R \) be a field and \( B \) be a vector space over \( R \). Then \( B \) is \( S \)-saturated for any multiplicitively closed subset \( S \) of \( R \) with \( 1 \in S \).

Proposition 2. If \( t_S(B) = 0 \) then the following statements are equivalent:
(i) \( B \) is \( S \)-saturated;
(ii) \( B \) is \( S \)-divisible;
(iii) \( B \) is an \( S^{-1}R \)-module.
Proof. (i) \implies (ii) Let \( y \in B \) and \( s \in S \). We consider the \( R \)-morphism 
\[
f : B \to B, \quad f(x) = sx \quad (\forall) x \in B.
\]
Since \( f \) is monic, \( sy = f(y) \in f(B) \), \( s \in S \) it results that \( y \in f(B) \), i.e. \( (\exists) x \in B \) such that \( y = f(x) \) and so \( B \) is \( S \)-divisible.

(ii) \implies (i) Let \( f : B \to B' \) be an \( R \)-monomorphism with \( t_S(B') = 0 \) and \( sb' \in f(B) \) with \( s \in S \), \( b \in B' \). \( B \) being \( S \)-divisible, we have that \( f(B) = S \)-divisible. It follows that \( (\exists) b'' \in f(B) \) such that \( sb' = sb'' \) and so \( s(b' - b'') = 0 \). Since \( t_S(B') = 0 \), we have \( b' = b'' \in f(B) \). Therefore \( B \) is \( S \)-saturated.

(ii) \implies (iii) It is very easy to verify that \( B \) has a structure of 
\[
S^{-1}R \times B \to B
\]
\[
\left( \frac{r}{s}, b \right) \mapsto \frac{r}{s}b = rb_1,
\]
where \( b_1 \in B \) such that \( b = sb_1 \), \( (\forall) \frac{r}{s} \in S^{-1}R \), \( (\forall)b \in B \).

(iii) \implies (ii) Let \( b \in B \) and let \( s \in S \). As \( B \) is an \( S^{-1}R \)-module, we have \( \frac{1}{s}b \in B \); it follows that there exists \( b' \in B \) such that \( b = sb' \). Therefore \( B \) is \( S \)-divisible.

Remark. Let \( E \) be an injective \( R \)-module. If \( S = R^* \) then \( E/t_S(E) \) is an injective \( R \)-module (this follows from (2)). If \( S \subset R^* \), \( S \neq R^* \) then, in general, \( E/t_S(E) \) is not an injective \( R \)-module.

In the following we will consider the class \( \mathcal{R} \) of the domains \( R \) satisfying the property:

\((*)\) for any injective \( R \)-module \( E \) and for any multiplicatively closed subset \( S \) of \( R \) with \( 1 \in S \), \( E/t_S(E) \) is an injective \( R \)-module.

Proposition 3. Let \( R \) be an element of \( \mathcal{R} \), \( S \) be a multiplicatively closed subset of \( R \) with \( 1 \in S \) and let \( B \) be an \( S \)-saturated \( R \)-module with \( t_S(B) = 0 \). Then \( B \) has an injective resolution:

\[
0 \to B \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \to \cdots
\]
with \( t_S(E_i) = 0 \), \( (\forall)i \in \mathbb{N} \).

Proof. We know that there exists an injective \( R \)-module \( E \) and an \( R \)-monomorphism \( f : B \to E \). From \( t_S(B) = 0 \) it results that

\[
\bar{f} : B \to E/t_S(E)
\]
\[
\bar{f}(b) = f(b) + t_S(E), \quad (\forall)b \in B
\]
is an \( R \)-monomorphism.

Taking \( E_0 = E/t_S(E) \), we have a short exact sequence:

\[
(1) \quad 0 \to B \xrightarrow{i_0} E_0 \xrightarrow{\pi_0} X_0 \to 0,
\]
\( i_0 = \bar{f} \), \( \pi_0 \) = the natural map, with \( E_0 \) = injective and \( t_S(E_0) = 0 \). We have:
Indeed, if $x \in t_S(X_0)$ then there exists $s \in S$ such that $sx = 0$; as $\pi_0$ is an epimorphism, it results that there exists $e_0 \in E_0$ such that $\pi_0(e_0) = x$. Then we have: $\pi_0(se_0) = s\pi_0(e_0) = sx = 0$, and so $se_0 \in \ker \pi_0 = i_0(B)$. But $B$ is $S$-saturated and $i_0 : B \to E_0$ is monic, therefore $e_0 \in i_0(B)$ and $x = \pi_0(e_0) = 0$.

We have:

\begin{equation}
X_0 \text{ is } S\text{-saturated.}
\end{equation}

Indeed, $X_0 = \pi_0(E_0)$ and $E_0$ is divisible; this implies that $X_0$ is divisible, so $X_0$ is $S$-divisible which gives us that $X_0$ is $S$-saturated (see the relation (2) and Proposition 2). Repeating the previous reasoning with $X_0$ as $B$, we obtain a short exact sequence:

$$0 \to X_0 \xrightarrow{i_1} E_1 \xrightarrow{\pi_1} X_1 \to 0,$$

where $E_1$ is injective, $t_S(E_1) = 0$, $X_1$ is $S$-saturated and $t_S(X_1) = 0$.

By induction, we obtain a short exact sequence:

$$0 \to X_{n-1} \to E_n \to X_n \to 0, \quad (\forall)n \in \mathbb{N},$$

where $E_n$ is injective, $t_S(E_n) = 0$, $X_n$ is $S$-saturated and $t_S(X_n) = 0$, $X_n$ is $S$-saturated and $t_S(X_n) = 0$, $x \in \mathbb{N}$.

If we take $d_n = i_{n+1} \circ \pi_n : E_n \to E_{n+1}, \quad (\forall)n \in \mathbb{N}$, we have an injective resolution of $B$:

$$0 \to B \xrightarrow{i_0} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \to \cdots$$

with $t_S(E_i) = 0, \quad (\forall)i \in \mathbb{N}$.

**Proposition 4.**

(i) If $P$ is a projective $R$-module then $S^{-1}P$ is a projective $S^{-1}R$-module.

(ii) If $E$ is an injective $R$-module with $t_S(E) = 0$ then $S^{-1}E$ is an injective $S^{-1}R$-module.

**Proof.** (i) Let $A, B$ be two $S^{-1}R$-modules and let $f : A \to B, \ h : S^{-1}P \to B$ be two $S^{-1}R$-morphisms with $f$ epimorphism.
We consider $\varphi : P \rightarrow S^{-1}P$ be the natural map $\left(p \mapsto \frac{p}{1}, \forall p \in P \right)$. We have that $A, B$ are two $R$-modules and $f, h$ are $R$-morphisms. From the fact that $P$ is projective it results that there is an $R$-morphism $h' : P \rightarrow A$ such that $f \circ h' = h \circ \varphi$.

Let $\bar{h} : S^{-1}P \rightarrow A$, $\bar{h} \left(\frac{p}{s}\right) = \frac{h'(p)}{s}$, $\forall p \in P$, $\forall s \in S$. It is easy to verify that:

(a) $\bar{h}$ is well defined;

(b) $\bar{h}$ is an $S^{-1}R$-morphism;

(iii) $f \circ \bar{h} = h$.

From the previous statements we have that $S^{-1}P$ is a projective $S^{-1}R$-module.

(ii) Let $A, B$ be two $S^{-1}R$-modules and let $\sigma : A \rightarrow B$, $h : A \rightarrow S^{-1}E$ be two $S^{-1}R$-morphisms with $\sigma$ monomorphism.

Because $E$ is divisible and $t_S(E) = 0$ we have that: $\forall e \frac{e}{s} \in S^{-1}E$, $\exists! e' \in E$ such that $\frac{e}{s} = e'$ and so we have a function $u : S^{-1}E \rightarrow E$, $\frac{e}{s} \mapsto e'$ with $e = se'$. It is easy to see that $u$ is an $S^{-1}R$-morphism and, in particular, $u$ is an $R$-morphism. Since $E$ is injective, there is an $R$-morphism $h' : B \rightarrow E$ such that $h' \circ \sigma = u \circ h$.

Let $\bar{h} : B \rightarrow S^{-1}E$, $\bar{h}(b) = \frac{h'(b)}{1}$, $\forall b \in B$. We have that:

(a) $\bar{h}$ is well defined;

(b) $\bar{h}$ is an $S^{-1}R$-morphism;

(c) $\bar{h} \circ \sigma = h$. 
By the previous statements, $S^{-1}E$ is an injective $S^{-1}R$-module.

**Proposition 5.** If $A$ is an $R$-module and $B$ is an $S$-saturated $R$-module with $t_S(B) = 0$, then we have the isomorphism of $S^{-1}R$-modules:

$$S^{-1}\text{Hom}_R(A, B) \cong \text{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B).$$

**Proof.** We define:

$$\psi : S^{-1}\text{Hom}_R(A, B) \longrightarrow \text{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B)$$

\[\frac{f}{s} \mapsto \psi_{\frac{f}{s}} : A^{-1}A \longrightarrow S^{-1}B\]

\[\frac{a}{s'} \mapsto \frac{f(a)}{ss'},\]

$(\forall) \frac{f}{s} \in S^{-1}\text{Hom}_R(A, B), \ (\forall) \frac{a}{s'} \in S^{-1}A.$

It is easy to verify that:

(a) $\psi$ is well defined;

(b) $\psi$ is an $S^{-1}R$-monomorphism.

We shall prove that $\psi$ is epic. Let $f' : S^{-1}A \longrightarrow S^{-1}B$ be an $S^{-1}R$-morphism; if $\varphi : A \longrightarrow S^{-1}A$ is the natural map $\left(a \mapsto \frac{a}{1}, \ (\forall) a \in A\right)$ and $u : S^{-1}B \longrightarrow B$ is the morphism which was constructed in the proof of Proposition 4, then we have $f = u \circ f' \circ \varphi \in \text{Hom}_R(A, B)$ and $\psi \left(\frac{f}{1}\right) = f'$, so that $\psi$ is epic. In conclusion, $\psi$ is an isomorphism of $S^{-1}R$-modules.

**Corollary.** If $A, B_1, B_2$ are $R$-modules, $h : B_1 \longrightarrow B_2$ is an $R$-morphism and $B_1, B_2$ are $S$-saturated with $t_S(B_1) = t_S(B_2) = 0$ then we have:

\[(*) \quad \text{Ker } S^{-1}\text{Hom}_R(A, h) \cong \text{Ker } \text{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}h)\]

\[\text{Im } S^{-1}\text{Hom}_R(A, h) \cong \text{Im } \text{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}h).\]

**Proof.** We have the commutative diagram:

$$\begin{array}{ccc}
S^{-1}\text{Hom}_R(A, B_1) & \xrightarrow{\psi_1} & S^{-1}\text{Hom}_R(A, B_2) \\
\text{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B_1) & \xrightarrow{\psi_2} & \text{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B_2)
\end{array}$$

where $\psi_1$ and $\psi_2$ are the isomorphisms given by Proposition 5.
From $\psi_2 \circ S^{-1}\text{Hom}_R(A, h) = \text{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}h) \circ \psi_1$, we obtain the isomorphisms (**)  

**Remark.** Let $A$ and $B$ be two $R$-modules.

(a) Following (1), d), §1, we have:

\[(\alpha) \quad S^{-1}\text{Tor}^R_0(A, B) \cong \text{Tor}^{S^{-1}R}_0(S^{-1}A, S^{-1}B).\]

(b) From Proposition 5, it results that, if $B$ is $S$-saturated with $t_S(B) = 0$, then we have:

\[(\beta) \quad S^{-1}\text{Ext}^0_R(A, B) \cong \text{Ext}^0_{S^{-1}R}(S^{-1}A, S^{-1}B).\]

We extend the isomorphisms (α) and (β) for an arbitrary $n \in \mathbb{N}$.

**Theorem.** Let $R$ be a domain, $S$ be a multiplicatively closed subset of $R$ with $1 \in S$ and $A, B$ be two $R$-modules. Then:

(i) $S^{-1}\text{Tor}^R_n(A, B) \cong \text{Tor}^{S^{-1}R}_n(S^{-1}A, S^{-1}B), \ (\forall)n \in \mathbb{N}$.

(ii) If $R$ is an element of $R$ and $B$ is $S$-saturated with $t_S(B) = 0$, then we have:

\[S^{-1}\text{Ext}^n_R(A, B) \cong \text{Ext}^n_{S^{-1}R}(S^{-1}A, S^{-1}B), \ (\forall)n \in \mathbb{N}.\]

**Proof.** (i) Let $\cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow B \rightarrow 0$ be a projective resolution of $B$. We have

\[\text{Tor}^R_n(A, B) = \frac{\text{Ker} \ 1_A \otimes d_n}{\text{Im} \ 1_A \otimes d_{n+1}}.\]

Following Proposition 4, it is easy to see that the sequence:

\[\cdots \rightarrow S^{-1}P_2 \xrightarrow{S^{-1}d_2} S^{-1}P_1 \xrightarrow{S^{-1}d_1} S^{-1}P_0 \rightarrow S^{-1}B \rightarrow 0\]

is a projective resolution of $S^{-1}R$-module $S^{-1}B$. Then

\[\text{Tor}^{S^{-1}R}_n(S^{-1}A, S^{-1}B) = \frac{\text{Ker} \ 1_{S^{-1}A} \otimes S^{-1}d_n}{\text{Im} \ 1_{S^{-1}A} \otimes S^{-1}d_{n+1}}.\]

From the facts that $S^{-1}$ is an exact functor and $S^{-1}(A/A') \cong S^{-1}A/S^{-1}A'$, it results:

\[S^{-1}\text{Ker} \ u \cong \text{Ker} \ S^{-1}u \quad \text{for any } R\text{-morphism } u.\]

\[S^{-1}\text{Im} \ u \cong \text{Im} \ S^{-1}u\]
Then
\[ S^{-1}\text{Tor}_n^R(A, B) = S^{-1}\frac{\text{Ker } 1_A \otimes d_n}{\text{Im } 1_A \otimes d_{n+1}} \cong S^{-1}\frac{\text{Ker } 1_A \otimes d_n \cong S^{-1}\text{Im } 1_A \otimes d_{n+1}}{\text{Ker } S^{-1}(1_A \otimes d_n)} \cong S^{-1}\frac{1_{S^{-1}A} \otimes S^{-1}d_n}{\text{Im } S^{-1}(1_A \otimes d_{n+1})} = \text{Tor}_{S^{-1}R}(S^{-1}A, S^{-1}B). \]

(ii) From Proposition 3, \( B \) has an injective resolution:
\[ 0 \rightarrow B \rightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \rightarrow \cdots \]
with \( t_S(E_i) = 0, \, (\forall) i \in \mathbb{N} \).
We have
\[ \text{Ext}^n_{R}(A, B) = \frac{\text{Ker } \text{Hom}(A, d_n)}{\text{Im } \text{Hom}(A, d_{n-1})}. \]
Following Proposition 4, it is easy to see that the sequence:
\[ 0 \rightarrow S^{-1}B \rightarrow S^{-1}E_0 \xrightarrow{S^{-1}d_0} S^{-1}E_1 \xrightarrow{S^{-1}d_1} S^{-1}E_2 \rightarrow \cdots \]
is an injective resolution of the \( S^{-1}R \)-module \( S^{-1}B \). Then
\[ \text{Ext}^{n}_{S^{-1}R}(S^{-1}A, S^{-1}B) = \frac{\text{Ker } \text{Hom}(S^{-1}A, S^{-1}d_n)}{\text{Im } \text{Hom}(S^{-1}A, S^{-1}d_{n-1})}. \]
We have:
\[ S^{-1}\text{Ext}^n_{R}(A, B) = S^{-1}\frac{\text{Ker } \text{Hom}(A, d_n)}{\text{Im } \text{Hom}(A, d_{n-1})} \cong S^{-1}\frac{\text{Ker } \text{Hom}(A, d_n) \cong S^{-1}\text{Im } \text{Hom}(A, d_{n-1})}{\text{Ker } S^{-1}\text{Hom}(A, d_n)} \cong S^{-1}\frac{\text{Ker } \text{Hom}(S^{-1}A, S^{-1}d_n)}{\text{Im } \text{Hom}(S^{-1}A, S^{-1}d_{n-1})} = \text{Ext}^{n}_{S^{-1}R}(S^{-1}A, S^{-1}B). \]

**Corollary 1.** In the hypothesis of the above theorem, we have:
\[ \text{Tor}^{S^{-1}R}_n(S^{-1}A, S^{-1}B) \cong S^{-1}R \otimes_R \text{Tor}^R_n(A, B) \]
\[ \text{Ext}^{n}_{S^{-1}R}(S^{-1}A, S^{-1}B) \cong S^{-1}R \otimes_R \text{Ext}^R_n(A, B). \]

**Proof.** We saw in (1), b), §1, that for any \( R \)-module \( A \) we have
\[ S^{-1}A \cong S^{-1}R \otimes_RA. \]

**Corollary 2.** If \( R \) is a domain, \( A \) and \( B \) are two \( R \)-modules and \( n \in \mathbb{N}^* \) then \( \text{Tor}^R_n(A, B) \) is a torsion \( R \)-module.
**Proof.** Taking $S = R^*$ in Corollary 1, we have
\[
(\gamma) \quad \text{Tor}^K_n(S^{-1}A, S^{-1}B) \cong K \otimes_R \text{Tor}^R_n(A, B).
\]
As $K$ is a semisimple ring, we have that every $K$-module is projective, and therefore
\[
(\delta) \quad \text{Tor}^K_n(S^{-1}A, S^{-1}B) = 0.
\]
Now, from $(\gamma)$, $(\delta)$ and (3), §1, we have that $\text{Tor}^R_n(A, B)$ is a torsion $R$-module.

**Remark.** In the hypothesis: "$B$ is saturated with $t(B) = 0$" we get that $B$ is injective and then $\text{Ext}^n_R(A, B) = 0$, for any $R$-module $A$ and any $n \in \mathbb{N^*}$. 
References


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