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The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers

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Abstract

The main goal of this note is to establish a connection between the fuzzy subgroups of a finite cyclic group with \( k \) direct factors and the lattice paths of \( \mathbb{Z}^k \). This leads us to an explicit formula for the well-known central Delannoy numbers.

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1. Introduction

One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite abelian group. This topic has undergone a rapid evolution in the last few years. Several papers have treated the particular case of finite cyclic groups. Thus, in [4] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [5] deals with this number for cyclic groups of order \( p^n q^m \) (\( p, q \) primes). The starting point of our discussion is given by the paper [8], where it is indicated a recurrence relation verified by the number of distinct fuzzy subgroups for two classes of finite abelian groups: finite cyclic groups and finite elementary abelian \( p \)-groups. For the first class this can be successfully used to obtain an explicit formula of the above number, which will play an essential role in our paper.

In the \( k \)-dimensional space \( \mathbb{Z}^k \) we consider a lattice path to be represented as a concatenation of directed steps. Fix nonnegative integers \( n_1, n_2, \ldots, n_k \) and let \( L(n_1, n_2, \ldots, n_k) \) denote the lattice consisting of all integer points \( (a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k \) satisfying \( 0 \leq a_i \leq n_i, \ i = 1, k \). Recall that \( L(n_1, n_2, \ldots, n_k) \) is partially ordered by the dominance relation “\( \leq \)”, defined by \( (a_1, a_2, \ldots, a_k) \leq (b_1, b_2, \ldots, b_k) \) if and only if \( a_i \leq b_i \) for all \( i = 1, k \). The

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Delannoy numbers $D(n_1, n_2, \ldots, n_k)$ count the number of lattice paths in $L(n_1, n_2, \ldots, n_k)$ from $(0, 0, \ldots, 0)$ to $(n_1, n_2, \ldots, n_k)$ in which only nonzero steps of the form $(x_1, x_2, \ldots, x_k)$ with $x_i \in \{0, 1, \ldots, k\}$, are allowed (such a path is sometimes referred to as a restricted king’s walk). When $n_i$, $i = 1, k$, have the same value $n$, we refer to $d^n_k = D(n, n, \ldots, n)$ as the central Delannoy numbers. They are used in counting a lot of mathematical configurations (see [7]) and many results to these numbers have been obtained. Here we recall only the following remarkable identity (see [6] for $k = 2$ and [2] for an arbitrary $k$)

\[ e_n^k = 2^{n-1}d_n^k, \quad (*) \]

where $e_n^k$ counts the lattice paths of $L(n_1, n_2, \ldots, n_k)$ from the origin to $(n, n, \ldots, n)$ that use nonzero steps of the form $(x_1, x_2, \ldots, x_k)$ with $x_i \geq 0, i = 1, k$. We also mention that there are explicit formulas of the central Delannoy numbers only for the 2-dimensional case in [1] and for the 3-dimensional case in [2].

This paper links the distinct fuzzy subgroups of a finite cyclic group of order $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ ($p_1, p_2, \ldots, p_k$ distinct primes), whose number was determined in [8], and the lattice paths of $L(n_1, n_2, \ldots, n_k)$. In particular, an explicit formula of $d_n^k$ for any $k$ will be obtained.

2. Fuzzy subgroups

In this section some basic notions and results of fuzzy subgroup theory are presented (for more details, see [3]). We also recall the main theorem of [8] concerning the number of distinct fuzzy subgroups of an arbitrary finite cyclic group.

Let $(G, \cdot, e)$ be a group ($e$ denotes the identity of $G$) and $\mu : G \rightarrow [0, 1]$ be a fuzzy subset of $G$. We say that $\mu$ is a fuzzy subgroup of $G$ if it satisfies the following two conditions:

(a) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in G$;
(b) $\mu(x^{-1}) \geq \mu(x)$, for any $x \in G$.

In this situation we have $\mu(x^{-1}) = \mu(x)$, for any $x \in G$, and $\mu(e) = \max \mu(G)$. For each $\alpha \in [0, 1]$, we define the level subset:

\[ \mu G_{\alpha} = \{x \in G \mid \mu(x) \geq \alpha\}. \]

These subsets allow us to characterize the fuzzy subgroups of $G$, as follows: $\mu$ is a fuzzy subgroup of $G$ if and only if its level subsets are subgroups in $G$.

The fuzzy subgroups of $G$ can be classified up to some natural equivalence relations on the set consisting of all fuzzy subsets of $G$. One of them (used in [8], too) is defined by

\[ \mu \sim \eta \iff (\mu(x) > \mu(y) \iff \eta(x) > \eta(y), \quad \text{for all } x, y \in G) \]

and two fuzzy subgroups $\mu, \eta$ of $G$ are called distinct if $\mu \not\sim \eta$.

Even for some particular classes of finite groups, as finite abelian groups, the problem of counting the number of distinct fuzzy subgroups is very difficult. An important step in order to solve this problem for finite elementary abelian $p$-groups is made in Section 3 of [8]. The largest class of finite groups for which this was completely solved consists of finite cyclic groups. The method used is founded on establishing a recurrence relation verified by the above number.

**Lemma 1.** Let $G$ be a finite cyclic group and $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ be the decomposition of $|G|$ as a product of prime factors. Then the number $f_k(n_1, n_2, \ldots, n_k)$ of all distinct fuzzy subgroups of
$G$ satisfies the following recurrence relation:

$$f_k(n_1, n_2, \ldots, n_k) = 2 \sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq k} f_k(n_1, \ldots, n_{i_1} - 1, \ldots, n_{i_2} - 1, \ldots, n_{i_r} - 1, \ldots, n_k).$$

By using generating functions, this gives us an explicit expression of the numbers $f_k(n_1, n_2, \ldots, n_k)$ (see Corollary 4 of [8]).

**Theorem 2.** Let $G$ be a finite cyclic group and $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ be the decomposition of $|G|$ as a product of prime factors. Then the number $f_k(n_1, n_2, \ldots, n_k)$ of all distinct fuzzy subgroups of $G$ is given by the equality

$$f_k(n_1, n_2, \ldots, n_k) = 2^{k} \sum_{i_2=0}^{n_2} \sum_{i_3=0}^{n_3} \cdots \sum_{i_k=0}^{n_k} \left( -\frac{1}{2} \right)^{r-2} \prod_{r=2}^{k} \binom{n_{i_r}}{i_r} \left( n_1 + \sum_{s=2}^{r} (n_s - i_s) \right),$$

where the above iterated sums are equal to 1 for $k = 1$.

### 3. The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers

In the following, let $G$ be a finite cyclic group of order $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ ($p_1, p_2, \ldots, p_k$ distinct primes) and denote by $L(n_1, n_2, \ldots, n_k)$ the lattice of integer points $(a_1, a_2, \ldots, a_k)$ satisfying $0 \leq a_i \leq n_i$, for all $i = 1, k$. Let $\mu : G \to [0, 1]$ be a fuzzy subgroup of $G$. Put $\mu(G) = \{a_1, a_2, \ldots, a_r\}$ and suppose that $a_1 > a_2 > \cdots > a_r$. Then the level subgroups of $\mu$ determines the following chain of subgroups of $G$ which ends in $G$:

$$\mu^G_{a_1} \subset \mu^G_{a_2} \subset \cdots \subset \mu^G_{a_r} = G. \quad (1)$$

Moreover, for any $x \in G$ and $i = 1, r$, we have

$$\mu(x) = a_i \iff i = \max\{j \mid x \in \mu^G_{a_j}\} \iff x \in \mu^G_{a_i} \setminus \mu^G_{a_{i-1}},$$

where, by convention, we set $\mu^G_{a_0} = \emptyset$.

Our first aim is to study the equivalence class of $\mu$ modulo $\sim$, more precisely what can be said about a fuzzy subgroup $\eta$ of $G$ which is equivalent to $\mu$. If $\eta(G) = \{\beta_1, \beta_2, \ldots, \beta_s\}$ with $\beta_1 > \beta_2 > \cdots > \beta_s$, then $\eta$ induces also a chain of subgroups of $G$ that terminates in $G$:

$$\eta^G_{\beta_1} \subset \eta^G_{\beta_2} \subset \cdots \subset \eta^G_{\beta_s} = G. \quad (1')$$

First of all, we shall prove that $s = r$. Assume that $s \neq r$ and, without loss of generality, that $s < r$. Choose the following elements: $x_1 \in \mu^G_{a_1}$, $x_2 \in \mu^G_{a_2} \setminus \mu^G_{a_1}$, $x_r \in \mu^G_{a_r} \setminus \mu^G_{a_{r-1}}$. Then $\mu(x_1) > \mu(x_2) > \cdots > \mu(x_r)$. Since $\mu \sim \eta$, it obtains $\eta(x_1) > \eta(x_2) > \cdots > \eta(x_r)$ and so $\eta(G)$ has at least $r$ elements, a contradiction. Thus $s = r$.

Now, let us consider an index $i \in \{1, 2, \ldots, r\}$ minimal with the property that $\mu^G_{a_i} \neq \eta^G_{\beta_i}$. For $i = 1$, by taking $x \in \mu^G_{a_1} \setminus \eta^G_{\beta_1}$ it results $\mu(x) = a_1 = \mu(e)$ and $\eta(x) < \beta_1 = \eta(e)$, which contradict the relation $\mu \sim \eta$. For $i \geq 2$, we distinguish the following two cases.

**Case 1.** $\mu^G_{a_i} \subset \eta^G_{\beta_i}$ or $\mu^G_{a_i} \supset \eta^G_{\beta_i}$. 


Suppose that \( \mu G_{\alpha_i} \subseteq \eta G_{\beta_i} \) (the other situation is similar) and take an element \( x \in \eta G_{\beta_i} \setminus \mu G_{\alpha_i} \).

Because \( \mu G_{\alpha_i-1} = \eta G_{\beta_i-1} \), \( x \) is not contained in \( \eta G_{\beta_i-1} \). It follows that \( \eta(x) = \beta_i \). For an arbitrary \( y \in \mu G_{\alpha_i} \), we get \( \mu(y) \geq \alpha_i > \mu(x) \) and \( \eta(y) \leq \beta_i = \eta(x) \), a contradiction.

**Case 2.** \( \mu G_{\alpha_i} \nsubseteq \eta G_{\beta_i} \) and \( \mu G_{\alpha_i} \not\supseteq \eta G_{\beta_i} \).

In this situation there exist \( x \in \mu G_{\alpha_i} \setminus \eta G_{\beta_i} \) and \( y \in \eta G_{\beta_i} \setminus \mu G_{\alpha_i} \). Therefore \( \mu(x) = \alpha_i > \mu(y) \) and \( \eta(x) < \beta_i = \eta(y) \), contradicting again our hypothesis.

In this way, we have shown that the relation \( \mu \sim \eta \) implies that the subgroup chains (1) and (1’) must coincide.

Conversely, if the chains (1) and (1’) are the same, then let \( x, y \in G \) such that \( \mu(x) > \mu(y) \).

This means that we have \( \mu(x) = \alpha_i \) and \( \mu(y) = \alpha_j \) with \( \alpha_i > \alpha_j \), for some \( i \) and \( j \). From the equalities \( \mu G_{\alpha_i} = \eta G_{\beta_i} \) and \( \mu G_{\alpha_j} = \eta G_{\beta_j} \), we easily obtain \( \eta(x) = \beta_i > \beta_j = \eta(y) \).

Thus \( \mu \sim \eta \) and therefore an equivalence class of fuzzy subgroups of \( G \) with respect to \( \sim \) (in other words, a distinct fuzzy subgroup of \( G \)) is uniquely determined by a chain of subgroups of \( G \) which ends in \( G \).

Next, in the chain (1) set \( |\mu G_{\alpha_i}| = p_1^{a_{i1}} p_2^{a_{i2}} \cdots p_k^{a_{ik}} \), for all \( i = 1, r \). Then one obtains the following lattice path in \( L(n_1, n_2, \ldots, n_k) \):

\[
(a_{11}, a_{21}, \ldots, a_{k1}) \leq (a_{12}, a_{22}, \ldots, a_{k2}) \leq \cdots \leq (a_{1r}, a_{2r}, \ldots, a_{kr}) = (n_1, n_2, \ldots, n_k).
\]

(2)

Clearly, the previous correspondence between the chains of type (1) and the lattice paths of type (2) is one-to-one and onto, and hence we have proved the following theorem.

**Theorem 3.** Let \( G \) be a finite cyclic group of order \( p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \) (\( p_1, p_2, \ldots, p_k \) distinct primes) and \( L(n_1, n_2, \ldots, n_k) \) be the lattice of integer points \((a_1, a_2, \ldots, a_k)\) satisfying \( 0 \leq a_i \leq n_i \), for all \( i = 1, k \). Then there exists a bijection between the set of distinct fuzzy subgroups of \( G \) and the set of lattice paths with nonzero arbitrary steps in \( L(n_1, n_2, \ldots, n_k) \) that terminate in \((n_1, n_2, \ldots, n_k)\).

Obviously, every lattice path \( \mathcal{L} \) in \( L(n_1, n_2, \ldots, n_k) \) from \((0, 0, \ldots, 0)\) to \((n_1, n_2, \ldots, n_k)\) determines two paths of type (2): \( \mathcal{L} \) itself and the path obtained from \( \mathcal{L} \) by eliminating the origin. So, Theorems 2 and 3 imply the following result.

**Theorem 4.** The number \( e^{k}_{n_1, n_2, \ldots, n_k} \) of all lattice paths with nonzero arbitrary steps in \( L(n_1, n_2, \ldots, n_k) \) from \((0, 0, \ldots, 0)\) to \((n_1, n_2, \ldots, n_k)\) satisfies the following equality

\[
e^{k}_{n_1, n_2, \ldots, n_k} = \frac{1}{2} f_k(n_1, n_2, \ldots, n_k),
\]

where \( f_k(n_1, n_2, \ldots, n_k) \) is the number computed in Theorem 2.

In particular, by taking \( n_1 = n_2 = \cdots = n_k = n \) (note that \( e^{k}_{n,n,\ldots,n} = e^{k}_{n} \)) and using the identity (*) we infer the following corollary, too.

**Corollary 5.** The central Delannoy numbers \( a^{k}_{n} \) are given by the equality

\[
a^{k}_{n} = \frac{1}{2^n} f_k(n, n, \ldots, n) = 2^{(k-1)n} \sum_{i_2, i_3, \ldots, i_k = 0}^{n} \left( \frac{1}{2} \right)^{\sum_{r=2}^{k} i_r} \prod_{r=2}^{k} \binom{n}{i_r} \left( rn - \sum_{s=2}^{r} i_s \right),
\]

where the above iterated sums are equal to \( 1 \) for \( k = 1 \).
This explicit expression of the central Delannoy numbers in an arbitrary dimension constitutes the main contribution of our paper. For \( k = 2 \) and \( k = 3 \), the equality in Corollary 5 becomes

\[
d^2_n = 2^n \sum_{m=0}^{n} \left( -\frac{1}{2} \right)^m \binom{n}{m} \binom{2n-m}{n}
\]

and

\[
d^3_n = 2^{2n} \sum_{u,v=0}^{n} \left( -\frac{1}{2} \right)^{u+v} \binom{n}{u} \binom{n}{v} \binom{2n-u}{n} \binom{3n-u-v}{n}
\]

respectively. We note that an equivalent way of writing \( d^2_n \) is the following

\[
d^2_n = 2^n \sum_{m=0}^{n} \frac{1}{2^m} \binom{n}{m}^2
\]

which seems to be the simplest formula to generate the well-known sequence of central Delannoy numbers in the 2-dimensional case: 1, 3, 13, 63, 321, 1683, 8989, 48639, … and so on.

Finally, we remark that from (4) we also obtain two immediate inequalities verified by the numbers \( d^2_n \):

\[
\left( \frac{2n}{n} \right) \leq d^2_n \leq 2^n \left( \frac{2n}{n} \right)
\]

and

\[
2^{n+1} - 1 \leq d^2_n \leq (2^{n+1} - 1) \left( \left\lfloor \frac{n}{2} \right\rfloor \right)^2
\]

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