Classifying fuzzy subgroups for a class of finite p-groups

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Abstract

In this note we give explicit formulas for the number of distinct fuzzy subgroups of finite p-groups having a cyclic maximal subgroup.

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Key words: fuzzy subgroups, chains of subgroups, maximal subgroups, recurrence relations.

1 Introduction

One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group. This topic has enjoyed a rapid development in the last few years. Several papers have treated the particular case of finite abelian groups. Thus, in [10] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [11], [12], [13] and [23] deal with this number for cyclic groups of order $p^n q^m$ (p, q primes). Also, recall here the paper [20], where a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups: (arbitrary) finite cyclic groups and finite elementary abelian p-groups. The explicit formula obtained for the first class leads in [18] to an expression of the well-known central Delannoy numbers.

The starting point for our discussion in given by the papers [19] and [21], where the above study has been extended to some important classes of finite nonabelian groups G: dihedral groups and hamiltonian groups, respectively. We have used the natural equivalence relation introduced in [20] and we have developed a method to determine the number and nature of fuzzy subgroups of G with respect to this equivalence. For a different approach for classification see [5] and [6]. In our case the corresponding equivalence classes of fuzzy subgroups are closely connected to the chains of subgroups in G. Note that an essential role in solving our counting problem is played again by the Inclusion-Exclusion Principle. It leads us to some recurrence relations, whose solutions have been easily found.

Since the structure of a finite group depends on the structure of its Sylow subgroups, it is natural to continue this study by focusing on finite p-groups. In order to apply the above method to such a group G we need to know the maximal subgroups of G, and therefore we must restrict only to several classes of finite p-groups. One of them consists of the finite p-groups that possess a cyclic maximal subgroup. Describing the fuzzy subgroup structure of these groups is the main goal of the current note.

Most of our notation is standard and will usually not be repeated here. Basic notions and results on lattices (respectively on groups) can be found in [2] (respectively in [16]). For subgroup lattice concepts we refer the reader to [14] and [17].

2 Preliminaries

Let $(G, \cdot, e)$ be a group (where e denotes the identity of G) and $\mathcal{F}(G)$ be the collection of all fuzzy subsets of G. An element μ of $\mathcal{F}(G)$ is said to be a *fuzzy subgroup* of G if it satisfies the following two conditions:

a) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in G$;

b) $\mu(x^{-1}) \geq \mu(x)$, for any $x \in G$.

In this situation we have $\mu(x^{-1}) = \mu(x)$, for any $x \in G$, and $\mu(e) = \sup \mu(G)$. As in the case of subgroups, the set $FL(G)$ consisting of all fuzzy subgroups of G forms a lattice with respect to the usual ordering of fuzzy set inclusion, called the *fuzzy subgroup lattice* of G. For each $\alpha \in [0, 1]$, we define the level subset:

$$_{\mu}G_{\alpha} = \{x \in G \mid \mu(x) \geq \alpha\}.$$

These subsets allow us to characterize the fuzzy subgroups of G, in the next manner: μ is a fuzzy subgroup of G if and only if its level subsets are subgroups in G. This well-known theorem gives a link between $FL(G)$ and the classical subgroup lattice $L(G)$ of G.

The fuzzy subgroups of G can be classified up to some natural equivalence relations on $\mathcal{F}(G)$. One of them (used in [20] and [23], too) is defined by

$$\mu \sim \eta \text{ iff } (\mu(x) > \mu(y) \Longleftrightarrow \eta(x) > \eta(y), \text{ for all } x, y \in G)$$

and two fuzzy subgroups μ, η of G will be called *distinct* if $\mu \not\sim \eta$. This equivalence relation generalizes that used in Murali's papers [9]-[13]. It is also closely connected to the concept of level subgroup. In this way, suppose that the group G is finite and let $\mu : G \to [0,1]$ be a fuzzy subgroup of G. Put $\mu(G) = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ and assume that $\alpha_1 > \alpha_2 > ... > \alpha_r$. Then μ determines the following chain of subgroups of G which ends in G:

(1) $$_{\mu}G_{\alpha_1} \subset {}_{\mu}G_{\alpha_2} \subset ... \subset {}_{\mu}G_{\alpha_r} = G.$$

Moreover, for any $x \in G$ and $i = \overline{1, r}$, we have

$$\mu(x) = \alpha_i \Longleftrightarrow i = \max\{j \mid x \in {}_{\mu}G_{\alpha_j}\} \Longleftrightarrow x \in {}_{\mu}G_{\alpha_i} \setminus {}_{\mu}G_{\alpha_{i-1}},$$

where, by convention, we set $_{\mu}G_{\alpha_0} = \emptyset$. A necessary and sufficient condition for two fuzzy subgroups μ, η of G to be equivalent with respect to $\sim$ has been identified in [23]: $\mu \sim \eta$ if and only if μ and η have the same set of level subgroups, that is they determine the same chain of subgroups of type (1). This result shows that *there exists a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G which end in G*. So, the problem of counting all distinct fuzzy subgroups of G can be translated into a combinatorial problem on the subgroup lattice $L(G)$ of G: finding the number of all chains of subgroups of G that terminate in G. Even for some particular classes of finite groups, as finite abelian groups, this problem is very difficult. The largest class of groups for which it was completely solved is constituted by finite cyclic groups (see Corollary 4 of [20]). If G is a finite cyclic group of order n (that is $G \cong \mathbb{Z}_n$) and $n = p_1^{m_1} p_2^{m_2} ... p_s^{m_s}$ is the decomposition of n as a product of prime factors, then the number $h(n)$ (which will be also denoted by $h(\mathbb{Z}_n)$) of all distinct fuzzy subgroups of G is given by the equality

(2)
$$h(n) = 2^{\sum\limits_{\alpha=1}^{s} m_\alpha} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{m_3} ... \sum_{i_s=0}^{m_s} \left(-\frac{1}{2}\right)^{\sum\limits_{\alpha=2}^{s} i_\alpha} \prod_{\alpha=2}^{s} \binom{m_\alpha}{i_\alpha} \binom{m_1 + \sum\limits_{\beta=2}^{\alpha} (m_\beta - i_\beta)}{m_\alpha},$$

where the above iterated sums are equal to 1 for $s = 1$. Mention that an important step in order to establish a similar explicit formula for finite elementary abelian p-groups is made in Section 3 of [20].

In the following we describe the method that will be used in counting the chains of subgroups of an arbitrary finite group G. Let $M_1, M_2, ..., M_k$ be the maximal subgroups of G and denote by $h(G)$ the number of chains of subgroups of G ended in G. The technique to compute $h(G)$ is founded on applying the Inclusion-Exclusion Principle. Let $\mathcal{C}$ be the set of chains in G of type

$$H_1 \subset H_2 \subset ... \subset H_r = G,$$

$\mathcal{C}'$ be the set of chains in G of type

$$H_1 \subset H_2 \subset ... \subset H_r \neq G$$

and $\mathcal{C}_i$ be the set of chains of $\mathcal{C}'$ which are contained in M_i, $i = \overline{1,k}$. Then

$$|\mathcal{C}| = 1 + |\mathcal{C}'| = 1 + \Big|\bigcup_{i=1}^{k} \mathcal{C}_i\Big| =$$

$$= 1 + \sum_{i=1}^{k} |\mathcal{C}_i| - \sum_{1 \le i_1 < i_2 \le k} |\mathcal{C}_{i_1} \cap \mathcal{C}_{i_2}| + ... + (-1)^{k-1} \Big|\bigcap_{i=1}^{k} \mathcal{C}_i\Big|.$$

Clearly, for every $1 \le \ell \le k$ and $1 \le i_1 < i_2 < ... < i_\ell \le k$, the set $\bigcap_{j=1}^{\ell} \mathcal{C}_{i_j}$ consists of all chains of $\mathcal{C}'$ which are included in $\bigcap_{j=1}^{\ell} M_{i_j}$. This shows that

$$\Big|\bigcap_{j=1}^{\ell} \mathcal{C}_{i_j}\Big| = 2h\Big(\bigcap_{j=1}^{\ell} M_{i_j}\Big) - 1$$

and therefore

$$|\mathcal{C}| = 1 + \sum_{i=1}^{k} (2h(M_i) - 1) - \sum_{1 \le i_1 < i_2 \le k} (2h(M_{i_1} \cap M_{i_2}) - 1) + ... + (-1)^{k-1} (2h\Big(\bigcap_{i=1}^{k} M_i\Big) - 1) =$$

$$= 2\left(\sum_{i=1}^{k} h(M_i) - \sum_{1 \le i_1 < i_2 \le k} h(M_{i_1} \cap M_{i_2}) + ... + (-1)^{k-1} h\Big(\bigcap_{i=1}^{k} M_i\Big)\right) + c,$$

where

$$c = 1 + \sum_{i=1}^{k}(-1) - \sum_{1\le i_1<i_2\le k}(-1) + ... + (-1)^{k-1}(-1) = (1-1)^k = 0.$$

Hence $h(G)$ $(= |\mathcal{C}|)$ is given by the following equality:

$$(3) \quad h(G) = 2\left(\sum_{i=1}^{k} h(M_i) - \sum_{1\le i_1<i_2\le k} h(M_{i_1}\cap M_{i_2}) + ... + (-1)^{k-1} h\Big(\bigcap_{i=1}^{k} M_i\Big)\right).$$

This has been used in [19] to obtain explicit formulas of $h(D_{2n})$ for some positive integers n, where D_{2n} denotes the dihedral group of order $2n$. Recall only that if $n = p^m$, then

$$(4) \qquad h\left(D_{2n}\right) = \frac{2^m}{p-1}\,\left(p^{m+1}+p-2\right).$$

We conclude that if the maximal subgroup structure of a finite group G is known (i.e. we know the number of maximal subgroups of G, their types and their intersections), then from (3) some recurrence relations can be inferred which permit us to determine explicitly $h(G)$.

3 The number of fuzzy subgroups of finite p-groups having a cyclic maximal subgroup

Let p be a prime, $n \ge 3$ be an integer and denote by $\mathcal{G}$ the class consisting of all finite p-groups of order p^n possessing a maximal subgroup which is cyclic. Clearly, $\mathcal{G}$ contains the abelian p-groups $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$. By Theorem 4.1 of [9], II, the nonabelian p-groups that belong to $\mathcal{G}$ are:

– $M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1,\ y^{-1}xy = x^{p^{n-2}+1}\rangle$,

when p is odd, and the next groups

– $M(2^n)$ $(n \ge 4)$,

– the dihedral group

$$D_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ yxy^{-1} = x^{-1}\rangle,$$

– the generalized quaternion group

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1,\ yxy^{-1} = x^{-1} \rangle,$$

– the quasi-dihedral group

$$S_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ y^{-1}xy = x^{2^{n-2}-1} \rangle \ (n \geq 4),$$

when $p = 2$.

In the following we will determine explicitly the number of distinct fuzzy subgroups of all groups contained in $\mathcal{G}$.

We observe first that $h(\mathbb{Z}_{p^n})$ can be directly computed by (2):

$$h(\mathbb{Z}_{p^n}) = h(p^n) = 2^n.$$

On the other hand, for every noncyclic p-group G in $\mathcal{G}$ the Frattini subgroup $\Phi(G)$ is cyclic of order p^{n-2}. Therefore G has $p+1$ maximal subgroups and all intersections of at lest two maximal subgroups of G are equal to $\Phi(G)$. This permit us to write the equality (3) in a more simple way, namely

$$h(G) = 2\left[\sum_{i=1}^{p+1} h(M_i) + \left(-\binom{p+1}{2} + \binom{p+1}{3} - \ldots + (-1)^p \binom{p+1}{p+1}\right) h(\Phi(G))\right] =$$

$$= 2\sum_{i=1}^{p+1} h(M_i) - 2p\, h(\Phi(G)).$$

Since

$$h(\Phi(G)) = h(\mathbb{Z}_{p^{n-2}}) = h(p^{n-2}) = 2^{n-2},$$

one obtains the following result.

Theorem 1. *Let G be a finite noncyclic p-group of order p^n having a cyclic maximal subgroup. Then*

$$h(G) = 2\sum_{i=1}^{p+1} h(M_i) - p \cdot 2^{n-1}, \tag{5}$$

where M_1, M_2, ..., M_{p+1} are the maximal subgroups of G.

Theorem 1 shows that the computation of the number of distinct fuzzy subgroups of a finite noncyclic p-group G that possesses a cyclic maximal subgroup is reduced to the computation of this number for the maximal subgroups of G. So, in the following we will focus on describing the maximal subgroups of all noncyclic groups in $\mathcal{G}$.

It is well-known that $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$ has one maximal subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$ and p cyclic maximal subgroups. By (5), it follows that

$$h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = 2\left(h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) + p \cdot h(\mathbb{Z}_{p^{n-1}})\right) - p \cdot 2^{n-1} =$$

$$= 2 \cdot h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) + p \cdot 2^{n-1}.$$

This recurrence relation easily leads to

$$h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = 2^{n-1}\left[(n-1)p + 2\right].$$

$M(p^n)$ has a similar maximal subgroup structure, more precisely its maximal subgroups are $\langle x^p, y\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$ and the cyclic subgroups $\langle x\rangle$, $\langle xy\rangle$, ..., $\langle x^{p-1}y\rangle$. Then

$$h(M(p^n)) = h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = 2^{n-1}\left[(n-1)p + 2\right]. \tag{6}$$

This equality leads us to the following remark.

Remark. There exist large classes of finite nonisomorphic groups with the same fuzzy subgroup structure. More exactly, the number of distinct fuzzy subgroups of a finite group G depends essentially on the structure of the subgroup lattice of G and not on the isomorphism class of G.

Next we will investigate the 2-groups contained in $\mathcal{G}$. Obviously, by putting $p = 2$ in (6), one obtains

$$h(M(2^n)) = n \cdot 2^n.$$

For D_{2^n} the number of distinct fuzzy subgroups can be directly computed from (4), namely

$$h(D_{2^n}) = 2^{2n-1},$$

while for Q_{2^n} and S_{2^n} we must look again on their maximal subgroup structure.

It is easy to cheek that Q_{2^n} has 3 maximal subgroups, two isomorphic with $Q_{2^{n-1}}$ and one isomorphic with $\mathbb{Z}_{2^{n-1}}$. In this way, (5) becomes

$$h(Q_{2^n}) = 2\,(2 \cdot h(Q_{2^{n-1}}) + h(\mathbb{Z}_{2^{n-1}})) - 2^n =$$

$$= 4 \cdot h(Q_{2^{n-1}}),$$

implying that

$$h(Q_{2^n}) = 2^{2n-2}.$$

S_{2^n} has also 3 maximal subgroups of type $\mathbb{Z}_{2^{n-1}}$, $D_{2^{n-1}}$ and $Q_{2^{n-1}}$. It results

$$\begin{aligned} h(S_{2^n}) &= 2\,(h(\mathbb{Z}_{2^{n-1}}) + h(D_{2^{n-1}}) + h(Q_{2^{n-1}})) - 2^n = \\ &= 2\left(2^{n-1} + 2^{2n-3} + 2^{2n-4}\right) - 2^n = \\ &= 3 \cdot 2^{2n-3}. \end{aligned}$$

Hence the following theorem holds.

Theorem 2. *The number of distinct fuzzy subgroups of a finite p-group of order p^n having a cyclic maximal subgroup is:*

a) $h(\mathbb{Z}_{p^n}) = 2^n$;

b) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M(p^n)) = 2^{n-1}\,[(n-1)p + 2]$;

c) $h(D_{2^n}) = 2^{2n-1}$;

d) $h(Q_{2^n}) = 2^{2n-2}$;

e) $h(S_{2^n}) = 3 \cdot 2^{2n-3}$.

We end this note by indicating two open problems with respect to the above results.

Problem 1. Determine the number of distinct fuzzy subgroups for other remarkable classes of finite p-groups.

Problem 2. Extend the problem of classifying fuzzy subgroups from finite p-groups to finite nilpotent groups, according to the fact that a finite nilpotent group can (uniquely) be written as a direct product of p-groups.

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