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Two classes of finite groups whose Chermak-Delgado lattice is a chain of length zero

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ABSTRACT
It is an open question in the study of Chermak-Delgado lattices precisely which finite groups \(G\) have the property that \(\text{CD}(G)\) is a chain of length 0. In this note, we determine two classes of groups with this property. We prove that if \(G = AB\) is a finite group, where \(A\) and \(B\) are abelian subgroups of relatively prime orders with \(A\) normal in \(G\), then the Chermak-Delgado lattice of \(G\) equals \(\{AC_B(A)\}\), a strengthening of earlier known results.

1. Introduction
Throughout this paper, \(G\) will denote a finite group. The relation between the structure of a group and the structure of its lattice of subgroups constitutes an important domain of research in group theory. The topic has enjoyed a rapid development starting with the first half of the 20\(^{th}\) century. Many classes of groups determined by different properties of partially ordered subsets of their subgroups (especially lattices of (normal) subgroups) have been identified. We refer to Schmidt’s book [11] for more information about this theory.

An important sublattice of the subgroup lattice \(L(G)\) of a finite group \(G\) is defined as follows. Denote by

\[ m_G(H) = |H||G_G(H)| \]

the Chermak-Delgado measure of a subgroup \(H\) of \(G\) and let

\[ m(G) = \max\{m_G(H) \mid H \leq G\} \quad \text{and} \quad \text{CD}(G) = \{H \leq G \mid m_G(H) = m(G)\}. \]

Then the set \(\text{CD}(G)\) forms a modular self-dual sublattice of \(L(G)\), which is called the Chermak-Delgado lattice of \(G\). It was first introduced by Chermak and Delgado [5] and revisited by Isaacs [9]. In the last years, there has been a growing interest in understanding this lattice (see, e.g., [1–3, 6, 10, 14]).

Most current study has focused on \(p\)-groups. Groups whose Chermak-Delgado lattice is a chain or a quasi-antichain have been studied. It is easy to see that if \(G\) is abelian, then \(\text{CD}(G) = \{G\}\) is a chain of length 0. An open question is to classify those groups \(G\) for which \(\text{CD}(G)\) is a chain of length 0. In this paper, we determine two classes of groups which have this property. Section 1 concerns semidirect products of abelian groups of coprime orders, and Section 2 concerns Frobenius groups.
Recall two important properties of the Chermak-Delgado lattice that will be used in our paper:

- if $H \in \mathcal{CD}(G)$, then $C_G(H) \in \mathcal{CD}(G)$ and $C_G(C_G(H)) = H$;
- the minimum subgroup $M(G)$ of $\mathcal{CD}(G)$ (called the Chermak-Delgado subgroup of $G$) is characteristic, abelian, and contains $Z(G)$.

### 2. Semidirect products of Abelian groups of coprime orders

The following appears in [8].

**Theorem 1.** Let $N$ be nilpotent and suppose that it acts faithfully on $H$, where $(|N|, |H|) = 1$. Let $\pi$ be a set of prime numbers containing all primes for which the Sylow subgroup of $N$ is nonabelian. Then there exists an element $x \in H$ such that $|C_N(x)|$ is a $\pi$-number, and if $\pi$ is nonempty, $x$ can be chosen, so that $|C_N(x)| \leq (|N|/p)^{1/p}$, where $p$ is the smallest member of $\pi$.

Taking $\pi = \emptyset$ we obtain:

**Corollary 2.** Let $N$ be abelian and suppose that it acts faithfully on $H$, where $(|N|, |H|) = 1$. Then there exists an element $x \in H$ such that $C_N(x) = 1$.

We note that Corollary 2 has an elementary proof using a variation on Brodkey’s Theorem, see [4].

**Theorem 3.** Let $G$ be a finite group which can be written as $G = AB$, where $A$ and $B$ are abelian subgroups of relatively prime orders and $A$ is normal. Then

$$m(G) = |A|^2|C_B(A)|^2 \quad \text{and} \quad \mathcal{CD}(G) = \{AC_B(A)\}.$$  

**Proof.** Let $M = M(G)$. Since $M$ is abelian and $M \cap B$ is a Hall $\pi$-subgroup of $M$, we have that $M \cap B$ is characteristic in $M$. And since $M$ is a characteristic subgroup of $G$, we have that $M \cap B$ is a characteristic subgroup of $G$. Thus $M \cap B$ centralizes $A \triangleleft G$, and so $M \cap B \leq C_B(A)$. Since $Z(G) \leq M$, we have that $C_B(A) = Z(G) \cap B \leq M \cap B$. Thus $M \cap B = C_B(A)$.

And so $M = UC_B(A)$ for some $U \leq A$. It suffices to show that $U = A$, as $AC_B(A)$ being self-centralizing would imply that $\mathcal{CD}(G) = \{AC_B(A)\}$.

Let $K$ be the kernel of the action of $C_B(U)$ on $A/U$. Suppose $p$ is a prime divisor of $|K|$, and let $P$ be a Sylow $p$-subgroup of $K$. Let $a \in A$ be arbitrary. Consider the action of $P$ on elements of the coset $aU$. Note that either $P$ centralizes $aU$, or every element of $aU$ has a nontrivial orbit under $P$. This is true because $P \leq C_B(U)$. But in the latter case, being that $P$ is a $p$-group, this would imply that $p$ divides $|aU| = |U|$ which is coprime to $p$. Thus, it must be the former case that $P$ centralizes $aU$. And since $a \in A$ was arbitrary, and since $A$ is the union of its $U$-cosets, we have that $P$ centralizes $A$. Now, $p$ was arbitrary, and $K$, being abelian, is a direct product of its Sylow subgroups, and it follows that $K$ centralizes $A$. Thus $K = C_B(A)$. And so $C_B(U)/C_B(A)$ acts faithfully on $A/U$.

By Corollary 2, an element of $A/U$ has a regular orbit under the action of $C_B(U)/C_B(A)$. And so

$$|A : U| \geq |C_B(U) : C_B(A)|.$$  

This leads to $m_G(AC_B(A)) \geq m_G(M)$ and thus $m_G(AC_B(A)) = m_G(M)$ by the maximality of $m_G(M)$. Then

$$|U||C_B(U)| = |A||C_B(A)|,$$

implying that

$$|U| = |A| \quad \text{and} \quad |C_B(U)| = |C_B(A)|$$  

because $A$ and $B$ are of coprime order. Hence $U = A$, as desired. \qed
Recall that a ZM-group is a finite group with all Sylow subgroups cyclic. Zassenhaus classified such groups (see [7, 15] for a good exposition): they are of type

\[ \text{ZM}(m, n, r) = \langle a, b \mid a^m = b^n = 1, \ b^{-1}ab = a^r \rangle, \]

where the triple \((m, n, r)\) satisfies the conditions

\[ \gcd(m, n) = \gcd(m, r - 1) = 1 \quad \text{and} \quad r^n \equiv 1 \pmod{m}. \]

Note that \(|\text{ZM}(m, n, r)| = mn\) and \(Z(\text{ZM}(m, n, r)) = \langle b^d \rangle\), where \(d\) is the multiplicative order of \(r\) modulo \(m\), i.e. \(d = \min(k \in \mathbb{N}^* \mid r^k \equiv 1 \pmod{m})\). The structure of \(CD(\text{ZM}(m, n, r))\) has been determined in the main theorem of [12] and can also be obtained from Theorem 3, taking \(G = \text{ZM}(m, n, r), A = \langle a \rangle,\) and \(B = \langle b \rangle\). Thus, Theorem 3 generalizes [12].

**Corollary 4.** Let \(G = \text{ZM}(m, n, r)\) be a ZM-group. Then

\[ m(G) = \frac{m^2n^2}{d^2} \quad \text{and} \quad CD(G) = \{\langle a, b^d \rangle\}. \]

There are many other classes of groups which fall under the umbrella of Theorem 3. A natural one occurs as follows. Let \(V\) be a vector space and let \(H\) be an abelian subgroup of \(Aut(V)\), so that \(H\) and \(V\) have coprime orders, and let \(G = V \rtimes H\) be the semidirect product. Then, by Theorem 3, we have that \(CD(G) = \{V\}\).

### 3. Frobenius groups

If \(A\) acts on a group \(N\) we say that the action is Frobenius if \(n^a \neq n\) whenever \(n \in N\) and \(a \in A\) are nonidentity elements. The following results on Frobenius actions can be found in [9].

**Proposition 5.** Let \(N\) be a normal subgroup of a finite group \(G\), and suppose that \(A\) is a complement for \(N\) in \(G\). The following are then equivalent.
1. The conjugation action of \(A\) on \(N\) is Frobenius.
2. \(A \cap A^g = 1\) for all elements \(g \in G \setminus A\).
3. \(C_G(a) \leq A\) for all nonidentity elements \(a \in A\).
4. \(C_G(n) \leq N\) for all nonidentity elements \(n \in N\).

If both \(N\) and \(A\) are nontrivial in the situation of Proposition 5, we say that \(G\) is a Frobenius group and that \(A\) and \(N\) are, respectively, the Frobenius complement and the Frobenius kernel of \(G\).

In 1959, Thompson proved that a Frobenius kernel is nilpotent, see [13].

**Theorem 6.** Suppose \(G = NA\) is a Frobenius group with kernel \(N\) and complement \(A\). Then \(CD(G) = CD(N)\).

**Proof.** First we argue that 1 and \(G\) are not in \(CD(G)\). Since \(G\) is Frobenius, we have \(Z(G) = 1\). On the other hand, since \(|N| \equiv 1 \pmod{|A|}\) and \(N\) is nontrivial, we have \(|N| > |A|\). By Thompson’s theorem, Frobenius kernels are nilpotent, and so \(Z(N) > 1\). Now the action of \(A\) on \(Z(N)\) is also Frobenius, therefore \(|Z(N)| \equiv 1 \pmod{|A|}\). This implies \(|Z(N)| > |A|\), and thus

\[ m_G(N) = |N||Z(N)| > |N||A| = |G| = m_G(G) = m_G(1). \]

Consequently, 1 and \(G\) are not in \(CD(G)\).

Let \(T\) be the greatest element of \(CD(G)\). If \(T \cap N = 1\), then since \(T \trianglelefteq G\), we have that \(T \leq C_G(N) \leq N\). So \(T = 1\), a contradiction. Therefore we have \(T \cap N > 1\). Let \(1 \neq n \in T \cap N\).
Then \( C_G(T) \leq C_G(T \cap N) \leq C_G(n) \leq N \). Now \( C_G(T) \in CD(G) \), implying that \( C_G(T) \neq 1 \). Thus, 
\[ T = C_G(G(T)) \leq N \text{.} \] Hence \( CD(G) = CD(T) = CD(N) \).

By Theorem 6, we infer that the class of Frobenius groups with abelian kernel provides another class of groups \( G \) for which \( CD(G) \) is a chain of length 0.

**Proposition 7.** Let \( P \) be a \( p \)-group which possesses an abelian subgroup \( A \), so that \(|P : A| = p\). Then \( CD(P) = \{A\} \) if and only if \(|P : Z(P)| > p^2\).

**Proof.** Let \(|P| = p^n\). Then \(|A| = p^{n-1}\). Suppose \( CD(P) = \{A\} \). \( P \) is non-abelian as otherwise \( CD(P) = \{P\} \). Note that \( C_P(A) = A \). This is true because otherwise \( C_P(A) = P \) implies that \( A = Z(P) \) and then \( P/Z(P) \) is cyclic, i.e. \( P \) is abelian, a contradiction. Thus, \( m_P(A) = |A|^2 = p^{2n-2} \). Now \( P \) is not in \( CD(P) \), and so \(|P|Z(P)| < m_P(A) = p^{2n-2} \). This implies that \(|P : Z(P)| > p^2\).

Conversely, suppose that \(|P : Z(P)| > p^2\). Then \( P \) is non-abelian, and as before, we conclude that \( C_P(A) = A \). Since \(|P : Z(P)| > p^2\), it follows that \(|P|Z(P)| < p^{2n-2} \), and so \( m_P(P) < p^{2n-2} = m_P(A) \), i.e. \( P \) is not in \( CD(P) \). Let \( T \) be the largest member of \( CD(P) \). If \( T \) were to have measure larger than \( A \), then \( T \) must have measure \( p^{2n-1} \), and it follows that \(|T| = p^{n-1} \) and \( C_P(T) = P \), but as before, this would imply that \( P \) is abelian, a contradiction. Thus, \( T \) has the same measure as \( A \). Since \( A \) is maximal in \( P \), we obtain \( T = A \). But \( A \) is abelian and self-centralizing, therefore it is also the least member of \( CD(P) \). Consequently, \( CD(P) = \{A\} \).

We end with an example of a Frobenius group, \( G \), with a non-abelian kernel, so that \( CD(G) \) is a chain of length 0.

**Example.** Let

\[
P = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in GF(7^2), c \in GF(7) \right\},
\]

where \( GF(7) \) is seen as a subgroup of \( GF(7^2) \). Then \( P \) is a non-abelian group of order \( 7^5 \). Clearly, its center

\[
Z(P) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \mid b \in GF(7^2) \right\}
\]

is of order \( 7^2 \) and so \(|P : Z(P)| = 7^3 > 7^2\). We remark that \( P \) has an abelian subgroup

\[
A = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in GF(7^2) \right\}
\]

of index 7 in \( P \). By Proposition 7, we have that \( CD(P) = \{A\} \).

Define an automorphism \( x \) of \( P \) such that

\[
\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^x = \begin{pmatrix} 1 & 2a & 4b \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{pmatrix}.
\]

Then \( x \) is fixed-point free of order 3 and so the semidirect product \( G = P \rtimes \langle x \rangle \) is a Frobenius group of order \( 3 \cdot 7^5 \) with Frobenius kernel \( P \), and by Theorem 6 we have

\[ CD(G) = CD(P) = \{A\}, \]

as desired.
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