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SUB-WEYL GEOMETRY AND ITS LINEAR CONNECTIONS

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The aim of this paper is to study from the point of view of linear connections the data $(M, \mathcal{D}, g, W)$ with M a smooth $(n+p)$-dimensional real manifold, $(\mathcal{D}, g)$ a n-dimensional semi-Riemannian distribution on M, $\mathcal{G}$ the conformal structure generated by g and W a Weyl substructure: a map $W : \mathcal{G} \to \Omega^1(M)$ such that $W(\overline{g}) = W(g) - du$, $\overline{g} = e^u g; u \in C^\infty(M)$. Compatible linear connections are introduced as a natural extension of similar notions from Weyl geometry and such a connection is unique if a symmetry condition is imposed. In the foliated case the local expression of this unique connection is obtained. The notion of Vranceanu connection is introduced for a pair (Weyl structure, distribution) and it is computed for the tangent bundle of Finsler spaces, particularly Riemannian, choosing as distribution the vertical bundle of tangent bundle projection and as one-form the Cartan form.

Keywords: Weyl substructure; compatible linear connection; Vranceanu connection; foliation; adapted frame; Finsler space.

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0. Introduction

After Einstein's approach to gravitation [9], several others theories have been developed as part of the efforts to cure problems arising when the gravitational field is coupled to matter fields. Thus, as soon as Einstein presented the General Relativity, Weyl [27, 28] proposed a new geometry in which a new scalar field accompanies the metric field and changes the scale of length measurements. The aim was to unify gravitation and electromagnetism, but this theory was briefly refuted by Einstein because the non-metricity had direct consequences over the spectral lines of elements which has never been observed, [19]. Now, the Weyl theory implies a semi-Riemannian manifold equipped with a conformal structure and a torsion-free linear connection which preserves the conformal structure. Its actual relationship with modern theories of Gravitation was pointed out in [10].

More precisely, let $\mathcal{G}$ be a conformal structure on the smooth manifold M_n i.e. an equivalence class of Riemannian metrics: $g \sim \overline{g}$ if there exists a smooth function $f \in C^\infty(M)$ such that $\overline{g} = e^f g$. Denoting by $\Omega^1(M)$ the $C^\infty(M)$-module of one-forms on M, a (Riemannian) Weyl structure is a map $W : \mathcal{G} \to \Omega^1(M)$ such that $W(\overline{g}) = W(g) - df$. In [11, 20] it is proved that for a Weyl manifold $(M, \mathcal{G}, W)$ there exists a unique torsion-free linear connection ∇ on M such that for every $g \in \mathcal{G}$ [26]:

$$\nabla g + W(g) \otimes g = 0 \tag{1}$$

called *Weyl connection*. The parallel transport induced by ∇ preserves the given conformal class $\mathcal{G}$. Also, the above theory can be expressed in terms of G-structures with G the conformal group $CO(n) = O(n) \times \mathbb{R}^+$.

The literature on Weyl structures is huge and the increasing interest in it is motivated in the last years by a new relationship with physics and gauge theory through the notion of *Weyl–Einstein manifold*, [12, 18]. Also, some interesting extensions of Weyl structures inspired by generalizations of Riemannian metrics have appeared: for Finsler metrics in [1, 14, 15] while for generalized Lagrange metrics in [8]. The infinite-dimensional case was treated in [3].

In this paper we propose another extension of Weyl structures and compatible connections (1) namely in the semi-Riemannian distributions framework. So, the first section is devoted to the proposed generalization and exactly as in the semi-Riemannian geometry the uniqueness of the compatible connection is obtained provided a symmetry condition holds. Also, the compatibility condition is rewritten in terms of quasi-connections. The second section deals with the foliated case through the local expression in an adapted frame and a new characterization of bundle-like metrics is obtained in terms of Weyl structures.

One concept used by many geometers in non-holonomic geometry is the *Vranceanu connection*. Locally, it was introduced in 1931 by Vranceanu [25]. In an invariant form, the Vranceanu connection was treated by Ianus in [13]. This notion knows a new approach in the last years especially in connection with Finsler geometry [5].

For Weyl structures on a manifold endowed with a distribution we introduce the notion of Vranceanu connection following a similar tool from the geometry of a pair (Riemannian manifold, distribution). This way, we obtain a generalization for some notions, results and relations of [4]. The global expression of this connection appears in the first section, while the local coefficients are given again for the foliated manifolds in the second part of the paper. Let us point out that we treat this connection considering global Weyl structures as examples of our theory.

We devote the third section to the Vranceanu linear connection for the tangent bundle of Finsler, particularly Riemannian spaces when we call it *Vranceanu–Cartan*, choosing as distribution the vertical bundle of tangent bundle projection and as one-form the Cartan form. The motivation for this name consists in the fact that Vranceanu in 1926 [24] and Cartan in 1928 [7] were the first who proposed a geometrization of non-holonomic mechanics. As final problems, the flatness of the Vranceanu–Cartan connection and the covariant derivative of the Liouville vector fields with respect to the Vranceanu connection are discussed.

At the end of these remarks let us point out that our work can also be considered as proposing a generalization of the sub-Riemannian geometry, [17]. So, we address a new theory, namely *sub-Weyl theory*, which seems to belong also to the paper [29]. This explains the present title.

1. Weyl Substructures and Compatible Connections

For a real manifold M we use the following notations: $C^\infty(M)$ is the ring of smooth real functions, $\chi(M)$ is the $C^\infty(M)$-module of vector fields on M.

Let M be a smooth $(n+p)$-dimensional real manifold and $\mathcal{D}$ an n-dimensional distribution on M. Suppose g is a semi-Riemannian metric on $\mathcal{D}$, that is, in the words of [4, p. 23], $(\mathcal{D}, g)$ is a *semi-Riemannian distribution* on M. Let $\mathcal{G} = \{\overline{g} = e^u g; u \in C^\infty(M)\}$ be the conformal structure generated by g.

Definition 1.1. A *Weyl substructure* is a map $W : \mathcal{G} \to \Omega^1(M)$ such that:

$$W(\overline{g}) = W(g) - du. \tag{2}$$

The data $(M, \mathcal{D}, g, W)$ will be called a *sub-Weyl manifold*.

Let us point out that a straightforward computation gives:

$$W(e^v \overline{g}) = W(\overline{g}) - dv.$$

It follows that if for some $g \in \mathcal{G}$ the one-form $W(g)$ is closed (or exact) then for every $\overline{g} \in \mathcal{G}$ the one-form $W(\overline{g})$ is closed (or exact).

We want a linear connection on $\mathcal{D}$ whose properties are similar to those of Weyl connection on a Riemannian manifold. To this end we consider a complementary distribution $\mathcal{D}'$ to $\mathcal{D}$ in TM:

$$TM = \mathcal{D} \oplus \mathcal{D}'. \tag{3}$$

Since M is supposed to be paracompact there exists such a distribution. Let Q and Q' be the corresponding projectors of this decomposition. Recall that a linear connection ∇ on $\mathcal{D}$ is said to be $\mathcal{D}'$*-torsion-free* if its $\mathcal{D}'$-torsion field vanishes i.e. [4, p. 23]:

$$\nabla_X QY = \nabla_{QY} QX + Q[X, QY], \quad \forall X, Y \in \mathcal{X}(M). \tag{4}$$

Definition 1.2. ∇ is *compatible* to the Weyl substructure if:

$$\nabla_{QX} g + W(g)(QX) \cdot g = 0, \quad \forall X \in \mathcal{X}(M). \tag{5}$$

Again it results that this relation has a geometrical meaning since:

$$\nabla_{QX}\overline{g} + W(\overline{g})(QX)\overline{g} = e^u(\nabla_{QX}g + W(g)(QX)g), \quad \forall X \in \mathcal{X}(M).$$

The aim of this section is to obtain a generalization of the results from Sec. 0.

Theorem 1.3. *Given a sub-Weyl manifold with a complementary distribution $\mathcal{D}'$ there exists an unique compatible linear connection ∇ on $\mathcal{D}$ such that ∇ is $\mathcal{D}'$-torsion-free.*

Proof. If such a connexion ∇ exists, from the compatibility conditions with the Weyl structures and the hypothesis that the $\mathcal{D}'$-torsion tensor field is vanishing, it results that:

$$\begin{aligned} 2g(\nabla_{QX}QY, QZ) = {} & QX(g(QY,QZ)) + QY(g(QZ,QX)) \\ & - QZ(g(QX,QY)) + g(Q[QX,QY],QZ) \\ & - g(Q[QY,QZ],QX) + g(Q[QZ,QX],QY) \\ & + W(g)(QX)g(QY,QZ) + W(g)(QY)g(QZ,QX) \\ & - W(g)(QZ)g(QX,QY). \end{aligned} \tag{6}$$

We applied the usual method for determining the Levi-Civita connection on a semi-Riemannian manifold. Because ∇ is $\mathcal{D}'$-torsion-free it follows that:

$$\nabla_{Q'X}QY = Q[Q'X, QY]. \tag{7}$$

The expressions (6) and (7) shows that ∇ is uniquely determined from the metric g and the hypothesis of being $\mathcal{D}'$-torsion free.

Hence, if it exists, it will be unique. To prove existence we just define ∇ by (6) and (7) and it satisfies the desired conditions. □

Remarks 1.4. (i) For the particular case $p = 0$ the result of [11, 20] from the Introduction is recovered.

(ii) In [6, p. 210] the notion of *constrained affine connection* is introduced. For our framework it is:

$$\bar{\nabla}_X QY = \nabla_X QY + (\nabla_X Q')(QX). \tag{8}$$

But it results immediately that $\bar{\nabla} = \nabla$ and then our ∇ can be an useful tool in a conformal non-holonomic mechanics as $\bar{\nabla}$ in the usual non-holonomic theory, see [6].

Example 1.5. Let (M, g, W) be a Weyl manifold [11] i.e. g is a global semi-Riemannian metric on M and W is a map with the property (1). It results the Weyl connection $\tilde{\nabla}$ given by (6) without Q which is a symmetric, compatible linear connection. Supposing that the given distribution $\mathcal{D}$ is semi-Riemannian with respect to $g|_{\mathcal{D}}$ then it has an orthogonal complementary distribution $\mathcal{D}^{\perp}$ with the corresponding projector $Q^\perp$. Therefore we get two Weyl substructures $(g_{|\mathcal{D}}, W)$ and $(g_{|\mathcal{D}^\perp}, W)$ with corresponding Weyl connections ∇ and $\nabla^\perp$. Using the terminology of [4, p. 96] let call ∇ *the intrinsic Weyl connection of* $\mathcal{D}$ and $\nabla^\perp$ *the transversal Weyl connection of* $\mathcal{D}$. Using the formula (2.4) from [4, p. 7] it results a linear connection ∇^* on M:

$$\nabla^*_X Y = \nabla_X QY + \nabla^\perp_X Q^\perp Y, \tag{9}$$

for $X, Y \in \chi(M)$. This linear connection is *adapted* to $\mathcal{D}$ i.e. for any $X \in \chi(M)$ and $U \in \Gamma(\mathcal{D})$ we have $\nabla^*_X U \in \Gamma(\mathcal{D})$, [4, p. 7]. The above formulae yield:

$$\nabla^*_X Y = Q\tilde{\nabla}_Q XQY + Q^\perp \tilde{\nabla}^\perp_Q XQ^\perp Y + Q[Q^\perp X, QY] + Q^\perp[QX, Q^\perp Y], \tag{10}$$

which compared with relation (3.16) from [4, p. 17] gives a similar result to Theorem 5.3. of [4, p. 26] namely that ∇^* is just *the Vranceanu connection* defined by the Weyl connection $\tilde{\nabla}$.

There are several features of the Vranceanu connection which makes it important:

- it is defined on the sections of whole TM not only of $\mathcal{D}$;
- if $\mathcal{D}$ is the tangent distribution of a foliation $\mathcal{F}$ (this case will be studied in the next section) then ∇^* is symmetric (torsion-free) if and only if the distribution $\mathcal{D}^\perp$ is integrable, Theorem 1.5. of [4, p. 100];
- if ∇^* is symmetric then the almost product structure $P = Q - Q^\perp$, naturally associated to the decomposition (3), is integrable which means that the Nijenhuis tensor field N_P vanishes.

Let us end this section with another form of the compatibility condition, more precisely one in terms of quasi-connections [21, p. 660]. Let $F \in \mathcal{T}^1_1(M)$ be a tensor field of $(1,1)$-type.

Definition 1.6. An application $D: \chi(M) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ is a *quasi-connection with respect to* F on $\mathcal{D}$ if, for all $X, Y \in \chi(M)$ and $Z \in \Gamma(\mathcal{D})$:

(i) $D_{fX+gY}Z = fD_X Z + gD_Y Z, D_{X+Y}Z = D_X Z + D_Y Z$,
(ii) $D_X(fZ) = fD_X Z + FX(f)Z$.

Let us remark that a linear connection ∇ on $\mathcal{D}$ yields a quasi-connection D^∇ through [21, p. 660]:

$$D^\nabla_X Z = \nabla_{FX} Z \tag{11}$$

and then we get the following proposition.

Proposition 1.7. *A linear connection is compatible with the Weyl substructure* W *if and only if the associated quasi-connection* (11) *with respect to the projector* Q *makes* g *a recurrent tensor with the recurrence factor* $-W(g) \circ Q$.

2. The Compatible Connection in Foliated Manifolds

Let (M, g) be an $(n + p)$-dimensional semi-Riemannian manifold and $\mathcal{F}$ be an n-foliation on M. We assume that $\mathcal{D}$, the tangent distribution of $\mathcal{F}$, is semi-Riemannian that is, the induced metric tensor field on $\mathcal{D}$ is non-degenerate and with constant index. The complementary orthogonal distribution $\mathcal{D}^\perp$ to $\mathcal{D}$ is semi-Riemannian too, [4, p. 95]; let call $\mathcal{D}$ and $\mathcal{D}^\perp$ *the structural* and *transversal distribution* respectively. Now, we want an expression of the compatible connection in local coordinates.

So, let $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha}\}$ be a frame field adapted to the decomposition:

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp, \tag{12}$$

i.e. $i \in \{1, \dots, n\}, \alpha \in \{n+1, \dots, n+p\}$ and $\frac{\partial}{\partial x^i} \in \Gamma(\mathcal{D})$. With:

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad g_{i\alpha} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha}\right) \tag{13}$$

it results an adapted basis for $\mathcal{D}^\perp$, [4, p. 98]:

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - A^i_\alpha \frac{\partial}{\partial x^i}, \tag{14}$$

where

$$A^i_\alpha = g^{ij} g_{j\alpha}. \tag{15}$$

Remark that $\{\frac{\delta}{\delta x^\alpha}\}$ is orthogonal to $\{\frac{\partial}{\partial x^i}\}$. Let us point that $A^i_\alpha g_{i\beta} = A^j_\beta g_{j\alpha}$. With respect to this adapted frame field we set:

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = C^k_{ij} \frac{\partial}{\partial x^k}, \nabla_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^i} = D^k_{i\alpha} \frac{\partial}{\partial x^k}. \tag{16}$$

Then a computation similar to that of [4, pp. 99–100] yields:

Proposition 2.1. *The local coefficients of the compatible connection* ∇ *with respect to the semi-holonomic frame field* $\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\}$ *are given by*:

$$C^k_{ij} = \Gamma^k_{ij} + \frac{1}{2}(\theta_i \delta^k_j + \theta_j \delta^k_i - g_{ij}\theta^k), \quad D^k_{i\alpha} = \frac{\partial A^k_\alpha}{\partial x^i} \tag{17}$$

where

$$W(g) = \theta_i \delta x^i + \rho_\alpha dx^\alpha, \tag{18}$$

A^i_α *are given by* (15), $\theta^k = g^{kl}\theta_l, \Gamma^k_{ij}$ *are the Christoffel symbols of* g *with respect to* $\mathcal{D}$:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) \tag{19}$$

and $\{\delta x^i, dx^\alpha\}$ *is the dual frame of the given semi-holonomic frame, with*:

$$\delta x^i = dx^i + A^i_\alpha dx^\alpha. \tag{20}$$

For $p = 0$ the well-known expression of [11, 20] are reobtained.

Example 2.2. Let us continue Example 1.5 with $\mathcal{D}$ the tangent distribution of a foliation $\mathcal{F}$. Consider the local expressions above to which we add:

$$g_{\alpha\beta} = g\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right) \tag{21}$$

and denotes by $[g^{\lambda\mu}]$ the inverse matrix of $[g_{\alpha\beta}]$. We express the Vranceanu connection ∇^* in local coordinates:

$$\begin{cases} \nabla^*_{\frac{\partial}{\partial x^j}} \dfrac{\partial}{\partial x^i} = C^k_{ij} \dfrac{\partial}{\partial x^k}, \quad \nabla^*_{\frac{\delta}{\delta x^\alpha}} \dfrac{\partial}{\partial x^i} = D^k_{i\alpha} \dfrac{\partial}{x^k}, \\ \nabla^*_{\frac{\partial}{\partial x^i}} \dfrac{\delta}{\delta x^\alpha} = \nabla^\perp_{\frac{\partial}{\partial x^i}} \dfrac{\delta}{\delta x^\alpha} = L^\gamma_{\alpha i} \dfrac{\delta}{\delta x^\gamma}, \\ \nabla^*_{\frac{\delta}{\delta x^\beta}} \dfrac{\delta}{\delta x^\alpha} = \nabla^\perp_{\frac{\delta}{\delta x^\beta}} \dfrac{\delta}{\delta x^\alpha} = F^\gamma_{\alpha\beta} \dfrac{\delta}{\delta x^\gamma}. \end{cases} \tag{22}$$

A similar calculus like in [4, p. 99] gives

$$L^\gamma_{\alpha i} = 0$$

and

$$F^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\mu}\left(\frac{\delta g_{\mu\beta}}{\delta x^\alpha} + \frac{\delta g_{\alpha\mu}}{\delta x^\beta} - \frac{\delta g_{\alpha\beta}}{\delta x^\mu}\right) + \frac{1}{2}(\rho_\alpha \delta^\gamma_\beta + \rho_\beta \delta^\gamma_\alpha - \rho^\gamma g_{\alpha\beta}), \tag{23}$$

with $\rho^\gamma = \rho_\alpha g^{\alpha\gamma}$.

From Proposition 1.4. of [4, p. 100] it results that the only non-null component of the torsion tensor field T^* of ∇^* is:

$$T^*\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) = T^{*k}_{\alpha\beta} \frac{\partial}{\partial x^k} = \left[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right] = \left(\frac{\delta A^k_\alpha}{\delta x^\beta} - \frac{\delta A^k_\beta}{\delta x^\alpha}\right) \frac{\partial}{\partial x^k}, \tag{24}$$

which, therefore, describes exactly how far is $\mathcal{D}^\perp$ from integrability.

Now, we consider the curvature tensor field R^* of ∇^* and take in attention the transversal part using the notation of [5, p. 104]:

$$\begin{cases} R^*\left(\dfrac{\delta}{\delta x^\gamma}, \dfrac{\delta}{\delta x^\beta}\right) \dfrac{\delta}{\delta x^\alpha} = R^{*\mu}_{\alpha\beta\gamma} \dfrac{\delta}{\delta x^\mu}, \\ R^*\left(\dfrac{\partial}{\partial x^i}, \dfrac{\delta}{\delta x^\beta}\right) \dfrac{\delta}{\delta x^\alpha} = R^{*\mu}_{\alpha\beta i} \dfrac{\delta}{\delta x^\mu}, \\ R^*\left(\dfrac{\partial}{\partial x^j}, \dfrac{\partial}{\partial x^i}\right) \dfrac{\delta}{\delta x^\alpha} = R^{*\mu}_{\alpha ij} \dfrac{\delta}{\delta x^\mu} \end{cases} \tag{25}$$

with

$$\begin{cases} R^{*\mu}_{\alpha\beta\gamma} = \dfrac{\delta F^{\mu}_{\alpha\beta}}{\delta x^{\gamma}} - \dfrac{\delta F^{\mu}_{\alpha\gamma}}{\delta x^{\beta}} + F^{\varepsilon}_{\alpha\beta} F^{\mu}_{\varepsilon\gamma} - F^{\varepsilon}_{\alpha\gamma} F^{\mu}_{\varepsilon\beta}, \\ R^{*\mu}_{\alpha\beta i} = \dfrac{\partial F^{\mu}_{\alpha\beta}}{\partial x^{i}}, \quad R^{*\mu}_{\alpha i j} = 0. \end{cases} \tag{26}$$

The structural components of the curvature of ∇^* are:

$$\begin{cases} R^*\left(\dfrac{\delta}{\delta x^{\alpha}}, \dfrac{\delta}{\delta x^{\beta}}\right) \dfrac{\partial}{\partial x^{i}} = R^{*h}_{i\alpha\beta} \dfrac{\partial}{\partial x^{h}}, \\ R^*\left(\dfrac{\partial}{\partial x^{k}}, \dfrac{\delta}{\delta x^{\alpha}}\right) \dfrac{\partial}{\partial x^{i}} = R^{*h}_{i\alpha k} \dfrac{\partial}{\partial x^{h}}, \\ R^*\left(\dfrac{\partial}{\partial x^{k}}, \dfrac{\partial}{\partial x^{j}}\right) \dfrac{\partial}{\partial x^{i}} = R^{*h}_{ijk} \dfrac{\partial}{\partial x^{h}}, \end{cases} \tag{27}$$

with [4, p. 100]

$$\begin{cases} R^{*h}_{i\alpha\beta} = \dfrac{\delta D^{h}_{i\beta}}{\delta x^{\alpha}} - \dfrac{\delta D^{h}_{i\alpha}}{\delta x^{\beta}} + D^{k}_{i\beta} D^{h}_{k\alpha} - D^{k}_{i\alpha} D^{h}_{k\beta} - T^{*k}_{\alpha\beta} C^{h}_{ik}, \\ R^{*h}_{i\alpha k} = \dfrac{\partial D^{h}_{i\alpha}}{\partial x^{k}} - \dfrac{\delta C^{h}_{ik}}{\delta x^{\alpha}} + D^{j}_{i\alpha} C^{h}_{jk} - C^{j}_{ik} D^{h}_{j\alpha} + D^{j}_{k\alpha} C^{h}_{ij}, \\ R^{*h}_{ijk} = \dfrac{\partial C^{h}_{ij}}{\partial x^{k}} - \dfrac{\partial C^{h}_{ik}}{\partial x^{j}} + C^{l}_{ij} C^{h}_{lk} - C^{l}_{ik} C^{h}_{lj}. \end{cases} \tag{28}$$

Let $X \in \chi(M)$ with the decomposition $X = X^i \frac{\partial}{\partial x^i} + X^\alpha \frac{\delta}{\delta x^\alpha}$. The covariant derivative of the metric g with respect to the Vranceanu connection is:

$$\begin{cases} (\nabla^*_X g)\left(\dfrac{\partial}{\partial x^{i}}, \dfrac{\partial}{\partial x^{j}}\right) = X^k \left(\dfrac{\partial g_{ij}}{\partial x^{k}} - C^{h}_{ik} g_{hj} - C^{h}_{kj} g_{ih}\right) + X^{\alpha} \dfrac{\delta g_{ij}}{\delta x^{\alpha}}, \\ (\nabla^*_X g)\left(\dfrac{\delta}{\delta x^{\alpha}}, \dfrac{\delta}{\delta x^{\beta}}\right) = X^i \dfrac{\partial g_{\alpha\beta}}{\partial x^{i}} + X^{\mu} \left(\dfrac{\delta g_{\alpha\beta}}{\delta x^{\mu}} - F^{\rho}_{\alpha\mu} g_{\rho\beta} - F^{\rho}_{\mu\beta} g_{\alpha\rho}\right), \\ (\nabla^*_X g)\left(\dfrac{\partial}{\partial x^{i}}, \dfrac{\delta}{\delta x^{\alpha}}\right) = 0 \end{cases} \tag{29}$$

and a straightforward computations using (23) yields:

$$(\nabla^*_X g)\left(\frac{\delta}{\delta x^{\alpha}}, \frac{\delta}{\delta x^{\beta}}\right) = X^i \frac{\partial g_{\alpha\beta}}{\partial x^{i}} - (X^{\mu} \rho_{\mu}) g_{\alpha\beta}, \tag{30}$$

which implies a generalization of equivalence of items (i) and (ii) of Theorem 3.3. from [4, p. 112] (obtained for $W(g) = 0$).

Proposition 2.3. *Let $(M, g, \mathcal{F})$ be a semi-Riemannian manifold, where $\mathcal{F}$ is a non-degenerate foliation. Then g is a bundle-like metric for $\mathcal{F}$ if and only if there exists an one-form $W(g)$ on M such that the induced metric g on $\mathcal{D}^{\perp}$ is a recurrent tensor with respect to the Vrănceanu connection of the Weyl manifold $(M, g, W : g \to W(g))$ with the recurrence factor $-W(g) \circ Q^{\perp}$.*

Also

$$(\nabla^*_X g)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = -(X^k\theta_k)g_{ij} + X^\alpha \frac{\delta g_{ij}}{\delta x^\alpha}. \quad (31)$$

3. Weyl Structures on Tangent Bundles of Finsler Manifolds

Let N be a real n-dimensional manifold and TN its tangent bundle. Then a local chart $x = (x^a)$ on N defines a local chart $(x, y) = (x^a, y^a)_{1\leq a\leq n}$ on TN. Denote by $0(N)$ the zero section of TM and consider $T^0N = TN\backslash 0(N)$.

Definition 3.1 ([22]). The pair (N, F) is a *Finsler manifold* if $F : TN \to [0, \infty)$ with the following conditions:

(F1) F is smooth on T^0N and vanishes only on $0(N)$,
(F2) F is positively homogeneous of degree one with respect to (y^a),
(F3) the matrix $[g_{bc}(x^a, y^a)] = [\frac{1}{2}\frac{\partial^2 F^2}{\partial y^b \partial y^c}]$ is positive definite.

The vertical bundle $V(N)$ of N is the tangent distribution to the foliation defined by the fibers of $\pi : TN \to N$. Then $V(N)$ is locally spanned by $\frac{\partial}{\partial y^a}$. Denote by $[g^{bc}]$ the inverse matrix of $[g_{bc}]$ and define:

$$G^a(x, y) = \frac{1}{4}g^{ab}\left(\frac{\partial^2 F^2}{\partial y^b \partial x^c}y^c - \frac{\partial F^2}{\partial x^b}\right). \quad (32)$$

There exists on TN an n-distribution $H(N)$, called *horizontal*, locally spanned by the vector fields:

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - G^b_a \frac{\partial}{\partial y^b}, \quad (33)$$

where

$$G^a_b = \frac{\partial G^a}{\partial y^b}. \quad (34)$$

It is easy to see that $H(N)$ is complementary to $V(N)$ in TN and using the decomposition:

$$T(TN) = H(N) \oplus V(N) \quad (35)$$

we define the Riemannian metric G on TN, called the *Sasaki–Finsler metric*:

$$G = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix}, \quad (36)$$

which means that with respect to the semi-holonomic frame field $\{\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^a}\}$ we have:

$$G\left(\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b}\right) = G\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right) = g_{ab}, \quad G\left(\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^b}\right) = 0. \quad (37)$$

The above discussion shows that on the Riemannian manifold (TN, G) we have a foliation $\mathcal{F}$ with $V(N)$ and $H(N)$ as structural and transversal distribution respectively, therefore we obtain the framework discussed in the previous section. Suppose given an one-form $W(G)$ on TN with the expression:

$$W(G) = \rho_a dx^a + \theta_a \delta y^a, \tag{38}$$

where $(dx^a, \delta x^a)$ is the dual frame of the given semi-holonomic frame:

$$\delta y^a = dy^a + G^a_b dx^b. \tag{39}$$

The aim of this section is to obtain the coefficients of the Vranceanu connection for the pair (Weyl structure $W : G \to W(G)$, distribution $V(N)$). These coefficients are given by:

$$\begin{cases} \nabla^*_{\frac{\partial}{\partial y^b}} \dfrac{\partial}{\partial y^a} = C^c_{ab} \dfrac{\partial}{\partial y^c}, & \nabla^*_{\frac{\delta}{\delta x^b}} \dfrac{\partial}{\partial y^a} = D^c_{ab} \dfrac{\partial}{\partial y^c}, \\ \nabla^*_{\frac{\partial}{\partial y^b}} \dfrac{\delta}{\delta x^a} = L^c_{ab} \dfrac{\delta}{\delta x^c}, & \nabla \dfrac{\delta}{\delta x^a} = F^c_{ab} \dfrac{\delta}{\delta x^c}. \end{cases} \tag{40}$$

Using Proposition 3.1 from [4, pp. 226–227] and the results of the previous section we derive:

Proposition 3.2. *The Vranceanu connection of a Weyl manifold* (TN, G, W) *has the local coefficients*:

$$\begin{cases} C^c_{ab} = \dfrac{1}{2}\left(g^{cd}\dfrac{\partial g_{ab}}{\partial y^d} + \theta_a \delta^c_b + \theta_b \delta^c_a - \theta^c g_{ab}\right), \\ D^c_{ab} = \dfrac{\partial^2 G^a}{\partial y^b \partial y^c}, \quad L^c_{ab} = 0, \\ F^c_{ab} = \dfrac{1}{2} g^{cd}\left(\dfrac{\delta g_{db}}{\delta x^a} + \dfrac{\delta g_{ad}}{\delta x^b} - \dfrac{\delta g_{ab}}{\delta x^d}\right) + \dfrac{1}{2}\left(\rho_a \delta^c_b + \rho_b \delta^c_a - \rho^c g_{ab}\right). \end{cases} \tag{41}$$

Example 3.3. (1) A Finsler manifold is a *Landsberg space* if, [2, p. 239]:

$$\frac{1}{2} g^{cd}\left(\frac{\delta g_{db}}{\delta x^a} + \frac{\delta g_{ad}}{\delta x^b} - \frac{\delta g_{ab}}{\delta x^d}\right) = \frac{\partial G^c_a}{\partial y^b}. \tag{42}$$

Therefore, the Vrănceanu connection for a Weyl manifold provided by a Landsberg space has:

$$F^c_{ab} = \frac{\partial^2 G^c}{\partial y^a \partial y^b} + \frac{1}{2}\left(\rho_a \delta^c_b + \rho_b \delta^c_a - \rho^c g_{ab}\right). \tag{43}$$

(2) A Finsler manifold is a *locally Minkowski space* if, [2, p. 239], there exists a covering by charts (U, x) of N such that $g_{ab} = g_{ab}(y)$. A locally Minkowski space is a Landsberg one with $G^a = 0$ and then one have:

$$F^c_{ab} = \frac{1}{2}\left(\rho_a \delta^c_b + \rho_b \delta^c_a - \rho^c g_{ab}\right). \tag{44}$$

Example 3.4. In the following we consider a natural one-form $W(G)$. The condition F3 of Definition 3.1 means that $F^2 : TN \to [0,\infty)$ is a regular Lagrangian in the sense of Analytical Mechanics and then, it defines a Legendre transform $L(F^2) : TN \to T^*N$, with T^*N the cotangent bundle of N. With coordinates (x^a) on TN we have induced coordinates (x^a, p_a) on T^*N. Also, on T^*N lives a global one-form, called Liouville, $\theta = p_a dx^a$. The pullback of the Liouville form through the Legendre transform, $\theta_F = L(F^2)^*(\theta)$, is called *the Cartan form* of F. Therefore, we define $W(G) = \theta_F = \frac{1}{2}\frac{\partial F^2}{\partial y^a} dx^a$ which yields the following proposition.

Proposition 3.5. *The Vranceanu connection of a Weyl manifold* (TN, F, θ_F) *has the coefficients*:

$$\begin{cases} C^c_{ab} = \dfrac{1}{2} g^{cd} \dfrac{\partial g_{ab}}{\partial y^d}, \\ D^c_{ab} = \dfrac{\partial^2 G^c}{\partial y^a \partial y^b}, \\ L^c_{ab} = 0, \\ 4F^c_{ab} = 2g^{cd}\left(\dfrac{\delta g_{db}}{\delta x^a} + \dfrac{\delta g_{ad}}{\delta x^b} - \dfrac{\delta g_{ab}}{\delta x^d}\right) + \dfrac{\partial F^2}{\partial y^a}\delta^c_b + \dfrac{\partial F^2}{\partial y^b}\delta^c_a - \dfrac{\partial F^2}{\partial y^u} g^{uc} g_{ab}. \end{cases} \tag{45}$$

But $\frac{\partial F^2}{\partial y^v} = 2g_{vu} y^u$ from (F2) and then:

$$F^c_{ab} = \frac{1}{2} g^{cd}\left(\frac{\delta g_{db}}{\delta x^a} + \frac{\delta g_{ad}}{\delta x^b} - \frac{\delta g_{ab}}{\delta x^d}\right) + \frac{1}{2} y^u \left(g_{ub}\delta^c_a + g_{au}\delta^c_b - g_{ab}\delta^c_u\right). \tag{46}$$

Corollary 3.6. *The Vranceanu connection of a Landsberg–Weyl manifold* (TN, F, θ_F) *has*:

$$F^c_{ab} = \frac{\partial^2 G^c}{\partial y^a \partial y^b} + \frac{1}{2} y^u \left(g_{ub}\delta^c_a + g_{au}\delta^c_b - g_{ab}\delta^c_u\right) \tag{47}$$

while for a locally Minkowski space:

$$F^c_{ab} = \frac{1}{2} y^u \left(g_{ub}\delta^c_a + g_{au}\delta^c_b - g_{ab}\delta^c_u\right). \tag{48}$$

Remarks 3.7. From now we use $i, j, k, \ldots$ for the vertical coordinates and $a, b, c, \ldots$ for the horizontal ones, for a better identification of the structural and respectively transversal components. But we keep the above notations for the coefficients of the Vranceanu connection.

The transversal components of the curvature R^* of ∇^* are given by (26) with:

$$R^{*c}_{abi} = \frac{1}{2}(g_{ib}\delta^c_a + g_{ai}\delta^c_b - g_{ab}\delta^c_i) \tag{49}$$

which never vanishes.

Example 3.8. Let $g = (g_{ab}(x))$ be a Riemannian metric on N with the Christoffel coefficients Γ^c_{ab}. Then $F = (g_{uv}y^u y^v)^{\frac{1}{2}}$ is a Finsler fundamental function on N.

Definition 3.9. We define the *Vranceanu–Cartan connection* on TN for the Riemannian manifold (N, g), the Vranceanu connection obtained from the process of Example 3.4. Namely it is associated to the Weyl manifold $(TN, W : G \to \theta_F)$ with the above F.

This Vranceanu connection is a particular case of Proposition 3.2 and then:

Proposition 3.10. *The Vranceanu–Cartan connection of the tangent bundle TN of a Riemannian manifold (N, g) is*:

$$\begin{cases} C^c_{ab} = 0, \quad D^c_{ab} = \Gamma^c_{ab}, \quad L^c_{ab} = 0, \\ F^c_{ab} = \Gamma^c_{ab} + \dfrac{1}{2} y^u \left(g_{ub}\delta^c_a + g_{au}\delta^c_b - g_{ab}\delta^c_u\right). \end{cases} \tag{50}$$

For $X = X^i \frac{\partial}{\partial y^i} + X^a \frac{\delta}{\delta x^a}$ the non-null covariant derivatives of the Sasaki–Riemann metric G with respect to the Vrănceanu–Cartan connection are:

$$\begin{cases} (\nabla^*_X G)\left(\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j}\right) = X^a \dfrac{\delta g_{ij}}{\delta x^a} = X^a \dfrac{\partial g_{ij}}{\partial x^a}, \\ (\nabla^*_X G)\left(\dfrac{\delta}{\delta x^a}, \dfrac{\delta}{\delta x^b}\right) = -(g_{ci}X^c y^i) g_{ab}. \end{cases} \tag{51}$$

Remarks 3.11. Denoting by Rg the $(1, 3)$-Riemannian curvature tensor field of g and using (26), we get that the non-vanishing transversal components of the curvature of Vranceanu–Cartan connection are:

- $R^{*d}_{abc} = (Rg)^d_{abc} +$ a very complicated expression in g and y,
- R^{*c}_{abi} from (49),

while the only non-null structural component of R^* is, using (28),

- $R^{*j}_{iab} = (Rg)^j_{iab}$.

Let us point also, that T^{*c}_{ab} from (24) is, [4, p. 233]:

- $T^{*c}_{ab} = (Rg)^c_{dab} y^d$.

Denote with V and H the vertical and horizontal projectors of TN. They correspond to Q respectively $Q^\perp$ in the notations of the first two sections. The above discussion about the curvature of ∇^* yields:

Proposition 3.12. *For the Vranceanu–Cartan connection and $X, Y, Z \in \chi(TN)$ we have*:

(1) ∇^* *is torsion-free if and only if the base manifold (N, g) is flat.*

(2) $R^*(HX, HY)VZ = 0$ *if and only if the base manifold* (N, g) *is flat. Moreover, the Vranceanu–Cartan is never vertical-horizontal flat but is vertical flat i.e.* $R^*(V\cdot, V\cdot) = 0$.

Using the equivalent conditions from [2, p. 237] we derive the following corollary.

Corollary 3.13. *The projection* $\pi_T : (TN, G) \to (N, g)$ *is totally geodesic i.e. the projection of any geodesic in* (TN, G) *is also a geodesic in* (N, g) *if and only if the Vranceanu–Cartan connection is torsion-free.*

Let us end this section with the covariant derivative of the Liouville vector fields with respect to the Vranceanu connection in the general (i.e. Finslerian) framework of this section. More precisely, define [4, p. 231]:

$$\begin{cases} L = y^i \dfrac{\partial}{\partial y^i}, \\ L^* = y^a \dfrac{\delta}{\delta y^a} \end{cases} \tag{52}$$

called *the Liouville vector field* on TN respectively the *transversal Liouville vector field* or *geodesic spray*. For $X = X^i \frac{\partial}{\partial y^i} + X^a \frac{\delta}{\delta x^a}$ it results:

$$\begin{cases} \nabla^*_X L = (X^i + X^j y^k C^i_{kj}) \dfrac{\partial}{\partial y^i} + X^c (D^a_{bc} y^b - G^a_c) \delta^i_a \dfrac{\partial}{\partial y^i}, \\ \nabla^*_X L^* = (X^i + X^j y^k L^i_{kj}) \delta^a_i \dfrac{\delta}{\delta x^a} + X^c (F^a_{bc} y^b - G^a_c) \dfrac{\delta}{\delta x^a}, \end{cases} \tag{53}$$

which, replacing the coefficients from (41) and using the relations (3.32_a) from [4, p. 231] and (3.39) from [4, p. 232], yields the following proposition.

Proposition 3.14. *The covariant derivative of the Liouville vector fields with respect to the Vranceanu connection of a Weyl manifold* (TN, G, W) *are*:

$$\begin{cases} \nabla^*_X L = \left[X^i + \dfrac{1}{2} X^j y^k (\theta_j \delta^i_k + \theta_k \delta^i_j - \theta^i g_{jk}) \right] \dfrac{\partial}{\partial y^i}, \\ \nabla^*_X L^* = \left[X^i \delta^a_i + \dfrac{1}{2} X^b y^c (\rho_b \delta^a_c + \rho_c \delta^a_b - \rho^a g_{bc}) \right] \dfrac{\delta}{\delta x^a}. \end{cases} \tag{54}$$

In particular, for the Weyl manifold (TN, F, θ_F) *we get*:

$$\begin{cases} \nabla^*_X L = X^i \dfrac{\partial}{\partial y^i}, \\ \nabla^*_X L^* = \left(X^i \delta^a_i + \dfrac{1}{2} X^a y^c y^u g_{uc} \right) \dfrac{\delta}{\delta x^a}. \end{cases} \tag{55}$$

In order to provide a global expression of these relations let us denote the vertical and horizontal components of $W(G)$:

$$W(G)^V = \theta_i \delta y^i, \quad W(G)^H = \rho_a dx^a \tag{56}$$

and then (54) becomes:

$$\begin{cases} 2\nabla^*_X L = 2VX + W(G)^V(VX)L + W(G)^V(L)VX - G(VX, L)W(G)^{V\#}, \\ 2\nabla^*_X L^* = 2\Theta(X) + W(G)^H(HX)L^* + W(G)^H(L^*)HX - G(HX, L^*)W(G)^{H\#}, \end{cases}$$

where $\#$ is the musical isomorphism defined by G:

$$W(G)^{V\#} = \theta^i \frac{\partial}{\partial y^i}, \quad W(G)^{H\#} = \rho^a \frac{\delta}{\delta x^a} \tag{57}$$

while (55) is:

$$\begin{cases} \nabla^*_X L = VX, \\ \nabla^*_X L^* = \Theta(X) + \frac{1}{2}F^2 HX, \end{cases} \tag{58}$$

with F the Finsler fundamental function of Definition 3.1 and Θ the *adjoint structure,* [16, p. 991], $\Theta = \frac{\delta}{\delta x^i} \otimes \delta y^i$.

A conclusion of the last formula is that vertical, horizontal projectors and the adjoint structure, which are main objects in Finsler geometry, can be recovered from $\nabla^* L$ and respectively $\nabla^* L^*$.

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