Slant curves in 3-dimensional f-Kenmotsu manifolds

Constantin Călin[a], Mircea Crasmareanu[b], Marian Ioan Munteanu[c]

[a] *Technical University 'Gh. Asachi', Department of Mathematics, 700049 Iaşi, Romania*
[b] *'Al.I. Cuza' University of Iaşi, Faculty of Mathematics, Bd. Carol I, no. 11, 700506 Iaşi, Romania*
web-page: `www.math.uaic.ro/~mcrasm`
[c] *'Al.I. Cuza' University of Iaşi, Faculty of Mathematics, Bd. Carol I, no. 11, 700506 Iaşi, Romania*
web-page: `www.math.uaic.ro/~munteanu`

Abstract

The aim of this paper is to study slant curves of three-dimensional f-Kenmotsu manifolds. These curves are characterized through the scalar product between the normal at the curve and the Reeb vector field. The classification of slant curves in the hyperbolic 3-dimensional space is provided as well as some remarkable cases. Slant curves with proper mean curvature vector field are characterized.

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Introduction

One of the fruitful notion of classical differential geometry of curves is that of *curve of constant slope*. Also called *cylindrical helix*, this is a curve in the Euclidean space $\mathbb{E}^3$ for which the tangent vector field makes a constant angle with a fixed direction called the *axis* of the curve; the second name is due to the fact that there exists a cylinder on which the curve moves in such a way as to cut each ruling at a constant angle. The well-known characterization of these curve is the Bertrand-Lancret-de Saint Venant Theorem ([2]): the curve γ in $\mathbb{E}^3$ is of constant slope if and only if the ratio of torsion and curvature is constant. Then, for a cylindrical helix we have *the Lancret invariant*:

$$Lancret(\gamma) = \frac{\tau}{\kappa}\ . \tag{1}$$

A very interesting generalization of this object is that of *slant curve* in almost contact metric geometry and it was introduced in [7], in the three-dimensional case, with the slant angle θ between the tangent and the structural vector field and. In the particular case $\cos\theta = 0$ we recover the *Legendre curves* of [1]. For slant curves in 3-dimensional Sasakian manifolds we have, from Theorem 3.1. of [7, p. 362]:

$$Lancret(\gamma) = \frac{\tau \pm 1}{\kappa}. \tag{2}$$

Although the Bibliography in Legendre curves is very rich ([3], [5], [6], [9], [14], [16], [19], [20]), slant curves are studied until now only in Sasakian geometry in [7] and contact pseudo-Hermitian geometry in [8]. Therefore, the purpose of this paper is to begin a study of slant curves in another important class of almost contact manifolds introduced by K. Kenmotsu in 1972. More precisely, we consider the most general case of Kenmotsu geometry, defined by a smooth strictly positive function on the given manifold.

Our work is structured as follows. The first section is a very brief review of Kenmotsu geometry and Frenet curves in general Riemannian geometry. The next section is devoted to the study of slant (particularly Legendre) curves in

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Email addresses: `c0nstc (at) yahoo.com` (Constantin Călin), `mcrasm (at) uaic.ro` (Mircea Crasmareanu), `marian.ioan.munteanu (at) gmail.com` (Marian Ioan Munteanu)

this generalized framework of f-Kenmotsu manifolds. Thus, we obtain a characterization of slant curves similar to Proposition 3.1. of [7, p. 362] and a Lancret invariant is derived only for Legendre curves in ($f \equiv constant =$)β-Kenmotsu manifold. Our Lancret expression is more complicated involving, in addition to curvature and torsion, a square root and the derivative of the curvature along the curve. An example of a slant curve is included for the β-Kenmotsu manifold of warped product type. A special attention is paid to the hyperbolic space $\mathbb{H}^3(-1)$ with its canonical Kenmotsu structure.

1. f-Kenmotsu manifolds and Frenet curves

Let M be a smooth $(2n+1)$-dimensional manifold endowed with an almost contact metric structure (φ, ξ, η, g):

$$\begin{cases} \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \\ \varphi(\xi) = 0, \quad \eta(X) = g(X, \xi), \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{cases} \tag{3}$$

for any vector fields $X, Y \in \mathcal{X}(M)$ where I is the identity of the tangent bundle TM, φ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is a metric (Riemannian) tensor field. Throughout the paper all objects are differentiable of class C^∞.

A classification result in almost contact metric geometry is due to S. Tanno, [17]:

Theorem A *Let M be a connected almost contact metric manifold of dimension $2n+1$. Then the maximum dimension of the automorphisms group of M is $(n+1)^2$ and this maximum is attained if and only if the sectional curvatures of planes containing the Reeb vector field ξ is a constant c. Then M is one of the following spaces*:

1) *a homogeneous Sasaki space form if $c > 0$,*

2) *a global Riemannian product of a line or a circle with a complex space form if $c = 0$,*

3) *the warped product $\mathbb{R} \times_\rho \mathbb{C}^n$ with $\rho(t) = \exp(\sqrt{-c}\, t)$ if $c < 0$.*

In this paper we are interested in the third case from the slant curves point of view.

We say that $(M, \varphi, \xi, \eta, g)$ is an *f-Kenmotsu manifold* if the Levi-Civita connection ∇ of g satisfy [15, p. 2]:

$$(\nabla_X \varphi)(Y) = f(g(\varphi X, Y)\xi - \varphi(X)\eta(Y)), \tag{4}$$

where $f \in C^\infty(M)$ is strictly positive and $df \wedge \eta = 0$ holds (for $n \geq 2$). If f is equal to a constant $\beta > 0$, we get a *β-Kenmotsu manifold* with the particular case $\beta = 1$ when the manifold is usually known as a *Kenmotsu manifold*, [12].

In a f-Kenmotsu manifold we have, [15, p. 3]:

$$\nabla_X \xi = f(X - \eta(X)\xi), \tag{5}$$

for all X tangent to M, and hence

$$\nabla_\xi \xi = 0 \tag{6}$$

which means that the integral curves of ξ are geodesics for the metric g.

Example 1. We present now an example of a β-Kenmotsu $(2n+1)$-dimensional manifold: let $\mathbb{K}^{2n+1}_\beta := \mathbb{R}^{2n+1}$ with global coordinates (t, x^i, y^i), $1 \leq i \leq n$ and $\xi = \frac{\partial}{\partial t}$, $\eta = dt$:

$$\begin{cases} \varphi(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}, \quad \varphi(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i}, \quad \varphi(\frac{\partial}{\partial t}) = 0, \\ g = dt^2 + \exp(2\beta t) \sum\limits_{i=1}^{n} \left[(dx^i)^2 + (dy^i)^2 \right]. \end{cases} \tag{7}$$

It follows that our manifold is the warped product $\mathbb{K}^{2n+1}_\beta = \mathbb{R} \times_\rho \mathbb{C}^n$ with $\rho(t) = \exp(\beta t)$.

In the last part of this section we recall the notion of *Frenet curve* after [4, p. 164]; let r be an integer with $1 \leq r \leq m = \dim M$. The curve $\gamma : I \subseteq \mathbb{R} \to M$ parametrized by the arc length s is called *r-Frenet curve* on M if there exist r orthonormal vector fields $(E_1 = \gamma', ..., E_r)$ along γ such that there exist positive smooth functions $\kappa_1, ..., \kappa_{r-1}$ of s with:

$$\begin{cases} \nabla_{\gamma'} E_1 & = & \kappa_1 E_2, \\ \nabla_{\gamma'} E_2 & = & -\kappa_1 E_1 + \kappa_2 E_3, \\ \quad \cdots & \cdots & \cdots \\ \nabla_{\gamma'} E_r & = & -\kappa_{r-1} E_{r-1}. \end{cases} \tag{8}$$

The function k_j is called *the* j-th *curvature* of γ. The curve γ is known as

1) a *geodesic* if $r = 1$; then we get the well-known equation $\nabla_{\gamma'}\gamma' = 0$,
2) a *circle* if $r = 2$ and κ_1 is a constant; then, we have: $\nabla_{\gamma'} E_1 = \kappa_1 E_2$ together with $\nabla_{\gamma'} E_2 = -\kappa_1 E_1$,
3) a *helix of order r* if $\kappa_1, ..., \kappa_{r-1}$ are constants.

The Frenet curve γ is called *non-geodesic* if $\kappa_1 > 0$ everywhere on I.

2. Slant and Legendre curves in 3-dimensional *f*-Kenmotsu manifolds

From now on we fix $n = 1$. Consider a Frenet curve γ and its Frenet frame denoted by $(T = \gamma', N, B)$. The Frenet equations (8) become:

$$\begin{cases} \nabla_T T = \kappa N, \\ \nabla_T N = -\kappa T + \tau B, \\ \nabla_T B = -\tau N. \end{cases} \tag{9}$$

Definition. The *contact angle* of γ is the function $\theta : I \to [0, 2\pi)$ given by:

$$\cos\theta(s) = g(T(s), \xi). \tag{10}$$

γ is a *slant curve* (or more precisely θ-*slant curve*) if θ is a constant function, [7, p. 361]. In the particular case of $\theta \equiv \frac{\pi}{2}$ (or $\frac{3\pi}{2}$), γ is called *Legendre curve*, [1].

Example 2. We mention the following examples:

i) The integral curves of the Reeb vector field ξ are slant curves with $\theta \equiv 0$.

ii) The Legendre curves of a Kenmotsu manifold with N parallel with ξ are circles, [18]. In fact, a direct analysis of (9) for the general case of f-Kenmotsu manifolds, gives that a Legendre curve with the above condition has: $N = -\xi$, $\kappa = f|_\gamma$ and $\tau = 0$. In particular, a Legendre curve in a β-Kenmotsu manifold is a circle.

iii) The tangent vector field of a Legendre curve belongs to *the contact distribution* $\mathcal{D} := \ker\eta$. A remarkable fact of 1-Kenmotsu geometry in any dimension is that $\mathcal{D}$ defines a Riemannian foliation since it is integrable and any leaf of $\mathcal{D}$ is totally umbilical, its mean curvature vector is $(-\xi)$ and has a natural Kähler structure.

In the following we suppose that γ is non-geodesic i.e. $\kappa > 0$ and then γ can not be an integral curve of ξ which means $\theta \neq 0, \pi$.

A first result of this note is the following characterization of slant curves:

Proposition 1. *The Frenet curve γ is a θ-slant curve if and only if, along γ the following relation holds:*

$$\eta(N) = -\frac{f}{\kappa}\sin^2\theta. \tag{11}$$

Then a necessary condition for γ to be θ-slant is:

$$|\sin\theta| \leq \min\left\{\frac{\kappa}{f}, 1\right\}. \tag{12}$$

Proof. Let us take the covariant derivative in the relation (10) along γ:

$$0 = -\theta' \sin\theta = g(\kappa N, \xi) + g(T, f(T - \eta(T)\xi)) = \kappa\eta(N) + f \sin^2\theta$$

which yields (11). The expression of ξ in the Frenet frame is:

$$\xi = (\cos\theta)T + \left(-\frac{f}{\kappa}\sin^2\theta\right)N + \eta(B)B$$

and since ξ is a unitary vector field we get from this decomposition of ξ that:

$$1 = \cos^2\theta + \frac{f^2}{\kappa^2}\sin^4\theta + \eta(B)^2$$

and then $\eta(B)^2 \geq 0$ implies $|\frac{f}{\kappa}\sin\theta| \leq 1$. This last equation together with $|\frac{f}{\kappa}\sin^2\theta| \leq 1$ given by (11) imply, since both f and κ are strictly positive, that $|\sin\theta| \leq \min\left\{\sqrt{\frac{\kappa}{f}}, \frac{\kappa}{f}, 1\right\}$. But we have either $1 \leq \sqrt{\frac{\kappa}{f}} \leq \frac{\kappa}{f}$, or $1 \geq \sqrt{\frac{\kappa}{f}} \geq \frac{\kappa}{f}$. Hence (12) is proved. □

Remark 1. The decomposition of ξ in the Frenet frame of a slant curve is:

$$\xi = (\cos\theta)T + \left(-\frac{f}{\kappa}\sin^2\theta\right)N + \left(\frac{|\sin\theta|}{\kappa}\sqrt{\kappa^2 - f^2\sin^2\theta}\right)B \tag{13}$$

and the equality $|\sin\theta| = \sqrt{\frac{\kappa}{f}}$ implies $\eta(N) = -1$ (and then $\eta(T) = \eta(B) = 0$) which means that $\cos\theta = 0$ and $|\sin\theta| = \frac{\kappa}{f} = 1$. It follows that γ is a Legendre curve with $N = -\xi$ and $\kappa = f|_\gamma$ as in Example 2 ii).

Remark 2. Suppose $\theta > 0$. If we add the condition: N is parallel with ξ, as in [18], then $N = \lambda\xi$ with $\lambda \in C^\infty(M)$ satisfying $|\lambda(\gamma(s))| = 1$ for all $s \in I$. The second Frenet equation reads: $T(\lambda)\xi + \lambda f(T - \cos\theta\xi) = -\kappa T + \tau B$ and:

1) the component on $N \parallel \xi$ yields: $T(\lambda) = \lambda f \cos\theta$,

2) the component on B yields $\tau = 0$,

3) the component on T gives: $\lambda f = -\kappa$ and from $|\lambda| = 1$ we get $\kappa = f_{|_\gamma}$ and $\lambda = -1$.

Returning to the above relation 1) we have $\cos\theta = 0$ i.e. γ is a Legendre curve discussed in Example 2 ii).

Remark 3. In [11, p. 155] it is introduced the following notion: a non-geodesic curve is called *a slant helix* if the principal normal lines of γ make a constant angle with a fixed direction. Therefore, a slant curve with $\kappa \equiv constant \neq 0$ in a 3-dimensional β-Kenmotsu manifold is a slant helix with ξ as fixed direction.

As consequence an important result is obtained:

Theorem 1. *Let γ be a non-geodesic θ-slant curve ($\theta \neq 0, \pi$) such that N is non-parallel with ξ. Then its torsion is:*

$$\tau = \cot\theta\sqrt{\kappa^2 - f^2\sin^2\theta} - \frac{\kappa\gamma'(\frac{f}{\kappa})\sin\theta}{\sqrt{\kappa^2 - f^2\sin^2\theta}}. \tag{14}$$

In particular, in a β-Kenmotsu manifold M the following statements hold:

1. *a non-geodesic slant curve with with N non-parallel with ξ and constant curvature κ has a constant torsion τ and so, is a helix of order* 3,
2. *a Legendre curve with $\theta = \frac{\pi}{2}$ and N non-parallel with ξ has the torsion:*

$$\tau = \frac{\beta\kappa'}{\kappa\sqrt{\kappa^2 - \beta^2}}. \tag{15}$$

In conclusion, a β-Kenmotsu manifold has the following Lancret invariant for Legendre curves with $\theta = \frac{\pi}{2}$ and N non-parallel with ξ:

$$Lancret(\gamma) = \frac{\kappa\tau\sqrt{\kappa^2 - \beta^2}}{\kappa'}. \tag{16}$$

Proof. We take the covariant derivative in (11) and use again the Frenet equations and (11). From (15) we obtain that: $Lancret(\gamma) = \beta$. $\square$

3. Classification of slant curves in the hyperbolic space $\mathbb{H}^3$

In the following we classify all θ-slant curve in the hyperbolic space $\mathbb{H}^3(-1)$. The angle θ will be considered in the interval $(0, \pi)$.

First of all, let us pointed out that the manifold $\mathbb{K}^3_1$ (see **Example 1**) has constant sectional curvature -1, and by changing the t-coordinate, one can obtain the upper half space model of the hyperbolic 3-space. More precisely, considering $z = \exp(-t)$ one gets that $(\mathbb{K}^3_1, g)$ is isometric to $(\mathbb{H}^3_+, g_{-1})$ where $\mathbb{H}^3_+ = \{(x, y, z) \in \mathbb{R}^3; z > 0\}$ and

$$g_{-1} = \frac{1}{z^2}\left(dx^2 + dy^2 + dz^2\right). \tag{17}$$

In the following we describe all slant curves in the hyperbolic space $\mathbb{H}^3$ thought as the warped product $\mathbb{K}^3_1 = \mathbb{R}\times_\rho\mathbb{C}$ described in Example 1 with $\beta = 1$.

Theorem 2. *Let γ be a θ-slant curve in $\mathbb{K}^3_1$. Then, γ is locally given, up to translations in $\mathbb{K}^3_1$, by*

$$\gamma(s) = \left(s\cos\theta, \sin\theta\int^s e^{-u\cos\theta}\zeta(u)du\right) \tag{18}$$

where $s \in I$ is the arc-length parameter of γ and $\zeta(u) = (\cos\alpha(u), \sin\alpha(u))$, $\alpha \in C^\infty(I)$, is an arbitrary parametrization of the circle $\mathbb{S}^1$. In particular, the Legendre curves of $\mathbb{K}^3_1$ are curves lying in the (xy)-plane.

Proof. It is easy to prove that the parametrization (18) furnishes a slant curve and θ is the constant angle. In order to prove the converse, let γ be parametrized by arc-length as

$$\gamma(s) = (t(s), x(s), y(s)).$$

Thus, we have $\dot{t}(s)^2 + e^{2t(s)}\left(\dot{x}(s)^2 + \dot{y}(s)^2\right) = 1$, for all $s \in I$. Here $\dot{x}$ represents the derivative of x with respect to s, and so on.
Since the Reeb vector field is $\xi = (1, 0, 0)$, the condition for γ to be θ-slant writes $\dot{t}(s) = \cos\theta$. Therefore, up to a translation along t-axis, we get $t(s) = s\cos\theta$.

We immediately obtain $\dot{x}(s)^2 + \dot{y}(s)^2 = e^{-2s\cos\theta}\sin^2\theta$. Subsequently, there exists a (smooth) function α such that

$$\dot{x}(s) = e^{-s\cos\theta}\cos\alpha(s)\sin\theta, \quad \dot{y}(s) = e^{-s\cos\theta}\sin\alpha(s)\sin\theta.$$

Hence the conclusion. $\square$

Easy computations yield

$$\kappa = \sin\theta\sqrt{1 + \alpha'(s)^2}, \quad \tau = \pm\left(\cos\theta\alpha'(s) + \frac{\alpha''(s)}{1 + \alpha'(s)^2}\right). \tag{19}$$

For Legendre curves, we have

$$\kappa = \sqrt{1 + \alpha'(s)^2}, \quad \tau = \pm\frac{\alpha''(s)}{1 + \alpha'(s)^2}. \tag{20}$$

See also [13, Th. 2.10].

Thus, particular situations could be interesting and we will draw some Euclidean pictures of them.

a We have $\kappa = 0$ if and only if $\theta \in \{0, \pi\}$, and hence γ is a geodesic. More precisely, up to translations of $\mathbb{K}^3_1$, it represents the vertical axis Ot.

b If $\theta \notin \{0, \pi\}$ then κ is constant if and only if α is an affine function $\alpha(s) = as + b$, with $a, b \in \mathbb{R}$.

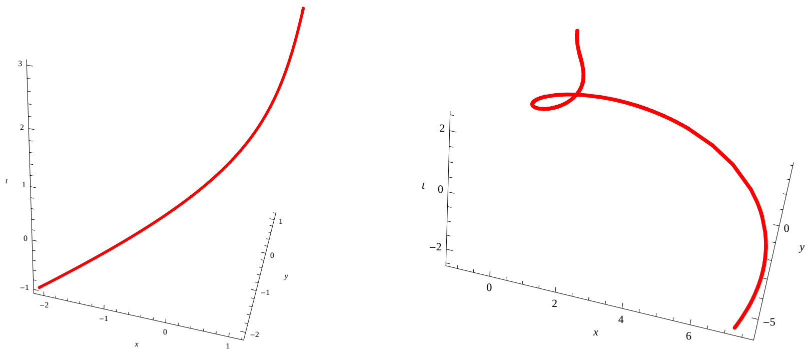

Figure 1: $\theta = \pi/3$ and $\alpha(s)$ =constant (left), resp. $\alpha(s) = s$ (right)

c In the general situation, namely $\theta \neq 0, \pi$ and α is not an affine function, we draw:

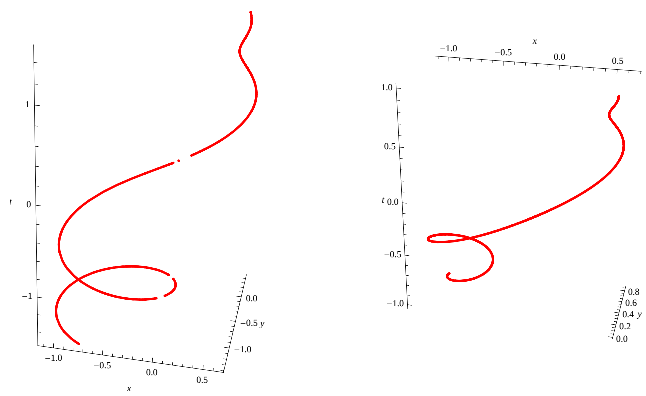

Figure 2: $\theta = \pi/3$ and $\alpha(s) = s^2$ (left), resp. $\alpha(s) = s^3$ (right)

We end this section with the following interesting result.

Proposition 2. *Let γ be a Legendre Bertrand curve in $\mathbb{H}^3$, i.e. $\theta = \frac{\pi}{2}$ and there exist $a, b \in \mathbb{R} \setminus \{0\}$ such that $a\kappa + b\tau = 1$. Then the curvature κ is the (unique) solution of the equation*

$$I(\kappa) = (s - s_0)/b, \quad s_0 \in \mathbb{R}$$

where

$$\begin{aligned}
I(\kappa) &= -\arctan\frac{1}{\sqrt{\kappa^2-1}} - \frac{a}{\sqrt{a^2-1}}\arctan\frac{\kappa-a}{\sqrt{a^2-1}\sqrt{\kappa^2-1}}, && \text{for } |a|>1,\\
I(\kappa) &= -\arctan\frac{1}{\sqrt{\kappa^2-1}} + \sqrt{\frac{\kappa+1}{\kappa-1}}, && \text{for } a=1,\\
I(\kappa) &= -\arctan\frac{1}{\sqrt{\kappa^2-1}} - \sqrt{\frac{\kappa-1}{\kappa+1}}, && \text{for } a=-1,\\
I(\kappa) &= -\arctan\frac{1}{\sqrt{\kappa^2-1}} + \frac{a}{\sqrt{1-a^2}}\log\left|\frac{\sqrt{1-a^2}\sqrt{\kappa^2-1}+\kappa-a}{1-a\kappa}\right|, && \text{for } |a|<1.
\end{aligned} \tag{21}$$

Proof. Since γ is Legendre we have $\tau = \frac{\kappa'}{\kappa\sqrt{\kappa^2-1}}$. The condition $a\kappa + b\tau = 1$ becomes

$$\frac{\kappa'}{\kappa(1-a\kappa)\sqrt{\kappa^2-1}} = \frac{1}{b}\,.$$

Integrating this ordinary differential equation, we get $I(\kappa) = (s-s_0)/b$, where $I(\kappa)$ is given by (21), depending of the values of a.

□

Remark 4. We can describe all Legendre Bertrand curves γ in the hyperbolic space $\mathbb{K}^3_1$ as follows: If κ is given as in Proposition 2, then γ is given by

$$\gamma(s) = \left(0, \int\cos(\alpha(s))ds, \int\sin(\alpha(s))ds\right),$$

with $\alpha(s) = \displaystyle\int \sqrt{\kappa(s)^2-1}\, ds$.

4. Slant curves in $\mathbb{H}^3$ with proper mean curvature vector field

Let M be the warped product $\mathbb{R}\times_\rho\mathbb{R}^2$, $\rho\in C^\infty(\mathbb{R})$, and let γ be a slant curve in M, as before. Denote by h the second fundamental form of γ and by H its mean curvature field. We know that

$$H = \mathrm{trace}(h) = h(T,T) = \nabla_T T. \tag{22}$$

Then γ is called *a curve with proper mean curvature vector field* if there exists $\lambda\in C^\infty(\gamma)$ such that

$$\Delta H = \lambda H. \tag{23}$$

In particular, if $\lambda = 0$ then γ is known as *a curve with harmonic mean curvature vector field*. Here the Laplace operator Δ acts on the vector valued function H and it is given by

$$\Delta H = -\nabla_T\nabla_T\nabla_T T. \tag{24}$$

Making use of Frenet equations, we can rewrite (23) as

$$-3\kappa'\kappa\, T + (\kappa'' - \kappa^3 - \kappa\tau^2)\, N + (2\kappa'\tau + \kappa\tau')\, B = -\lambda\kappa N. \tag{25}$$

It follows that both κ and τ are constants, and the function λ becomes a constant too, namely

$$\lambda = \kappa^2 + \tau^2 \tag{26}$$

(see Theorem 1.1 in [10]).

As consequence, it follows that there is no θ-slant, non-geodesic curves in a 3-dimensional manifold with harmonic mean curvature vector field.

For our framework we state the following

Proposition 3. *A non-geodesic θ-slant curve in an f-Kenmotsu manifold has a proper mean curvature vector field if and only if its curvature and torsion are constants and then*

$$\lambda = \frac{\kappa^2}{\sin^2\theta} - f^2\cos^2\theta + \frac{f'^2\sin^2\theta}{\kappa^2 - f^2\sin^2\theta} - 2f'\cos\theta. \tag{27}$$

Proof. We compute λ of (23) by using (14) and (26). Let us pointed out that the formulae (27) yields that a θ-slant curve with proper mean curvature field has:

$$f^2\cos^2\theta - \frac{f'^2\sin^2\theta}{\kappa^2 - f^2\sin^2\theta} + 2f'\cos\theta = constant. \tag{28}$$

□

Corollary 1. *A Legendre curve in a f-Kenmotsu manifold has proper mean curvature vector field if and only if*

1) its curvature κ is a constant,

2) there exists a constant μ such that $f = \kappa\sin(\mu s)$, $s \in \left(0, \frac{\pi}{2\mu}\right)$.

Then $\tau = -\mu$ and $\lambda = \kappa^2 + \mu^2$.

Proof. The statement 1) follows from Proposition 3. Concerning 2) let us remark that from (28) we have that a Legendre curve with proper mean curvature vector field fullfils $\frac{f'^2}{\kappa^2 - f^2} = constant$. The last equation can be integrated; namely let $\mu > 0$ and from

$$\int \frac{f'(s)ds}{\sqrt{\kappa^2 - f^2(s)}} = \mu$$

we derive that $f(s) = \kappa\sin(\mu s)$ which is strictly positive for $s \in (0, \frac{\pi}{2\mu})$; we use that κ is already constant. Returning to (14) we get $\tau = -\mu$ and then $\lambda = \kappa^2 + \mu^2$.

□

Theorem 3. *Let γ be a θ-slant curve in $\mathbb{K}^3_1$ with proper mean curvature vector field. Then, γ is given, up to translations in $\mathbb{K}^3_1$, by*

$$\gamma(s) = \left(s\cos\theta, \frac{\sin\theta\, e^{-s\cos\theta}}{a^2 + \cos^2\theta}\,(a\sin(as) - \cos\theta\cos(as), -a\cos(as) - \cos\theta\sin(as))\right), \tag{29}$$

where a is an arbitrary constant.

Proof. Since γ is a θ-slant curve, it can be parametrized as in Theorem 2. We know the curvature and the torsion from (19). On the other hand, γ has proper mean curvature vector field, thus κ and τ should be constants. Therefore, α is an affine function. A translation in the parameter s allows us to consider $\alpha(s) = as$, with $a \in \mathbb{R}$. Hence:

$$\gamma(s) = \left(s\cos\theta, \sin\theta\int^s e^{-u\cos\theta}(\cos(au), \sin(au))du\right), \tag{30}$$

Computing the integrals we get the conclusion.

Moreover, $\lambda = a^2 + \sin^2\theta$.

□

Remark 5. The Legendre curves with proper mean curvature vector field in $\mathbb{H}^3$ correspond to Euclidean circles in the (xy)-plane, namely they may be parametrized as

$$\gamma(s) = \left(0, \frac{1}{a}\,\sin(as), \frac{1}{a}\,(1 - \cos(as))\right), \quad a \neq 0. \tag{31}$$

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