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Abstract	We introduce and study a Darboux mate of a given spherical Legendre curve LC for the Euclidean S^2. Also, a triple sequence of curvatures is provided by the higher order derivatives of the last Frenet equation for the frontal of LC. These curvatures are expressed by a recurrence starting with the pair $(-k_1, -k_2, 0)$, where (k_1, k_2) is the classical curvature function of LC. Several examples are discussed, some of them in relationship with the usual theory of regular space curves. The case of Lorentz–Minkowski sphere S^2_1 is sketched only from the point of view of the geodesic curvature.

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The Darboux Mate and the Higher Order Curvatures of Spherical Legendre Curves

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2010 Mathematics Subject Classification 53A04 · 53A45 · 53A55

1 Introduction

The geometry of spherical curves is an important part of the classical theory of space curves. Also, a fresh research is being published, some of it with graphical applications, such as the class of spherical p-elastica curves. If the dimension of the involved sphere permits additional structures (Hermitian or almost contact) then the features of these curves become more interesting.

The present work concerns mainly with spherical Legendre curves related to the Euclidean unit sphere S^2 and we follow mainly the approach of Takahashi Masatomo from [10]. More precisely, we are interested in new curvatures which are naturally associated. There are two classes of such new curvatures: the first class (containing two elements) is provided by the Darboux mate of a given spherical Legendre curve; (2) the second class (containing three elements) is provided by the higher order Frenet equations. All these new curvatures are the contents of the first section below where for obtaining the higher order curvatures we follow the approach of [4]. We point out that a special attention is given to examples, some of them related to the usual theory of regular space curves.

The second section concerns an introduction to the Lorentz–Minkowski case provided by $\mathbb{R}^3$ with the metric of signature $(-, +, +)$. We remark that the computations become more difficult and hence we restrict to only two examples for which the geodesic curvature is computed.

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2 Euclidean Spherical Legendre Curves

The setting of this section is provided by the space $\mathbb{R}^3$ which is an Euclidean vector space with respect to the canonical inner product:

$$\langle u, v\rangle = u^1v^1 + u^2v^2 + u^3v^3, u = (u^1, u^2, u^3), v = (v^1, v^2, v^3) \in \mathbb{R}^3, 0 \leq \|u\|^2 = \langle u, u\rangle. \tag{2.1}$$

Let S^2 be the unit sphere of $\mathbb{E}^3 := (\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ and its unit spherical bundle:

$$T_1S^2 := \{(u, v) \in S^2 \times S^2; \langle u, v\rangle = 0\}$$

which is a compact three-dimensional contact metric manifold; see, for example, [3]. There is a natural action of $O(3)$ on T_1S^2:

$$(A, (u, v)) \in O(3) \times T_1S^2 \to (Au, Av) \in T_1S^2, \quad \langle Au, Av\rangle = \langle u, v\rangle = 0. \tag{2.2}$$

The general notion of Legendre curves associated to a contact form is well-known (cf. [5]), but we will work directly in our framework using the approach of [10] (see also [8]).

Definition 2.1 The smooth map $LC := (\gamma, \nu) : I \subseteq \mathbb{R} \to T_1S^2$, $t \in I \to (\gamma(t), \nu(t))$ is *a spherical Legendre curve* if $\langle\gamma'(t), \nu(t)\rangle = 0$ for all t in the open interval I. The map γ is called *the frontal* and ν is *the dual of* γ.

Since $\mathbb{R}^3$ is endowed also with the cross product $\times$, we define $\mu = \gamma \times \nu$ and hence the triple $\mathcal{F} := \{\gamma, \nu, \mu\}^t$ is a positively oriented *moving frame* along the frontal γ; here t means the transposition, so $\mathcal{F}$ is a column matrix. Its moving equation is provided by the Proposition 2.2. of [10]:

$$\frac{d}{dt}\mathcal{F}(t) = \begin{pmatrix} 0 & 0 & k_1(t) \\ 0 & 0 & k_2(t) \\ -k_1(t) & -k_2(t) & 0 \end{pmatrix}\mathcal{F}(t), \quad \begin{pmatrix} 0 & 0 & k_1(t) \\ 0 & 0 & k_2(t) \\ -k_1(t) & -k_2(t) & 0 \end{pmatrix} \in o(3). \tag{2.3}$$

The pair of smooth functions (k_1, k_2) is called *the curvature* of the spherical Legendre curve $LC = (\gamma, \nu)$. Sometimes, it is more useful to denote a given LC with all its elements as $LC = (\gamma, \nu; \mu)$.

Remark 2.2 (1) For the given LC, there are other three spherical Legendre curves ([10]): (i) $LC^{-1} := (-\gamma, \nu)$ with the curvature $(k_1, -k_2)$; (ii) $LC_{-1} := (\gamma, -\nu)$ with the curvature $(-k_1, k_2)$; (iii) $LC^{-1}_{-1} := (\nu, \gamma)$ with the curvature $(-k_2, -k_1)$.
(2) A matrix $A \in SO(3)$ acts on the set of all curves LC following (2): $LC = (\gamma, \nu) \to LC(A) := (A\gamma, A\nu)$. Since A is special we have ([6, p. 134]) $(A\gamma) \times (A\nu) = A\mu$ and then the curvatures of LC and $LC(A)$ are equal. The reciprocal of this statement is the Uniqueness Theorem of [10].

(3) As in the theory of regular space curves, the matrix Eq. (2.3) can be expressed into a vectorial form:

$$\frac{d}{dt}\mathcal{F}(t) = \Omega(t) \times \mathcal{F}(t) \tag{2.4}$$

with *the Darboux vector field*:

$$\Omega = k_2\gamma - k_1\nu = \mu \times \mu'. \tag{2.5}$$

Its norm $\omega := \sqrt{k_1^2 + k_2^2}$ is strictly positive and then every LC has a *Darboux mate*: $DLC = \left(\frac{1}{\omega}\Omega, \mu; -\frac{1}{\omega}(k_1\gamma + k_2\nu)\right)$. We remark that the frontal $\frac{1}{\omega}\Omega$ of DLC is exactly *the* $+$*-evolute* $Ev(\gamma)$ of the frontal γ as is defined in the relation (4.1) of [10, p. 347] and the fact that DLC is a spherical Legendre curve appears in the proof of Proposition 4.8. of [10, p. 349]. The curvature of the Darboux mate is

$$((Dk)_1, (Dk)_2) = \left(\frac{k_1'k_2 - k_1k_2'}{\omega^2}, \omega\right) \tag{2.6}$$

and if $k_1 \neq 0$, then

$$(Dk)_1 = -\left(\frac{k_2}{k_1}\right)'\left(\frac{k_1}{\omega}\right)^2. \tag{2.7}$$

□

This short note defines new curvature functions for LC. In the following, $n \in \mathbb{N}^*$ is a fixed positive integer.

Definition 2.3 The *n-order Frenet equation* of LC is

$$\frac{d^n}{dt^n}\mu(t) = K_1^n(t)\gamma(t) + K_2^n(t)\nu(t) + K_3^n(t)\mu(t). \tag{2.8}$$

and the triple (K_1^n, K_2^n, K_3^n) is called *the n-curvature of* LC. The *n-Darboux vector field* of LC is

$$\Omega(n) := -K_2^n\gamma + K_1^n\nu. \tag{2.9}$$

We note that the new functions $K_{1;2;3}^n$ are smooth, and from (2.2), it results the first ones are $(K_1^1, K_2^1, K_3^1) = (-k_1, -k_2, 0)$. For the general n, these curvatures are obtained through a recurrence process expressed by (Fig. 1). AQ1

Theorem 2.4 *The higher order curvatures are provided by the following recurrence relation:*

$$\begin{pmatrix} K_1^{n+1} \\ K_2^{n+1} \\ K_3^{n+1} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} K_1^n \\ K_2^n \\ K_3^n \end{pmatrix} - \begin{pmatrix} 0 & 0 & k_1(t) \\ 0 & 0 & k_2(t) \\ -k_1(t) & -k_2(t) & 0 \end{pmatrix}\begin{pmatrix} K_1^n \\ K_2^n \\ K_3^n \end{pmatrix}. \tag{2.10}$$

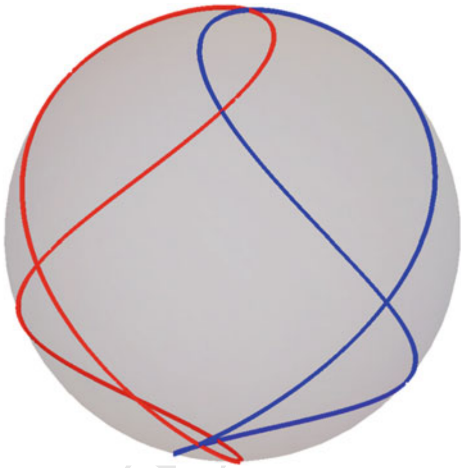

Fig. 1 The first two curves

Hence, the 2*- and* 3*-curvatures are*

$$\begin{cases} (K^2_1, K^2_2, K^2_3) = (-k'_1, -k'_2, -(k^2_1 + k^2_2)) = (-k'_1, -k'_2, -\omega^2), \\ (K^3_1, K^3_2, K^3_3) = (-k''_1 + k_1(k^2_1 + k^2_2), -k''_2 + k_2(k^2_1 + k^2_2), -3(k_1k'_1 + k_2k'_2)). \end{cases} \tag{2.11}$$

The derivative of the n-Darboux vector field is

$$\Omega'(n) = -(K^n_2)'\gamma + (K^n_1)'\nu + (k_2K^n_1 - k_1K^n_2)\mu. \tag{2.12}$$

Example 2.5 Let γ be an usual Frenet curve in $\mathbb{E}^3$ having the curvature function k and the torsion function τ. After [1, p. 2], the curve γ is a Mannheim one if there exist $\lambda \in \mathbb{R}^*$ such that $k = \lambda(k^2 + \tau^2)$. Following this case, we call a spherical Legendre curve LC as being λ-*Mannheim* if

$$k_1 = \lambda(k^2_1 + k^2_2) \tag{2.13}$$

and then the sign of k_1 is that of λ. The curvatures (2.10) become

$$(K^2_1, K^2_2, K^2_3) = \left(-k'_1, -k'_2, -\frac{k_1}{\lambda}\right), (K^3_1, K^3_2, K^3_3) = \left(-k''_1 + \frac{k^2_1}{\lambda}, -k''_2 + \frac{k_1k_2}{\lambda}, -3\frac{k'_1}{\lambda}\right). \tag{2.14}$$

Suppose that $\lambda > 0$. Then $K^3_1 \equiv 0$ if and only if $k_1(t) = \frac{6\lambda}{t^2} > 0$ for $t \in (0, \sqrt{6}\lambda]$ since $k_2(t) = \pm\frac{\sqrt{6(t^2-6\lambda^2)}}{t^2}$. Let us remark that if LC is λ-Mannheim then LC^{-1} is also λ-Mannheim and LC_{-1} is $(-\lambda)$-Mannheim. □

Example 2.6 Let us consider Example 2.8. of [10]. Starting with the natural numbers (k, m, n) satisfying $m = k + n$, the author defines the LC with

$$\gamma(t) = \frac{1}{\sqrt{1+t^{2n}+t^{2m}}}(1, t^n, t^m), \ \nu(t) = \frac{1}{\sqrt{n^2+m^2t^{2k}+k^2t^{2m}}}(kt^m, -mt^k, n), \ t \in \mathbb{R} \tag{2.15}$$

with the associated curvature:

$$k_1(t) = -\frac{t^{n-1}\sqrt{n^2+m^2t^{2k}+k^2t^{2m}}}{1+t^{2n}+t^{2m}}, \quad k_2(t) = \frac{kmnt^{k-1}\sqrt{1+t^{2n}+t^{2m}}}{n^2+m^2t^{2k}+k^2t^{2m}}. \tag{2.16}$$

All the new curvatures are extremely complicated and then we express only

$$\omega^2 = k_1^2(t) + k_2^2(t) = \frac{t^{2n-2}(n^2+m^2t^{2k}+k^2t^{2m})^3 + k^2m^2n^2t^{2k-2}(1+t^{2n}+t^{2m})^3}{(n^2+m^2t^{2k}+k^2t^{2m})^2(1+t^{2n}+t^{2m})^2}. \tag{2.17}$$

□

Example 2.7 Let us consider Example 2.7. of [10]. A spherical curve $\gamma : I \to S^2 \subset \mathbb{R}^3$ yields the spherical Legendre curve $LC(\gamma) = (\gamma, \mathbf{n}; -\mathbf{t})$ with $\mathbf{n} = \gamma \times \mathbf{t}$ for $\mathbf{t}(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$. Then the curvature of $LC(\gamma)$ is related to the geodesic curvature k_g of γ through $k_2 = k_g|k_1|$ for

$$k_g(t) = \frac{\det(\gamma(t), \gamma'(t), \gamma''(t))}{\|\gamma'(t)\|^3} \tag{2.18}$$

while the usual curvature k of γ as space curve is: $k = \sqrt{k_g^2+1}$; if $k_g > 0$ then γ is *convex*. Then we get $k_1(t) = -\|\gamma'(t)\| < 0$ which yields

$$(K_1^2, K_2^2, K_3^2) = \left((\|\gamma'(t)\|)', (\|\gamma'(t)\|k_g)', -\|\gamma'(t)\|^2(1+k_g^2)\right), \quad (Dk)_1 = \frac{k_g'}{1+k_g^2} \tag{2.19}$$

Inspired by the case of regular curves, a spherical Legendre curve will be a λ-*helix* if $k_2 = \lambda k_1$ and then $(Dk)_1 \equiv 0$.

Three remarkable spherical curves are

$$r^1(t) = (-\sin^2 t, \sin t\cos t, \cos t), r^2(t) = (\sin t\cos t, -\cos^2 t, \sin t), r^3(t) = (\cos t, \sin t, 0), \tag{2.20}$$

since these three-dimensional unit vectors are orthogonal two by two: $\langle r_1, r_2\rangle = \langle r_2, r_3\rangle = \langle r_3, r_1\rangle = 0$; the last curve is the equator of S^2. So, these maps can be considered together as the columns of a symmetric matrix Γ from $SO(3)$. With the spinorial formalism of [11, p. 28], there exists two complex-valued curves $\psi_1 = \psi_1(t)$, $\psi_2 = \psi_2(t)$ such that

$$\begin{cases} r^1 + ir^2 = (\psi_1^2 - \psi_2^2, i(\psi_1^2+\psi_2^2), -2\psi_1\psi_2), \\ r^3 = (\overline{\psi_1}\psi_2 + \psi_1\overline{\psi_2}, i(\psi_1\overline{\psi_2} - \overline{\psi_1}\psi_2), |\psi_1|^2 - |\psi_2|^2). \end{cases} \tag{2.21}$$

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A straightforward computation gives the expression of these curves:

$$\psi_1 = constant = \frac{i}{\sqrt{2}}, \quad \psi_2(t) = \frac{i}{\sqrt{2}}e^{it} = \frac{1}{\sqrt{2}}e^{i\left(\frac{\pi}{2}+t\right)}. \tag{2.22}$$

We remark that we have a new spherical curve $t \to (\psi_1, \psi_2(t)) \in S^3$ and then we recall the famous Hopf bundle $H : S^3 \subset \mathbb{C}^2 \to S^2\left(\frac{1}{2}\right) \subset \mathbb{R} \times \mathbb{C}$:

$$H(z, w) = \left(\frac{1}{2}(|z|^2 - |w|^2), z\bar{w}\right). \tag{2.23}$$

It results in the curve:

$$t \to H(\psi_1, \psi_2(t)) = \left(0, \frac{1}{2}e^{-it}\right) \in S^2\left(\frac{1}{2}\right). \tag{2.24}$$

The geodesic and the usual curvatures for the columns of Γ are

$$\begin{cases} k^1_g(t) = \frac{\cos t(2+\sin^2 t)}{(1+\sin^2 t)^{\frac{3}{2}}}, \quad k^2_g(t) = \frac{\sin t(2+\cos^2 t)}{(1+\cos^2 t)^{\frac{3}{2}}}, \quad k^3_g = 0, \\ k^1(t) = \frac{\sqrt{5+3\sin^2 t}}{(1+\sin^2 t)^{\frac{3}{2}}}, \quad k^2(t) = \frac{\sqrt{5+3\cos^2 t}}{(1+\cos^2 t)^{\frac{3}{2}}}, \quad k^3 = 1. \end{cases} \tag{2.25}$$

The lengths of these curves are

$$L^1 = \int_0^{2\pi} \sqrt{1+\sin^2 t}dt \simeq 7.6404, \quad L^2 = \int_0^{2\pi} \sqrt{1+\cos^2 t}dt \simeq 7.6404, \quad L^3 = 2\pi \simeq 6.28$$

while their total curvatures are

$$(Tk)^1 = \int_0^{2\pi} k^1(t)dt \simeq 9.4174, \quad (Tk)^2 = \int_0^{2\pi} k^2(t)dt \simeq 9.4174, \quad (Tk)^3 = L^3 = 2\pi.$$

A spherical curve with prescribed constant geodesic curvature $k_g = K$ and parametrized by the arc length s is

$$r_K(s) = \frac{1}{\sqrt{1+K^2}}\left(\cos(\sqrt{1+K^2}s), \sin(\sqrt{1+K^2}s), K\right) = \frac{1}{k}\left(\cos(ks), \sin(ks), K\right). \tag{2.26}$$

Following the approach of [7, p. 143] for a spherical curve parametrized by arc length, we define *the elastic curvature*

$$k_e := k''_g + \frac{1}{2}(k^3_g + k_g) \tag{2.27}$$

and a curve with a vanishing elastic curve will be called *elastic curve*. Then the elastic curvature of the curve (2.26) will be constant $k_e = \frac{K}{2}(K^2 + 1)$.

We recall also the characterization of a space curve γ to be a spherical one (not necessary on S^2) in terms of classical curvature and torsion:

$$\frac{\tau}{k}+\frac{d}{dt}\left[\frac{1}{\tau}\frac{d}{dt}\left(\frac{1}{k}\right)\right]=0 \tag{2.28}$$

as well as the usual parametrization of the unit sphere S^2:

$$S^2: r(u,v)=(\cos u\cos v, \sin u\cos v, \sin v), \quad u\in[0,2\pi], \quad v\in[-\pi,\pi]. \tag{2.29}$$

Then the above three curves are

$$r^1(t)=r\left(\frac{\pi}{2}+t,\frac{\pi}{2}-t\right), \quad r^2(t)=r\left(\frac{3\pi}{2}+t,t\right), \quad r^3(t)=r(t,0). \tag{2.30}$$

Also, the first curve is connected with the Viviani curve $\mathcal{V}$ which is the intersection of S^2 with the cylinder $x^2+y^2=x$. Since its parametrization is

$$\mathcal{V}(u)=(u,-\sqrt{u-u^2},-\sqrt{1-u})\subset S^2, \quad u\in[0,1] \tag{2.31}$$

it results $r^1(t)=-\mathcal{V}(\sin^2 t)$. The geodesic and the usual curvature of the Viviani curve are

$$k_g^{\mathcal{V}}(u)=-\frac{(2+u)\sqrt{1-u}}{(1+u)^{\frac{3}{2}}}<0, \quad k^{\mathcal{V}}(u)=\frac{\sqrt{5+3u}}{(1+u)^{\frac{3}{2}}} \tag{2.32}$$

and then $\mathcal{V}$ is not convex. The total geodesic and the total curvature of the Viviani curve are

$$\int_0^1 k_g^{\mathcal{V}}(u)du=-1, \quad \int_0^1 k^{\mathcal{V}}(u)du\simeq 1.4602.$$

□

Example 2.8 The simplest LC curve is that containing a constant dual curve:

$$\gamma(t)=(\cos t,\sin t,0)=(e^{it},0), \nu(t)=(0,0,1)=\Omega, \mu(t)=(\sin t,-\cos t,0)=-i\gamma \tag{2.33}$$

with i the complex unit of the plane $\mathbb{R}^2=\mathbb{C}$. It has the constant curvature $(-1,0)$ and then $K_2^n\equiv 0$ for all n. The Darboux mate $DLC=(\nu,\mu;\gamma)$ has the curvature $(0,1)$. Following the Darboux procedure, we have $DDLC=D^{(2)}LC=(\nu,\gamma;-\mu)=LC_{-1}^{-1}$ with the same curvature $(0,1)$. Another steps: $D^{(3)}LC=(\nu,-\mu;-\gamma)$ with the curvature $(0,1)$; $D^{(4)}LC=(\nu,-\gamma;\mu)$ with the curvature $(0,-1)$. It follows a Darboux periodicity of order 4: $D^{(5)}=D$.

The first two curves of the previous example can be joined to obtain a curve in $\mathbb{R}^6=\mathbb{C}^3$ which, in turn, can be put in correspondence with a special class of Legendre immersions. Namely, the Corollary 15 of [9, p. 301] expresses an arbitrary nonminimal biharmonic Legendre immersion $f_L: M^2\to S^5\subset\mathbb{C}^3$ as

$$f_L(x,y) = \frac{1}{\sqrt{2}}\left(e^{ix}, ie^{-ix}\sin(\sqrt{2}y), ie^{-ix}\cos(\sqrt{2}y)\right). \tag{2.34}$$

Then we have

$$(r^1, r^2)(t) = \sqrt{2}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} f_L\left(t, -\frac{t}{\sqrt{2}}\right) \in S^5 \tag{2.35}$$

and the 6×6 matrix above belongs to $SO(6)$.

Moreover, all three curves of Example 2.7 can be joined through the Veronese map:

$$V : S^2 \subset \mathbb{R}^3 \to S^4 \subset \mathbb{R}^5, V(u,v,w) := \left(\sqrt{3}vw, \sqrt{3}wu, \sqrt{3}uv, \frac{\sqrt{3}}{2}(u^2 - v^2), w^2 - \frac{u^2+v^2}{2}\right). \tag{2.36}$$

Then:

$$\begin{cases} V^1(t) := V \circ r^1(t) = \sqrt{3}\left(\sin t\cos^2 t, -\sin^2 t\cos t, -\sin^3 t\cos t, -\frac{1}{2}\sin^2 t\cos 2t, \frac{2\cos^2 t - \sin^2 t}{2\sqrt{3}}\right), \\ V^2(t) := V \circ r^2(t) = \sqrt{3}\left(-\sin t\cos^2 t, \sin^2 t\cos t, -\sin t\cos^3 t, -\frac{1}{2}\cos^2 t\cos 2t, \frac{2\sin^2 t - \cos^2 t}{2\sqrt{3}}\right), \\ V^3(t) := V \circ r^3(t) = \left(0, 0, \frac{\sqrt{3}}{2}\sin 2t, \frac{\sqrt{3}}{2}\cos 2t, -\frac{1}{2}\right), \end{cases} \tag{2.37}$$

and hence: $V^1 + V^2 + V^3 = 0 \in \mathbb{R}^5$. The angle between two distinct unit vectors V^i, $i = 1, 2, 3$ is constant, namely $\frac{2\pi}{3}$.

Following [2] for a given spherical curve γ, we consider the spherical angular momentum with respect to the vertical axis:

$$\mathcal{K}(t) := -\det(\gamma(t), \mathbf{t}(t), \bar{k}) = \frac{x'(t)y(t) - x(t)y'(t)}{\|\gamma'(t)\|}. \tag{2.38}$$

This momentum for the curves (2.20) is

$$\mathcal{K}^1(t) = -\frac{\sin^2 t}{\sqrt{1+\sin^2 t}} \le 0, \quad \mathcal{K}^2(t) = -\frac{\cos^2 t}{\sqrt{1+\cos^2 t}} \le 0, \quad \mathcal{K}^3 = -1. \tag{2.39}$$

□

Example 2.9 Return to Example 2.7. Fix a regular curve $r : I \to \mathbb{R}^3$ parametrized by arc-length s and having the curvature function k and the torsion function τ; let $(T = r', N, B)$ its Frenet frame. Then its *tangent indicatrix* $\gamma := r'$ will be a spherical curve with the geodesic curvature $k_g^T = \frac{\tau}{k}$; then γ is convex if and only if $\tau > 0$. Following Example 2.7, we get the spherical Legendre curve $TLC := (\gamma = r' = T, \mathbf{n} = B; -\mathbf{t} = -N)$ with the curvature:

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$$k_1^T = -k < 0, \quad k_2^T = \tau. \tag{2.40}$$

The Darboux vector field (2.5) is exactly the Darboux vector field of r and if this initial curve r is λ-Mannheim (or λ-helix) then TLC is $(-\lambda)$-Mannheim (respectively, $(-\lambda)$-helix). □

3 A Lorentz–Minkowski Variant of Spherical Legendre Curves

The setting of this work is provided by the Lorentz–Minkowski 3-space $E_1^3 := (\mathbb{R}^3, \langle u, v \rangle_L, \times_L)$ which is a pseudo-Euclidean vector space with

$$\langle u, v \rangle_L = -u^1 v^1 + u^2 v^2 + u^3 v^3, \quad u \times_L v = \begin{vmatrix} -\bar{i} & \bar{j} & \bar{k} \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{vmatrix}. \tag{3.1}$$

It follows two types of unit spheres:

$$S_1^2 := \{u \in \mathbb{R}^3; \langle u, u \rangle_L = +1\}, \quad H_1^2 := \{u \in \mathbb{R}^3; \langle u, u \rangle_L = -1\} \tag{3.2}$$

and the norm is defined through: $0 \leq \|u\|_L^2 = |\langle u, u \rangle_L|$. Following the model of [10] and looking at Example 3.2 below, we denote $\Delta^+ := \{(u, v) \in S_1^2 \times H_1^2; \langle u, v \rangle_L = 0\}$ and then we introduce.

Definition 3.1 A *positive spherical Lorentz–Legendre curve* is a pair $LLC^+ := (\gamma, \nu) : I \subseteq \mathbb{R} \to \Delta^+$ of smooth positive spherical curves with $\langle \gamma'(t), \nu(t) \rangle_L = 0$ for all t in the open interval I.

Fix now a smooth positive spherical curve $\gamma : I \to S_1^2$ which we call *regular* if $\|\gamma'(t)\|_L > 0$ for all $t \in I$. As in the Euclidean case, its unit tangent vector field and normal vector field are

$$\mathbf{t}(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|_L}, \quad \mathbf{n}(t) := \gamma(t) \times_L \mathbf{t}(t) \tag{3.3}$$

which yields the moving frame $\mathcal{F} := \{\gamma, \mathbf{t}, \mathbf{n}\}$ with the moving equations:

$$\frac{d}{dt}\mathcal{F}(t) = \|\gamma'(t)\|_L \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & k_g^L(t) \\ 0 & k_g^L(t) & 0 \end{pmatrix} \mathcal{F}(t), \quad k_g^L(t) := \frac{\langle \mathbf{t}'(t), \mathbf{n}(t) \rangle_L}{\|\gamma'(t)\|_L \|\mathbf{n}(t)\|_L^2} \tag{3.4}$$

since $a \times_L (b \times_L c) = -< a, c >_L b + < a, b >_L c$ for any $a, b, c \in E_1^3$; we suppose also $\|\mathbf{n}(t)\|_L^2 \neq 0$. We call k_g^L as being *the Lorentz geodesic curve* of γ. It follows a positive spherical Lorentz–Legendre curve $LLC = (\gamma, \mathbf{n})$.

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Example 3.2 Inspired by the curve r^1 of Example 2.7, let

$$\gamma(t) = (\sin t \sinh t, \sin t \cosh t, \cos t) \in S^2_1 \tag{3.5}$$

having then:

$$\begin{cases} \gamma'(t) = (\cos t \sinh t + \sin t \cosh t, \cos t \cosh t + \sin t \sinh t, -\sin t), \\ \langle \gamma'(t), \gamma'(t) \rangle_L = \cos^2 t \end{cases} \tag{3.6}$$

and then it is regular for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; both γ and $\mathbf{t}$ belong to S^2_1. Then $\|\gamma'(t)\|_L = \cos t$ and a straightforward computation gives that the normal is a timelike unit vector field, i.e., belongs to H^2_1:

$$\begin{cases} \mathbf{n}(t) = (\frac{\cosh t}{\cos t} + \sin t \sinh t, \frac{\sinh t}{\cos t} + \sin t \cosh t, -\sin t \tan t) \\ \langle \mathbf{n}(t), \mathbf{n}(t) \rangle_L = -1 \end{cases} \tag{3.7}$$

giving the geodesic curvature:

$$k^L_g(t) = \frac{2}{\cos t} \geq 0. \tag{3.8}$$

□

Example 3.3 A last example is

$$\gamma(t) = (\sin t, \cos t, \sqrt{2} \sin t) \in S^2_1, \quad t \in \mathbb{R}, \tag{3.9}$$

with

$$\begin{cases} \gamma'(t) = (\cos t, -\sin t, \sqrt{2} \cos t), \quad < \gamma'(t), \gamma'(t) >_L = 1, \\ \mathbf{n} = constant = (-\sqrt{2}, 0, -1), \quad < \mathbf{n}, \mathbf{n} >_L = -1, \quad k^L_g = 0. \end{cases} \tag{3.10}$$

□

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