The Hopf-Levi-Civita Data of Two-Dimensional Metrics

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Abstract The aim of this chapter is to study three 2-dimensional geometries, namely Riemannian, Kähler and Hessian, in a unitary way by using three (local) Hermitian matrices. One of these matrices corresponds to the symmetric matrix of metric while the other two Hermitian matrices is provided by the Christoffel symbols. The secondary diagonal of these Hermitian matrices are generated by the Hopf invariant and its conjugate, where for this notion we adopt the definition of Jensen et al. (Surfaces in classical geometries. A treatment by moving frames. Springer, Cham, 2016 [13]). In the Riemannian case a special view is towards an expression of the Gaussian curvature in terms of these data while in Kähler and Hessian geometry we use the corresponding potential function and a new (again local) differential operator of first order, similar to ∂.

Keywords Two-dimensional riemannian manifold · Hermitian data · Hopf-Levi-Civita data · Kähler and Hessian geometry

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1 Introduction

In the very recent paper [5] we associate to a symmetric (real) matrix of order two:

$$\Gamma = \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix} \tag{1.1}$$

the Hermitian matrix of the same order:

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$$\Gamma^c = \begin{pmatrix} B & 2\bar{A} \\ 2A & B \end{pmatrix}, \quad A = \frac{r_{11} - r_{22}}{4} - i\frac{r_{12}}{2}, \quad 2B = r_{11} + r_{22} = Tr\Gamma. \tag{1.2}$$

From the linear algebra point of view the bijective association $\Gamma \leftrightarrow \Gamma^c$ is useful since it preserves both the trace and the determinant of the initial matrix:

$$Tr\Gamma = Tr\Gamma^c, \quad \det\Gamma = \det\Gamma^c \tag{1.3}$$

and then these matrices are equivalent.

In fact, if Γ is the matrix of the second fundamental form of a surface then the expression $2A$ was considered by H. Hopf in his 1956 Lectures from Stanford University and appears at page 137 in the book [12]. The Lemma 2.2 from page 140 of Hopf's book is the characterization of CMC surfaces in terms of holomorphy of this complex valued function. For this reason, in [13, p. 56] this expression is called the Hopf invariant and is denoted by h. Therefore we define for the general symmetric matrix Γ:

(i) *the Hopf invariant*: $h(\Gamma) := 2A$, (ii) *the mean curvature*: $H(\Gamma) := B$, and hence the associated Hermitian matrix becomes:

$$\Gamma^c = \begin{pmatrix} H & \bar{h} \\ h & H \end{pmatrix}. \tag{1.4}$$

The aim of this chapter is to extend this approach to two-dimensional Riemannian, Kähler and Hessian geometries on a manifold M. We obtain three Hermitian matrices of smooth (local) functions associated to a given metric g and its Levi-Civita connection ∇. As an application, in the Riemannian case we produce new expressions of the Gaussian curvature $K(g)$ in terms of these complex-valued functions. The isothermal and warped metrics are the main Riemannian examples and a special formula is obtained in normal coordinates around a fixed point $p \in M$. In the isothermal case the holomorphy of Hopf-Levi-Civita invariants h^1, h^2 is characterized by the flatness of the metric g. In the warped case the real valued Gaussian curvature is obtained from the product of two pure complex functions in Formula (2.26). Also, the Hamilton cigar soliton, a projective metric and generalized Poincaré metrics are considered.

The second section concerns with the regular oriented surfaces M in the Euclidean 3-dimensional geometry. In addition to the symmetric matrices of induced metric and corresponding Christoffel symbols we have the symmetric matrices of the second fundamental form b and the shape (Weingarten) operator S. Since the Hopf characterization of CMC surfaces through the holomorphy of the Hopf invariant $h(b)$ is proved by using isothermal coordinates we work in a general (non-isothermal) chart and obtain only a necessary condition for the holomorphy of both $h(g)$ and $h(b)$. The surfaces with holomorphic h^g are obtained and the transport of this holomorphy from h^g to $\overline{h^g}$ through the (vertical) Wick rotation $y \to iy$ is studied.

This technique extends naturally to Kähler and Hessian geometries. For our computations we use their common feature to be generated by a potential ρ and the two dimensional unit disk is treated in both geometries. We remark the necessity to define a (local) differential operator of first operator similar to ∂ used in complex analysis of a single variable.

Inspired by the Hessian geometry we return to the general two-dimensional metrics by studying the Hopf invariants of other two remarkable symmetric covariant fields: the Hessian of a given smooth function [7] and the Lie derivative of the metric with respect to a given vector field. In the last case we establish, via the Hopf invariant, a relationship between (conformal) Killing vector fields $X = (X^1, X^2)$ and the holomorphy of the function $f_X = X^1 + iX^2$ for the case of isothermal metrics. We compute these holomorphic f_X for all constant curvature metrics and for the Hamilton cigar soliton.

The last section studies the transformation of the Christoffel-Hopf data in the case of two Riemannian metrics (or general symmetric linear connections) in subgeodesics correspondence. As important particular cases we mention the geodesic or projective correspondence and the conformal correspondence. We finish with the determination of a flat and projective Euclidean connection.

2 The Hopf-Levi-Civita Data for Two-Dimensional Riemannian Metrics

Fix a two-dimensional Riemannian manifold (M^2, g) and an atlas on it providing the local coordinates $(x, y) = (x^1, x^2)$ on any local chart U. Then the restricted metric on U is expressed as $g|_U = (g_{ij}(x, y))_{i,j=1,2}$ with $g_{ij} \in C^\infty(U, \mathbb{R})$. The discussion above suggests the following notion:

Definition 2.1 The *Hermitian data* of the pair $(U, g|_U)$ is:

$$g^c := \left(h^g = \frac{g_{11} - g_{22}}{2} - i g_{12} \in C^\infty(U, \mathbb{C}),\ H^g = \frac{g_{11} + g_{22}}{2} \in C^\infty(U, \mathbb{R}) \right). \tag{2.1}$$

The Hermitian data is useful to express the Beltrami coefficient μ_g of g. Namely, after [25, p. 142] the expression of metric g in the complex coordinate $z := x + iy$ is:

$$g = \lambda_g |dz + \mu_g d\bar{z}|^2, 4\lambda_g = g_{11} + g_{22} + 2\sqrt{\det g} > 0, \mu_g = \frac{g_{11} - g_{22} + 2i g_{12}}{g_{11} + g_{22} + 2\sqrt{\det g}} \tag{2.2}$$

with $\|\mu_g\|_{L^\infty(U)} < 1$. Then:

$$2\lambda_g = H^g + \sqrt{\det g}, \quad \mu_g = \frac{\overline{h^g}}{2\lambda_g}. \tag{2.3}$$

Let ∇ be the Levi-Civita connection of g and its Christoffel symbols $(\Gamma^k_{ij}(x, y))_{i,j,k=1,2}$ on the open $U \subseteq M$. Similar to the object above we introduce:

Definition 2.2 The *Hopf-Levi-Civita data* of $(U, g|_U)$ is the pair:

$$(\Gamma^k)_{k=1,2} = (\Gamma^1, \Gamma^2) := \left((h^1, H^1), (h^2, H^2)\right),$$

$$\begin{cases} h^k := \frac{1}{2}(\Gamma^k_{11} - \Gamma^k_{22}) - i\Gamma^k_{12} \in C^\infty(U, \mathbb{C}), \\ H^k := \frac{1}{2}(\Gamma^k_{11} + \Gamma^k_{22}) \in C^\infty(U, \mathbb{R}). \end{cases} \tag{2.4}$$

The pair (h^1, h^2) is *the Christoffel-Hopf data of* g while the pair (H^1, H^2) is *the Christoffel-mean curvature data of* g.

A first application of this data concerns with the geodesics of g which are provided by the well-known equation, [8, p. 5660]:

$$\frac{d^2y}{dx^2} = \Gamma^1_{22}\left(\frac{dy}{dx}\right)^3 + \left(2\Gamma^1_{12} - \Gamma^2_{22}\right)\left(\frac{dy}{dx}\right)^2 + \left(\Gamma^1_{11} - 2\Gamma^2_{12}\right)\frac{dy}{dx} - \Gamma^2_{11}. \tag{2.5}$$

It follows:

$$\frac{d^2y}{dx^2} = (H^1 - \Re h^1)\left(\frac{dy}{dx}\right)^3 + \left(\Re h^2 - 2\Im h^1 - H^2\right)\left(\frac{dy}{dx}\right)^2 +$$
$$+\left(H^1 + \Re h^1 + 2\Im h^2\right)\frac{dy}{dx} - \left(H^2 + \Re h^2\right)$$

where as usual $\Re z$ and $\Im z$ denotes the real and complex part respectively of a complex number z. The presence in a projective geometry of quantities Γ_{ijk} symmetric in the last two indices is pointed out in [23, p. 347].

For practical applications we restrict our formulae to some particular cases of the metric. Namely, we consider the diagonal case of metric, i.e. $g_{12} = 0$, which implies that h^g is also real-valued as H^g:

$$h^g(x, y) = \frac{1}{2}\left(g_{11}(x, y) - g_{22}(x, y)\right) \tag{2.6}$$

and the Christoffel symbols are provided by the following relations where, as usually, the subscript denotes the derivative with respect to the corresponding variable:

$$\Gamma^1_{11} = \frac{(\ln g_{11})_x}{2}, \quad \Gamma^1_{12} = \frac{(\ln g_{11})_y}{2}, \quad \Gamma^1_{22} = -\frac{(g_{22})_x}{2g_{11}}, \tag{2.7}$$

$$\Gamma^2_{11} = -\frac{(g_{11})_y}{2g_{22}}, \quad \Gamma^2_{12} = \frac{(\ln g_{22})_x}{2}, \quad \Gamma^2_{22} = \frac{(\ln g_{22})_y}{2} \tag{2.8}$$

which yield by a straightforward computation:

Proposition 2.3 *The Hopf-Levi-Civita data on U of a diagonal metric g is given by:*

$$\begin{cases} h^1 = \frac{1}{2g_{11}}\left[H^g_x - i(g_{11})_y\right], & H^1 = \frac{h^g_x}{g_{11}}, \\ h^2 = \frac{-1}{2g_{22}}\left[H^g_y + i(g_{22})_x\right], & H^2 = \frac{-h^g_y}{g_{22}}. \end{cases} \tag{2.9}$$

In particular, if g_{kk} depends only of x^k for $k \in \{1, 2\}$ then the Christoffel-Hopf data is also real-valued:

$$h^1 = \frac{H^g_x}{2g_{11}} = \frac{(g_{11})_x}{4g_{11}} = \frac{H^1}{2}, \quad h^2 = \frac{-H^g_y}{2g_{22}} = \frac{-(g_{22})_y}{4g_{22}} = \frac{H^2}{2}. \tag{2.10}$$

Remark 2.4 The Formulae (2.9) can be unified in:

$$h^k = \frac{(-1)^{k+1}}{2g_{kk}}\left[H^g_{x^k} + (-1)^k i(g_{kk})_{x^{k+1}}\right], \quad H^k = \frac{(-1)^{k+1}h^g_{x^k}}{g_{kk}} \tag{2.11}$$

where at the derivative with respect to x^{k+1} from h^k the power $k+1$ is considered modulo 2. We remark that if g_{11} depends only on x and g_{22} depends only on y then the diagonal metric g can be reduced to the Euclidean form. Indeed, with $g_{11} = A^2(x)$ and $g_{22} = B^2(y)$ we change the coordinates according to $du = Adx$ and $dv = Bdv$ obtaining $g = du^2 + dv^2$. □

We discuss now large classes of examples and apply the new tools in their study.

Example 2.5 Suppose that the given atlas is an *isothermal* or *conformal* one i.e. $g_{11} = g_{22} = E(x, y) = H^g$. Then $h^g = H^1 = H^2 = 0$ which means that the Christoffel-mean curvature data vanishes. Maybe this fact offers an explanation for the suitability of isothermal coordinates in the geometry of minimal surfaces. The Hopf parts are:

$$h^1 = \frac{1}{2E}\left(E_x - iE_y\right), \quad h^2 = \frac{-1}{2E}\left(E_y + iE_x\right) = -ih^1. \tag{2.12}$$

The complex coordinate $z = x + iy$ together with its conjugate $\bar{z} = x - iy$ yields *the Wirtinger derivatives*:

$$\partial = \partial_z := \frac{1}{2}\left(\partial_x - i\partial_y\right), \quad \bar{\partial} = \partial_{\bar{z}} := \frac{1}{2}\left(\partial_x + i\partial_y\right). \tag{2.13}$$

It follows for E considered as a function of $(z, \bar{z})$:

$$h^1(z, \bar{z}) = \frac{E_z}{E} = (\ln E)_z, \quad h^2(z, \bar{z}) = \frac{-iE_z}{E} \tag{2.14}$$

and hence an isothermal anti-holomorphic metric, i.e. satisfying $E_z = 0$, has a vanishing Hopf-Levi-Civita data.

Recall that the Gaussian (sectional) curvature K of g expressed in isothermal coordinates reads:

$$K = -\frac{\Delta \ln E}{2E} = -\frac{2\partial\overline{\partial}(\ln E)}{E} \tag{2.15}$$

with Δ being the usual Laplacian in real coordinates, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 = 4\partial\overline{\partial} = 4\overline{\partial}\partial$. In conclusion:

$$K(z, \bar{z}) = -\frac{2\overline{\partial}h^1}{E} \tag{2.16}$$

and hence flat metrics are characterized by holomorphic $h^1 = h^1(z)$, and consequently holomorphic h^2. In conclusion, a flat isothermal metric has a holomorphic Christoffel-Hopf data.

Returning to the general real case we note the remarkable particular case of *generalized Poincaré metric* $E = E(y)$ for which:

$$h^1 = \frac{-iE_y}{2E} = \frac{-i}{2}(\ln E)_y, \quad h^2 = \frac{-E_y}{2E} = \frac{-1}{2}(\ln E)_y, \quad K = \frac{h^2_y}{E}. \tag{2.17}$$

For the classical example of Poincaré metric $E(y) = y^{-2}$ we get: $h^1 = \frac{i}{y}, h^2 = \frac{1}{y}$.

Another remarkable example of diagonal isothermal metric is the Hamilton cigar metric which is a steady gradient Ricci soliton supported on $M = \mathbb{R}^2$ by the metric g_{cigar}: $g_{11} = g_{22} = \frac{1}{1+x^2+y^2}$. Its Christoffel symbols are computed in [1, p. 24] and then it results $h^1 = \frac{-\bar{z}}{1+|z|^2}$. □

Example 2.6 We remark that the equality $h^2 = -ih^1$ of (2.12) holds if we ask the vanishing of right-hand side of (2.5). This means that all geodesics of (M^2, g) are lines $x = constant$ and $y = ax + b$ and hence (M^2, g) is a *projective Euclidean space*. Indeed, from:

$$\Gamma^1_{22} = \Gamma^2_{11} = 0, \quad \Gamma^2_{22} = 2\Gamma^1_{12}, \quad \Gamma^1_{11} = 2\Gamma^2_{12} \tag{2.18}$$

it results:

$$h^1 = \Gamma^2_{12} - i\Gamma^1_{12}, \quad h^2 = -\Gamma^1_{12} - i\Gamma^2_{12} = -ih^1. \tag{2.19}$$

As an example of projective Euclidean space we have the unit sphere $S^2 = S^2(1)$ with its round metric provided by the central projection:

$$\begin{cases} g_{proj} = \frac{1+y^2}{(1+x^2+y^2)^2}dx^2 - \frac{2xy}{(1+x^2+y^2)^2}dxdy + \frac{1+x^2}{(1+x^2+y^2)^2}dy^2, \\ h^{g_{proj}} = \frac{-\bar{z}^2}{2(1+|z|^2)^2} \end{cases} \tag{2.20}$$

with the only non-zero Christoffel symbols [16, p. 471] (the case [3.4]):

$$\Gamma^1_{11} = 2\Gamma^2_{12} = \frac{-2x}{1+x^2+y^2}, \quad \Gamma^2_{22} = 2\Gamma^1_{12} = \frac{-2y}{1+x^2+y^2} \tag{2.21}$$

and hence $h^1(z, \bar{z}) = \frac{-\bar{z}}{1+|z|^2}$, exactly as the cigar soliton. This means that in spite of different Hermitian data the metrics g_{proj} and g_{cigar} have the same Christoffel-Hopf data (h^1, h^2). The Eq. (2.5) for the cigar soliton is decomposable:

$$\frac{d^2y}{dx^2} = \frac{1}{1+x^2+y^2}\left(x\frac{dy}{dx} - y\right)\left[\left(\frac{dy}{dx}\right)^2 + 1\right] \tag{2.22}$$

and it follows that the lines through origin, of equations $y = ax$ with a real parameter a, are geodesics of g_{cigar}. The study of geodesics for g_{cigar} in polar coordinates (r, θ) appears in [1, p. 25] and we note that in polar coordinates g_{proj} is:

$$g_{proj} = \frac{dr^2}{(1+r^2)^2} + \frac{r^2 d\theta^2}{1+r^2}, \quad K_{g_{proj}} = 1. \tag{2.23}$$

A relationship between g_{proj} and the potential f^{cigar} of Hamilton cigar soliton will be discussed in Example 6.4. We also point out that g_{proj} can be written as:

$$g_{proj} = \frac{1}{(1+x^2+y^2)^2}[(dx^2+dy^2) + (ydx - xdy)^2] \tag{2.24}$$

and we remark the occurrence of the numerator of classical 1-form:

$$\eta = \frac{xdy - ydx}{x^2+y^2} = -d\theta = \frac{i}{2}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right)$$

which is closed but not exact on the punctured plane $\mathbb{R}^2 \setminus \{O(0,0)\}$. But the restriction of η to, say, the right half-plane $x > 0$ is an exact form.

The classical differential system of geodesics for g_{proj} integrates to:

$$\dot{x} = C_1(1+x^2+y^2), \quad \dot{y} = C_2(1+x^2+y^2), \quad C_1, C_2 \in \mathbb{R}$$

and its general solution is:

$$x(t) = \tan(Ct)\cos\varphi, \quad y(t) = \tan(Ct)\sin\varphi, \quad C, \varphi \in \mathbb{R}.$$

□

Example 2.7 Suppose that the given metric is a *warped* one:

$$g_{11} = 1, \quad g_{22} = f^2(x), \quad f \in C^\infty(I \subseteq \mathbb{R}, \mathbb{R}^*_+). \tag{2.25}$$

Then its Hopf-Levi-Civita data is:

$$2h^1 = ff_x = -2H^1, \quad h^2 = \frac{-if_x}{f}, \quad H^2 = 0. \tag{2.26}$$

Its Gaussian curvature is:

$$K(x, y) = \left(\frac{f_x}{f}\right)^2 - \frac{2h^1_x}{f^2} = -(h^2)^2 - ih^2(\ln h^1)_x = -h^2[h^2 + i(\ln h^1)_x] \tag{2.27}$$

and then the real-valued function K is a product of two pure complex-valued functions; as product of real-valued functions we have: $K = \Im h^2[\Im h^2 + (\ln h^1)_x]$. Some interesting particular cases are:

(2.6.1) the Euclidean plane geometry is provided in polar coordinates by $f(x) = x$ and hence: $h^1 = \frac{x}{2}$, $h^2 = \frac{-i}{x}$,
(2.6.2) the canonical metric of the sphere S^2 is provided by $f(x) = \sin x$ and hence: $h^1 = \frac{\sin 2x}{4}$, $h^2 = -i \cot x$,
(2.6.3) we generalize the above two metrics following the approach of rotationally symmetric metrics from [21, pp. 17–18] based on the functions sn_k and $cs_k = sn'_k$ for a general parameter $k \in \mathbb{R}$; also $cs'_k = -ksn_k$. The warped metric is given by $f(x) = sn_k(x)$ and then:

$$h^1 = \frac{1}{2}sn_k(x)cn_k(x), \quad h^2 = \frac{-icn_k(x)}{sn_k(x)}. \tag{2.28}$$

□

Example 2.8 Another example of diagonal metric is provided by metrics in geodesics polar coordinates when $g_{11} = 1$ and $g_{22} = G(x, y)$. With the computations of [8, p. 5661] we have:

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^2_{11} = 0, \quad \Gamma^1_{22} = -\frac{G_x}{2}, \quad \Gamma^2_{12} = \frac{G_x}{2G}, \quad \Gamma^2_{22} = \frac{G_y}{2G}. \tag{2.29}$$

Then:

$$h^1 = \frac{G_x}{4} = -H^1, \quad h^2 = \frac{-(G_y + 2iG_x)}{4G}, \quad H^2 = \frac{G_y}{4G}. \tag{2.30}$$

The Gaussian curvature is expressed in terms of h^1 as:

$$K = \left(\frac{2h^1}{G}\right)^2 - \frac{2h^1_x}{G}. \tag{2.31}$$

□

Example 2.9 The most simple Riemannian metric is the Euclidean one. Suppose that M^2 is parallelizable with the basis $E = \{E_1, E_2\}$ and define the metric g making orthonormal this frame. If the structure coefficients C are defined by $[E_i, E_j] = C^k_{ij} E_k$ then the Levi-Civita connection is:

$$\Gamma^k_{ij} = \frac{1}{2}[C^k_{ij} - C^i_{jk} + C^j_{ki}]. \tag{2.32}$$

But only $[E_1, E_2]$ can be different to zero; then $\Gamma^1_{11} = \Gamma^2_{22} = \Gamma^2_{12} = 0$ and:

$$\Gamma^1_{22} = C^2_{12}, \Gamma^1_{12} = -\Gamma^2_{11} = C^1_{12}, \quad h^1 = -\frac{C^2_{12}}{2} - iC^1_{12}, h^2 = -\frac{C^1_{12}}{2} \tag{2.33}$$

and hence its Gaussian curvature is:

$$K = R^1_{122} = (C^2_{12})_x - (C^1_{12})_y - (C^1_{12})^2. \tag{2.34}$$

The Eq. (2.5) of its geodesics is:

$$\frac{d^2y}{dx^2} = C^2_{12}\left(\frac{dy}{dx}\right)^3 + 2C^1_{12}\left(\frac{dy}{dx}\right)^2 - C^1_{12}. \tag{2.35}$$

For the example of solvable non-Abelian case i.e. $C^1_{12} = 1$, $C^2_{12} = 0$ we have the pseudospherical metric with $K = -1$ and the Riccati equation:

$$\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2 + 1. \tag{2.36}$$

With WolframAlpha it results the solution:

$$y(x) = c_2 - \frac{1}{2}\ln[\cos(\sqrt{2}x + c_1)] \tag{2.37}$$

where c_1 and c_2 are real constants. The classical differential system of geodesics is:

$$\ddot{x} + 2\dot{x}\dot{y} = 0, \quad \ddot{y} - \dot{x}^2 = 0. \tag{2.38}$$

with the quadratic first integral: $\dot{x}^2 + 2\dot{y}^2$ =constant. □

Returning to the general case let $(x^1, x^2) \to (\tilde{x}^1, \tilde{x}^2)$ be a change of local coordinates on M. Recall the change of Christoffel symbols:

$$\tilde{\Gamma}^k_{ij}\frac{\partial x^d}{\partial \tilde{x}^k} = \frac{\partial^2 x^d}{\partial \tilde{x}^i \partial \tilde{x}^j} + \Gamma^d_{ab}\frac{\partial x^a}{\partial \tilde{x}^i \partial \tilde{x}^j}. \tag{2.39}$$

It follows:

$$2\tilde{H}^k \frac{\partial x^d}{\partial \tilde{x}^k} = \tilde{\Delta}_{can} x^d + \Gamma^d_{ab} \left(\frac{\partial x^a}{\partial \tilde{x}^1} \frac{\partial x^b}{\partial \tilde{x}^1} + \frac{\partial x^a}{\partial \tilde{x}^2} \frac{\partial x^b}{\partial \tilde{x}^2} \right) \tag{2.40}$$

where $\tilde{\Delta}_{can}$ is the Laplacian (of Euclidean type) with respect to tilde coordinates: $\tilde{\Delta}_{can} = \frac{\partial^2}{\partial(\tilde{x}^1)^2} + \frac{\partial^2}{\partial(\tilde{x}^2)^2}$. Hence the possible tensorial character of data (H^1, H^2) remains an open problem.

Since we arrive at the necessity to discuss special types of coordinates we recall after the Exercise 6 of pp. 56–57 from the second edition of [21] that fixed a point $p \in M$ and its *normal coordinates* then the metric g is expressed as:

$$g_{11} = 1 - K(p)y^2, \quad g_{12} = K(p)xy, \quad g_{22} = 1 - K(p)x^2. \tag{2.41}$$

It follows that in normal coordinates around p we have a geometrical interpretation of the pair (h^g, H^g) in terms of Gaussian curvature:

$$h^g = \frac{K(p)}{2}(x - iy)^2 = \frac{K(p)}{2}\bar{z}^2, H^g = 1 - \frac{K(p)}{2}(x^2 + y^2) = 1 - \frac{K(p)}{2}|z|^2 \tag{2.42}$$

and the Beltrami data is:

$$4\lambda_g = (1 + \sqrt{1 - K(p)|z|^2})^2 \geq 1, \mu_g = K(p) \left[\frac{z}{1 + \sqrt{1 - K(p)|z|^2}} \right]^2. \tag{2.43}$$

3 The Second-Hopf and Hopf-Shape Data for a Regular Surface

Suppose now that (M^2, g, i) is an oriented regular surface in Euclidean 3-geometry i.e. $i : (M, g) \to \mathbb{E}^3 = (\mathbb{R}^3, g_{can})$ is an isometric immersion. Let $b = (b_{ij})_{i,j=1,2}$ be the second fundamental form and $S = (s_{ij})_{i,j=1,2}$ the corresponding shape operator. In addition to the Hermitian and Hopf-Levi-Civita data we have:

Definition 3.1 The *second-Hopf data* of $(M, g)|_U$ is:

$$b^c := \left(h^b = \frac{b_{11} - b_{22}}{2} - i b_{12} \in C^\infty(U, \mathbb{C}), H^b = \frac{b_{11} + b_{22}}{2} \in C^\infty(U, \mathbb{R}) \right). \tag{3.1}$$

The *Hopf-shape data* of $(M, g)|_U$ is:

$$S^c := \left(h^S = \frac{s_{11} - s_{22}}{2} - is_{12} \in C^\infty(U, \mathbb{C}), H^S = \frac{s_{11} + s_{22}}{2} \in C^\infty(U, \mathbb{R}) \right). \tag{3.2}$$

Then the mean curvature $H(g)$ of (M^2, g) is exactly H^S. We note that both h^g and h^b are real valued if and only if the parametric lines are exactly the curvatures lines of M. Suppose now that g is expressed in isothermal coordinates; for example, the Enneper surface and the catenoid are minimal surfaces and usually expressed in isothermal coordinates. The Hopf characterization of CMC surfaces through the holomorphy of the function h^b is also Proposition 1.4 from [15, p. 21]. Moreover, on a non-minimal CMC surface there exists a special type of isothermal coordinates provided by Theorem 1.5 of the same book (p. 22); in this system of coordinates we have:

$$g_{11} = g_{22} = \frac{e^\omega}{2H}, \quad b_{11} = e^\omega \cosh \omega, \quad b_{12} = 0, \quad b_{22} = e^\omega \sinh \omega \tag{3.3}$$

with ω a solution of the sinh-Gordon equation. It follows immediately the Hopf parts:

$$h^b = \frac{1}{2}, \quad h^S = He^{-2\omega}, \quad H^b = \frac{e^{2\omega}}{2}. \tag{3.4}$$

For the general case of non-isothermal coordinates we can derive a necessary condition for the holomorphy of h^b as well as the (Euclidean) gradient vector field of H^b.

Proposition 3.2 *Suppose that the local chart (x, y) on M^2 is not isothermal and the function h^b is holomorphic. Then:*

$$(\Gamma^k_{12} b_{k2} - \Gamma^k_{22} b_{k1})_y = (\Gamma^k_{12} b_{k1} - \Gamma^k_{11} b_{k2})_x, \tag{3.5}$$

and the Euclidean gradient $\nabla_{can} H^b := (H^b_x, H^b_y)$ of H^b is:

$$\nabla H^b = (\Gamma^k_{12} b_{k2} - \Gamma^k_{22} b_{k1}, \Gamma^k_{12} b_{k1} - \Gamma^k_{11} b_{k2}). \tag{3.6}$$

Proof The Cauchy-Riemann equations for h^b are:

$$\frac{1}{2}(b_{11} - b_{22})_x = -(b_{12})_y, \quad \frac{1}{2}(b_{11} - b_{22})_y = (b_{12})_x \tag{3.7}$$

and the general Codazzi equations are:

$$(b_{12})_x - (b_{11})_y = \Gamma^k_{11} b_{k2} - \Gamma^k_{12} b_{k1}, \quad (b_{22})_x - (b_{12})_y = \Gamma^k_{12} b_{k2} - \Gamma^k_{22} b_{k1}. \tag{3.8}$$

It results the partial derivatives of H^b:

$$H^b_x = \Gamma^k_{12} b_{k2} - \Gamma^k_{22} b_{k1}, \quad H^b_y = \Gamma^k_{12} b_{k1} - \Gamma^k_{11} b_{k2} \tag{3.9}$$

which means (3.6). The commuting condition $H^b_{xy} = H^b_{yx}$ is exactly (3.5). □

Example 3.3 Suppose that $b_{11} = b_{22} = 0$ and $b_{12} \neq 0$. The Codazzi formulae (3.8) reduces to:

$$(b_{12})_x = (\Gamma^1_{11} - \Gamma^2_{12}) b_{12}, \quad (b_{12})_y = (\Gamma^2_{22} - \Gamma^1_{12}) b_{12} \tag{3.10}$$

while (3.5)–(3.6) means:

$$[(\Gamma^1_{12} - \Gamma^2_{22}) b_{12}]_y = [(\Gamma^2_{12} - \Gamma^1_{11}) b_{12}]_x, \quad \nabla_{can} H^b = -((b_{12})_y, (b_{12})_x). \tag{3.11}$$

But $H^b = 0$ and then b_{12} is a constant and hence the holomorphic h^b is a constant: $h^b = -i b_{12}$. □

In the same manner we treat the holomorphy of the function h^g even for the general case of previous section:

Proposition 3.4 *Suppose that (M^2, g) is a non-diagonal Riemannian two-dimensional manifold and its h^g is holomorphic. Then:*

$$(H^a_x + \Gamma^a_{12} H^2 + \Gamma^a_{k1} H^k) g_{a2} + (H^a_y + \Gamma^a_{12} H^1 + \Gamma^a_{k2} H^k) g_{a1} = 0. \tag{3.12}$$

Also, the Euclidean gradient of the function $\Re h^g$ is:

$$\nabla_{can} (\Re h^g) = (-(\Gamma^k_{12} g_{k2} + \Gamma^k_{22} g_{k1}), \Gamma^k_{11} g_{k2} + \Gamma^k_{12} g_{k1}). \tag{3.13}$$

Proof The Cauchy-Riemann equations for h^g are:

$$\frac{1}{2}(g_{11} - g_{22})_x = -(g_{12})_y, \quad \frac{1}{2}(g_{11} - g_{22})_y = (g_{12})_x \tag{3.14}$$

and the metrizability of Levi-Civita connection means:

$$(g_{12})_y = \Gamma^k_{12} g_{k2} + \Gamma^k_{22} g_{k1}, \quad (g_{12})_x = \Gamma^k_{11} g_{k2} + \Gamma^k_{12} g_{k1}. \tag{3.15}$$

Hence:

$$\frac{1}{2}(g_{11} - g_{22})_x = -(\Gamma^k_{12} g_{k2} + \Gamma^k_{22} g_{k1}), \quad \frac{1}{2}(g_{11} - g_{22})_y = \Gamma^k_{11} g_{k2} + \Gamma^k_{12} g_{k1} \tag{3.16}$$

which gives (3.13). Again the commutativity of second order derivatives means:

$$(\Gamma^k_{11} g_{k2} + \Gamma^k_{12} g_{k1})_x = -(\Gamma^k_{12} g_{k2} + \Gamma^k_{22} g_{k1})_y \tag{3.17}$$

and then, a straightforward computation yields the equation:

$$\left[(\Gamma^a_{11}+\Gamma^a_{22})_x + 2\Gamma^2_{12}(\Gamma^a_{11}+\Gamma^a_{22}) - R^a_{122} + \Gamma^a_{k1}(\Gamma^k_{11}+\Gamma^k_{22})\right] g_{a2} + \\ + \left[(\Gamma^a_{11}+\Gamma^a_{22})_y + 2\Gamma^1_{12}(\Gamma^a_{11}+\Gamma^a_{22}) + R^a_{121} + \Gamma^a_{k2}(\Gamma^k_{11}+\Gamma^k_{22})\right] g_{a1} = 0, \tag{3.18}$$

where R is the curvature tensor field of g. Due to the equalities $g_{a2}R^a_{122} = g_{a1}R^a_{121} = 0$ and using the notations of first section we get (3.12). □

Example 3.5 If M^2 is the graph $\mathbb{R}^3 \ni z = f(x, y)$ then:

$$h^g = \frac{f_x^2 - f_y^2}{2} - i f_x f_y \tag{3.19}$$

and hence, its holomorphy is expressed by the Cauchy-Riemann equations:

$$(f_x^2 - f_y^2)_x = -2(f_x f_y)_y, \quad (f_x^2 - f_y^2)_y = 2(f_x f_y)_x. \tag{3.20}$$

Equivalently:

$$f_x(f_{xx} + f_{yy}) = 0, \quad f_y(f_{xx} + f_{yy}) = 0 \tag{3.21}$$

and from the regularity of M^2 it results that f is (Euclidean) harmonic function:

$$f_{xx} + f_{yy} = 0 \tag{3.22}$$

and then we have an infinite number of graphs with holomorphic h^g; for example, these are provides by the real and imaginary part of the natural powers z^n, [11, p. 88]. Also, we recall that given two graphs $G_u : z = u(x, y)$, $G_v : z = v(x, y)$ if $f = u + iv$ is a holomorphic function then G_u and G_v share the same Gaussian curvature $K_f = -\left[\frac{|f''|}{1+|f'|^2}\right]^2 \leq 0$, [14, p. 128]. We present only two graphs for which $h^g(z)$ is proportional to z^2 and z^{-2}.

Example 3.6 The hyperbolic paraboloid $(hP)_c : \mathbb{R}^3 \ni z = c(x^2 - y^2)$ with $c \neq 0$ has the holomorphic h^g: $h^g(z = x + iy) = 2c^2 z^2$. The gradient of its real part is: $\nabla(\Re h^g)(x, y) = 4c^2(x, -y) = 4c^2\bar{z}$. The condition (3.12) is satisfied since: $H^1 = H^2 = 0$. Also:

$$h^1 = \frac{4c^2 x}{1 + 4c^2|z|^2}, \quad h^2 = \frac{-4c^2 y}{1 + 4c^2|z|^2}, \quad h^b = \frac{2c}{\sqrt{1 + 4c^2|z|^2}}, \tag{3.23}$$

$$h^S = \frac{2c}{[1 + 4c^2|z|^2]^{\frac{3}{2}}}[1 + c^2(2|z|^2 + \bar{z}^2 - z^2)]. \tag{3.24}$$

If the hyperbolic paraboloid is expressed as $(hP)^c : \mathbb{R}^3 \ni z = cxy$ then $h^g(z) = -\frac{c^2}{2}z^2$ and $\nabla(\Re h^g)(x, y) = c^2(y, -x) = -c^2(iz)$.

Example 3.7 The (funnel) revolution surface $M^2 : \mathbb{R}^3 \ni z = \ln(x^2 + y^2)$ has the holomorphic $h^g(z = x + iy) = 2z^{-2}$. Again the condition (3.12) is satisfied with $H^1 = H^2 = 0$. The gradient of its real part is:

$$\nabla(\Re h^g)(x, y) = \frac{4}{|z|^6}\left(x(-x^2 + 3y^2), y(x^2 - 3y^2)\right) = \frac{4\bar{z}(z\bar{z} - z^2 - \bar{z}^2)}{|z|^6}. \quad (3.25)$$

Also:

$$h^1 = \frac{-4x}{z^2(|z|^2 + 4)}, \quad h^2 = \frac{-4y}{z^2(|z|^2 + 4)}, \quad h^b = \frac{-2}{z^2|z|\sqrt{|z|^2 + 4}}, \quad (3.26)$$

$$h^S = \frac{-2(|z|^2 + 2)\bar{z}^2}{|z|^3(|z|^2 + 4)^{\frac{3}{2}}}. \quad (3.27)$$

□

Example 3.8 Since the Beltrami coefficient involves the conjugate $\overline{h^g}$ we ask for graphs with anti-holomorphic h^g. With the same approach as above we obtain the conditions:

$$f_{xx} = f_{yy}, \quad f_{xy} = 0 \quad (3.28)$$

and it results the (elliptic) paraboloid of revolution $(eP)_c : \mathbb{R}^3 \ni z = c(x^2 + y^2)$ with $\overline{h^g}(z = x + iy) = 2c^2z^2$. □

Example 3.9 We remark that $(eP)_c$ can be obtained from $(hP)_c$ through a Wick rotation $y \to iy$, [10]. The same transformation applied to the surface of Example 3.7 gives the surface $M^2 : \mathbb{R}^3 \ni z = \ln(x^2 - y^2)$ for $|x| > |y|$. Its Hermitian data is:

$$h^g(z = x + iy) = \frac{8z^2}{(z^2 + \bar{z}^2)^2}, H^g(x, y) = 2 + \frac{4(x^2 + y^2)}{(x^2 - y^2)^2} = 2 + \frac{16|z|^2}{(z^2 + \bar{z}^2)^2}. \quad (3.29)$$

Hence the Wick rotation does not transport the holomorphic property from h^g to $\overline{h^g}$. □

Remark 3.10 Fix $\tilde{\nabla}$ an arbitrary linear connection on M^2 with the Christoffel symbols $\tilde{\Gamma}^i_{jk}$. Suppose that the $(0, 3)$-tensor field $\tilde{\nabla}g$ is totally symmetric, [19, p. 155]:

$$\tilde{\nabla}g(\partial_x, \partial_y, \partial_x) = \tilde{\nabla}g(\partial_y, \partial_x, \partial_x), \quad \tilde{\nabla}g(\partial_x, \partial_y, \partial_y) = \tilde{\nabla}g(\partial_y, \partial_x, \partial_y).$$

AQ3 These two equations mean the Cauchy-Kowaleski differential system (31) of the cited paper which inserted in our Cauchy-Riemann equations (3.14) yields the following conditions for the holomorphy of h^g:

$$H^g_x = \left(\tilde{\Gamma}^1_{12} - \tilde{\Gamma}^2_{22}\right) g_{12} + \tilde{\Gamma}^2_{12} g_{22} - \tilde{\Gamma}^1_{22} g_{11}, \quad H^g_y = \left(\tilde{\Gamma}^2_{21} - \tilde{\Gamma}^1_{11}\right) g_{12} + \tilde{\Gamma}^1_{21} g_{11} - \tilde{\Gamma}^1_{11} g_{22}.$$

□

Another setting in which the Hopf invariant is useful consists in *hyperbolic surfaces*, which have a negative Gaussian curvature. Such a surface M^2 can be parametrized by its (real) asymptotic coordinates (α, β) and from [22, p. 89] we have:

$$\Gamma^1_{12} = -\frac{1}{4}[\ln(-K)]_\beta, \quad \Gamma^2_{12} = -\frac{1}{4}[\ln(-K)]_\alpha \tag{3.30}$$

and hence, a hyperbolic surface is a *pseudospherical* one, i.e. K is a negative constant, if and only if both h^1 and h^2 are real numbers. The pseudospherical metric of solvable non-Abelian case from Example 2.8 is not expressed in asymptotic coordinates since its $\Gamma^1_{12} = C^1_{12} = 1$.

We finish this section with a matrix approach, following the notations from Introduction and inspired by the relation $S = b \cdot g^{-1}$. We point out that the commuting relation $g^{-1} \cdot b = b \cdot g^{-1}$ means:

$$g_{12}(b_{11} - b_{22}) = b_{12}(g_{11} - g_{22}) \tag{3.31}$$

or, equivalently:

$$\Im h^g \cdot \Re h^b = \Im h^b \cdot \Re h^g \tag{3.32}$$

which says that the complex functions h^g, h^b have the same argument. As example, the relation (3.31) holds for an isothermal first fundamental form g while for a graph surface it reads:

$$f_x f_y (f_{xx} - f_{yy}) = f_{xy}(f^2_x - f^2_y). \tag{3.33}$$

Proposition 3.11 *For a surface $(M^2, g) \subset \mathbb{R}^3$ satisfying the commuting relation $g^{-1} \cdot b = b \cdot g^{-1}$ we have the matrix equality:*

$$(b \cdot g^{-1})^c = S^c = b^c \cdot (g^c)^{-1} \tag{3.34}$$

and then the mean curvature and Gaussian curvature of g are:

$$\begin{cases} H(g) = \frac{H^g H^b - \overline{h^g} h^b}{\det g} = \frac{H^g H^b - h^g \overline{h^b}}{\det g}, \\ K(g) = \frac{(H^g H^b - \overline{h^g} h^b)^2 - |H^g h^b - H^b h^g|^2}{(\det g)^2}. \end{cases} \tag{3.35}$$

Proof From:

$$g^c = \begin{pmatrix} H^g & \overline{h^g} \\ h^g & H^g \end{pmatrix}, \quad \det g^c = \det g > 0, \quad b^c = \begin{pmatrix} H^b & \overline{h^b} \\ h^b & H^b \end{pmatrix} \tag{3.36}$$

it results:

$$b^c \cdot (g^c)^{-1} = \frac{1}{\det g} \begin{pmatrix} H^g H^b - h^g \overline{h^b} & H^g \overline{h^b} - H^b \overline{h^g} \\ H^g h^b - H^b h^g & H^g H^b - \overline{h^g} h^b \end{pmatrix} \tag{3.37}$$

The equality of entries from the principal diagonal in (3.37):

$$\overline{h^g} h^b = h^g \overline{h^b} \in \mathbb{R} \tag{3.38}$$

is exactly (3.32). A straightforward computation of S^c from:

$$\begin{aligned} S &= \frac{1}{\det g} \begin{pmatrix} g_{22}b_{11} - g_{12}b_{12} & g_{22}b_{12} - g_{12}b_{22} \\ g_{11}b_{12} - g_{12}b_{11} & g_{11}b_{22} - g_{12}b_{12} \end{pmatrix} \\ &= \frac{1}{\det g} \begin{pmatrix} g_{22}b_{11} - g_{12}b_{12} & g_{11}b_{12} - g_{12}b_{11} \\ g_{22}b_{12} - g_{12}b_{22} & g_{11}b_{22} - g_{12}b_{12} \end{pmatrix} \end{aligned}$$

gives the conclusion (3.34). □

Example 3.12 The relation (3.33) is satisfied for quadratic graphs: $z = c(ax + by)^2$.

We return to the formula of Example 3.5 in order to compute the Gaussian curvature of a Blaschke factor B_a defined by $a \in \mathbb{C}$ with module $|a| < 1$:

$$B_a(z) = \frac{z - a}{1 - \bar{a}z}. \tag{3.39}$$

We obtain:

$$K_{B_a}(z) = -\left[\frac{2|a|(1 - |a|^2)|1 - \bar{a}z|}{(1 - |a|^2)^2 + |1 - \bar{a}z|^4}\right]^2 \tag{3.40}$$

and in particular:

$$K_{B_a}(0) = -\left[\frac{2|a|(1 - |a|^2)}{1 + (1 - |a|^2)^2}\right]^2, \quad K_{B_a}(a) = -\left[\frac{2|a|}{1 + (1 - |a|^2)^2}\right]^2. \tag{3.41}$$

Hence, we can define the smooth *Gauss-Blaschke functions* $GB^{1,2} : [0, 1) \to [0, +\infty)$:

$$GB^1(t) := \frac{2t(1-t^2)}{2-2t^2+t^4}, \quad GB^1(\cos\varphi) := \frac{2\cos\varphi\sin^2\varphi}{2\sin^2\varphi+\cos^4\varphi},$$

$$GB^2(t) := \frac{2t}{2-2t^2+t^4}. \tag{3.42}$$

4 The Hopf-Levi-Civita Data for Two Dimensional Kähler Geometry

Let now M^2 be a complex manifold of complex dimension 2 with local coordinates (z^1, z^2). Fix $(J, g = (g_{i\bar{j}}))$ a Kähler structure on M and let $\nabla = (\Gamma^k_{ij})$ be the Levi-Civita connection of g. We extend ∇ in a $\mathbb{C}$-linear way and then the only possible non-zero Christoffel symbols are Γ^k_{ij} and $\Gamma^{\bar{k}}_{\bar{i}\,\bar{j}} = \overline{\Gamma^k_{ij}}$. It is well-known that the Kähler condition gives:

$$\Gamma^k_{ij} = g^{k\bar{l}}\frac{\partial g_{i\bar{l}}}{\partial z^j} = g^{k\bar{l}}\frac{\partial g_{j\bar{l}}}{\partial z^i} \tag{4.1}$$

with $(g^{i\bar{j}})$ the matrix inverse to $(g_{i\bar{j}})$. Recall that the fundamental 2-form of the Kähler structure is defined as:

$$\omega(X, Y) := g(JX, Y) \tag{4.2}$$

on two arbitrary vector fields X, Y. Using the same Definition 2.2 we derive immediately:

Proposition 4.1 *The Hopf-Levi-Civita data of a complex two-dimensional Kähler metric g is:*

$$h^k = g^{k\bar{l}}\left[\frac{1}{2}\left(\frac{\partial g_{1\bar{l}}}{\partial z^1} - \frac{\partial g_{2\bar{l}}}{\partial z^2}\right) - i\frac{\partial g_{1\bar{l}}}{\partial z^2}\right], \quad B^k = \frac{g^{k\bar{l}}}{2}\left(\frac{\partial g_{1\bar{l}}}{\partial z^1} + \frac{\partial g_{2\bar{l}}}{\partial z^2}\right). \tag{4.3}$$

Suppose that ρ is a (local) Kähler potential for g which means that $\omega = \frac{i}{2}\partial\overline{\partial}\rho$. Then:

$$h^k = g^{k\bar{l}}\frac{\partial}{\partial \overline{z^l}}\left[\frac{1}{2}\left(\frac{\partial^2\rho}{\partial(z^1)^2} - \frac{\partial^2\rho}{\partial(z^2)^2}\right) - i\frac{\partial^2\rho}{\partial z^1\partial z^2}\right] = \frac{g^{k\bar{l}}}{2}\frac{\partial}{\partial\overline{z^l}}\left(\frac{\partial}{\partial z^1} - i\frac{\partial}{\partial z^2}\right)^2\rho, \tag{4.4}$$

$$H^k = \frac{g^{k\bar{l}}}{2} \frac{\partial}{\partial \overline{z^l}} \left[\frac{\partial^2 \rho}{\partial (z^1)^2} + \frac{\partial^2 \rho}{\partial (z^2)^2} \right]. \tag{4.5}$$

Example 4.2 The *Bergmann metric* on the two-dimensional unit disk $\Delta^2 = \{z = (z^1, z^2) \in \mathbb{C}^2; f(z) := 1 - |z^1|^2 - |z^2|^2 > 0\}$ is, [17, p. 14]:

$$g_{i\bar{j}} = \frac{\delta_{ij}}{f(z)} + \frac{\overline{z^i} z^j}{f^2(z)}, \quad g^{i\bar{j}} = f(z)(\delta^{ij} - z^i \overline{z^j}), \quad \rho = -\ln f. \tag{4.6}$$

A long but straightforward computation gives:

$$f(z) h^k = (\delta_1^k - i\delta_2^k)(\overline{z^1} - i\overline{z^2}), \quad f(z) H^k = \overline{z^k}. \tag{4.7}$$

□

Example 4.3 The *Fubini-Study metric* on the complex projective space $P^2(\mathbb{C})$ is, [17, p. 16]:

$$g_{i\bar{j}} = \frac{\delta_{ij}}{f} - \frac{\overline{z^i} z^j}{f^2}, \quad f = 1 + |z^1|^2 + |z^2|^2, \quad g^{i\bar{j}} = f(z)(\delta^{ij} + z^i \overline{z^j}), \quad \rho = \ln f. \tag{4.8}$$

Then:

$$f^2(z) h^k = z^k (\overline{z^1} - i\overline{z^2})^2, \quad f(z) H^k = \overline{z^k} + 2z^k [(\overline{z^1})^2 + (\overline{z^2})^2]. \tag{4.9}$$

□

Returning to the general Formula (4.4) we note the birth on M^2 of a remarkable differential operator:

$$\partial := \frac{1}{2} \left(\frac{\partial}{\partial z^1} - i \frac{\partial}{\partial z^2} \right) \tag{4.10}$$

yielding a (formal) Laplacian-type operator and a Carlson-Griffiths operator:

$$\Delta = 4\partial\bar{\partial} = 4\bar{\partial}\partial = \frac{\partial^2}{\partial (z^1)^2} + \frac{\partial^2}{\partial (z^1)^2}, \quad d^c = \frac{i}{2\pi}(\bar{\partial} - \partial). \tag{4.11}$$

It follows another expression of (4.5) and (4.6):

$$h^k = 2g^{k\bar{l}} \frac{\partial(\partial^2 \rho)}{\partial \overline{z^l}}, \quad H^k = 2g^{k\bar{l}} \frac{\partial(\Delta \rho)}{\partial \overline{z^l}}. \tag{4.12}$$

5 The Hopf-Levi-Civita Data for Two-Dimensional Hessian Geometry

The real counter-part of Kähler geometry is Hessian geometry and following the previous section we treat here this case. Suppose that our real manifold M^2 is endowed with a flat (and symmetric) connection D. We adopt, see also the Eq. (14) of [6, p. 3338]:

Definition 5.1 ([24, p. 14]) The Riemannian metric g is called *D-Hessian* if there exists (a potential) $\rho \in C^\infty(M)$ such that g is exactly the D-Hessian of ρ: $g = D(d\rho)$.

Let $(x, y) = (x^1, x^2)$ be an affine coordinates system with respect to D. The above definition means:

$$g_{ij} = \frac{\partial^2 \rho}{\partial x^i \partial x^j} \tag{5.1}$$

which yields the (local) *Codazzi equation*:

$$\frac{\partial g_{ia}}{\partial x^j} = \frac{\partial g_{ja}}{\partial x^i}. \tag{5.2}$$

The Christoffel symbols of g are, [24, p. 15]:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \frac{\partial g_{ij}}{\partial x^l} \tag{5.3}$$

and then we derive:

Proposition 5.2 *The Hopf-Levi-Civita data of the Hessian metric g is:*

$$h^k = \frac{g^{kl}}{2}\left[\frac{1}{2}\left(\frac{\partial g_{1l}}{\partial x} - \frac{\partial g_{2l}}{\partial y}\right) - i\frac{\partial g_{1l}}{\partial y}\right], \quad H^k = \frac{g^{kl}}{4}\left(\frac{\partial g_{1l}}{\partial x} + \frac{\partial g_{2l}}{\partial y}\right). \tag{5.4}$$

With respect to the potential ρ it follows:

$$h^k = \frac{g^{kl}}{4}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^2 \rho, \quad H^k = \frac{g^{kl}}{4}\frac{\partial}{\partial x^l}(\rho_{xx} + \rho_{yy}). \tag{5.5}$$

Remark 5.3 A more detailed version of Formulae (5.4) is:

$$4(\det g)h^1 = g_{22}\left[(g_{11})_x - (g_{12})_y - 2i(g_{11})_y\right] - g_{12}\left[(g_{12})_x - (g_{22})_y - 2i(g_{12})_y\right], \tag{5.6}$$

$$4(\det g)h^2 = -g_{12}\left[(g_{11})_x - (g_{12})_y - 2i(g_{11})_y\right] + g_{11}\left[(g_{12})_x - (g_{22})_y - 2i(g_{12})_y\right], \tag{5.7}$$

$$4(\det g)H^1 = g_{22}\left[(g_{11})_x + (g_{12})_y\right] - g_{12}\left[(g_{12})_x + (g_{22})_y\right], \tag{5.8}$$

$$4(\det g)H^2 = -g_{12}\left[(g_{11})_x + (g_{12})_y\right] + g_{11}\left[(g_{12})_x + (g_{22})_y\right]. \tag{5.9}$$

□

Again we note the birth on M of the differential operators of first order:

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{5.10}$$

and hence:

$$h^k = g^{kl}\frac{\partial(\partial^2\rho)}{\partial x^l}, \quad H^k = g^{kl}\frac{\partial(\partial\bar{\partial}\rho)}{\partial x^l}. \tag{5.11}$$

Example 5.4 Let us consider the Hessian unit disk $\Delta^1 = \{(x, y) \in \mathbb{R}^2; f(x, y) = 1 - x^2 - y^2 > 0\}$ with potential $\rho = -\ln f$. Then:

$$g_{ij} = \frac{2}{f}\left(\delta^{ij} + \frac{2}{f}x^i x^j\right), g_{11} = \frac{2(1 + x^2 - y^2)}{f^2}, g_{12} = \frac{4xy}{f^2}, g_{22} = \frac{2(1 - x^2 + y^2)}{f^2} \tag{5.12}$$

and then the Hermitian data of the metric g is:

$$h^g = \frac{2(x^2 - y^2)}{f^2} - i\frac{4xy}{f^2} = 2\frac{(x - iy)^2}{f^2} = 2\left(\frac{\bar{z}}{f}\right)^2, \quad H^g = \frac{2}{f^2}. \tag{5.13}$$

In the same unit disk we can consider the metric g_n of [20, p. 246]:

$$\begin{cases} g_n = \frac{1-y^2}{(1-x^2-y^2)^{n-2}}dx^2 + \frac{2xy}{(1-x^2-y^2)^{n-2}}dxdy + \frac{1-x^2}{(1-x^2-y^2)^{n-2}}dy^2, \\ h^{g_n} = \frac{\bar{z}^2}{2(1-|z|^2)^{n-2}}. \end{cases} \tag{5.14}$$

Note that g_3 is exactly the round metric of the sphere S^2 expressed as a graph and if we apply the double Wick rotation $(x \to ix, y \to iy)$ to g_4 we get the metric g_{proj}. Equivalently, g_3 is the round metric of S^2 provided by orthogonal projection. Also, the Funk metric on M is of Randers type with the Riemannian metric g_4 and the 1-form $\beta = d\rho$, as is expressed in Example 3.8 of [3, p. 88]. The only non-zero Christoffel symbols of g_4 are:

$$\Gamma^1_{11} = 2\Gamma^2_{12} = \frac{2x}{1 - x^2 - y^2}, \quad \Gamma^2_{22} = 2\Gamma^1_{12} = \frac{2y}{1 - x^2 - y^2}$$

and then the dual of β is the vector field $\xi = -2(1 - x^2 - y^2)^2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$. In polar coordinates we have:

$$g_4 = \frac{dr^2}{(1-r^2)^2} + \frac{r^2 d\theta^2}{1-r^2}, \quad K_{g_4} = -1.$$

In conclusion, g_{proj} and g_4 can be unified in the metric:

$$g_\varepsilon = \frac{dr^2}{(1+\varepsilon r^2)^2} + \frac{r^2 d\theta^2}{1+\varepsilon r^2}, \quad K_{g_\varepsilon} = \varepsilon, \quad \varepsilon \in \{+1, -1\} \leftrightarrow \{proj, 4\}.$$

The cited paper [20] studies the geodesics of g_n trough *the support function* $u = u(t)$ via the equations: $x = u \sin t + u' \cos t$, $y = -u \cos t + u' \sin t$. Performing the same approach to g_{proj} we arrive at the differential equation:

$$\frac{1}{u + u''} = \frac{u}{1 + u^2 + (u')^2} \tag{5.15}$$

which reduces to $uu'' - (u')^2 = 1$ with solution $u(t) = \cosh t$. □

We return now to the general case of a Riemannian metric g on M^2. The Hessian of a smooth scalar field $f \in C^\infty(M)$ with respect to the metric g is another symmetric covariant tensor field:

$$H_g(f)_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \tag{5.16}$$

and hence its Hopf invariant and mean curvature are:

$$h(H_g(f)) = \frac{f_{11} - f_{22}}{2} - i f_{12} - h^k f_k, \quad H(H_g(f)) = \frac{f_{11} + f_{22}}{2} - H^k f_k. \tag{5.17}$$

Let us remark that the condition (3.22) for $f \in C^\infty(\mathbb{R}^2, g_{can})$ means exactly $H(H_{can}(f)) = \frac{1}{2}\Delta_{can} f = 0$ while the conditions (3.28) means exactly $h(H_{can}(f)) = 0$. If (M^2, g) is a projective Euclidean space then the Hessian of f has the components:

$$H_g(f)_{11} = \frac{\partial^2 f}{\partial x^2} - 2\Gamma^2_{12}\frac{\partial f}{\partial x}, \quad H_g(f)_{12} = \frac{\partial^2 f}{\partial x \partial y} - \Gamma^1_{12}\frac{\partial f}{\partial x} - \Gamma^2_{12}\frac{\partial f}{\partial y},$$

$$H_g(f)_{22} = \frac{\partial^2 f}{\partial y^2} - 2\Gamma^1_{12}\frac{\partial f}{\partial y}.$$

Another symmetric covariant tensor field on (M, g) is the Lie derivative of g with respect to a fixed vector field $X = X^k \frac{\partial}{\partial x^k} = (X^1, X^2)$:

$$(\mathcal{L}_X g)_{ij} := X^k g_{ij,k} + X^k_{,i} g_{kj} + X^k_{,j} g_{ki}. \tag{5.18}$$

We associate to X the complex-valued function:

$$f_X := X^1 + iX^2 \tag{5.19}$$

and we establish a relationship between the Hopf invariant $h(\mathcal{L}_X g)$ and f_X:

Theorem 5.5 *Suppose that the metric g is an isothermal one. Then the Hopf invariant of the Lie derivative is:*

$$h(\mathcal{L}_X g) = 2E\overline{\partial} f_X \tag{5.20}$$

and consequently if X is a conformal Killing vector field then f_X is holomorphic.

Proof We have:

$$\begin{aligned} h(\mathcal{L}_X g) &= \frac{1}{2}[X^k(g_{11,k} - g_{22,k}) + 2X^k_{,1}g_{k1} - 2X^k_{,2}g_{k2}] \\ &\quad - i[X^k g_{12,k} + X^k_{,1}g_{k2} + X^k_{,2}g_{k1}] = \\ &= X^k h^g_{,k} + g_{k1}[X^k_{,1} - iX^k_{,2}] - g_{k2}[X^k_{,2} + iX^k_{,1}]. \end{aligned}$$

Taking into account the isothermal hypothesis it results:

$$h(\mathcal{L}_X g) = E[(X^1_{,1} - X^2_{,2}) - i(X^1_{,2} + X^2_{,1})] \tag{5.21}$$

which yields the conclusion. □

Example 5.6 We discuss the class of two-dimensional space-forms given by $K=$ constant. The isometry group $Isom(g)$ has the maximal dimension $\frac{n(n+1)}{2} = \frac{2\cdot 3}{2} = 3$.

(5.6.1) $K = c^2 > 0$. The round metric of the sphere $S^2\left(\frac{1}{c}\right)$ provided by the stereographic projection is:

$$g^p_c(x, y) = \frac{4(dx^2 + dy^2)}{[c(1 + x^2 + y^2)]^2} = \left(\frac{2|dz|}{c(1 + z^2)}\right)^2 \tag{5.22}$$

and the basis of its Lie algebra $isom(g)$ of Killing vector fields is:

$$\begin{cases} X^p_1(x, y) = \left(xy, \frac{y^2-x^2+1}{2}\right) = \frac{(r^2+1)\sin\theta}{2}\frac{\partial}{\partial r} + \frac{(1-r^2)\cos\theta}{2r}\frac{\partial}{\partial\theta}, \\ X^p_2(x, y) = \left(\frac{x^2-y^2+1}{2}, xy\right) = \frac{(r^2+1)\cos\theta}{2}\frac{\partial}{\partial r} + \frac{(r^2-1)\sin\theta}{2r}\frac{\partial}{\partial\theta}, \\ X^p_3(x, y) = (-y, x) = r(-\sin\theta, \cos\theta) \end{cases} \tag{5.23}$$

where p means positive. The corresponding holomorphic functions are:

$$f^p_1(z) = \frac{i(1 - z^2)}{2}, \quad f^p_2(z) = \frac{z^2 + 1}{2}, \quad f^p_3(z) = iz. \tag{5.24}$$

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Recall that the Weierstrass data of the Enneper surface is [13, p. 234]: $M = \mathbb{C}$, $g(z) = z$, $\eta = dz$ and then:

$$\alpha = \frac{1}{2}\begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} dz.$$

Hence we have:

$$i\alpha = \begin{pmatrix} f_1^p(z) \\ -f_2^p(z) \\ f_3^p(z) \end{pmatrix} dz.$$

(5.6.2) $K = -c^2 < 0$. The Poincaré expression of the metric concerns with the upper half-plane model of hyperbolic geometry $\mathbb{C}_+ = \{z \in \mathbb{C}; \Im z > 0\}$:

$$g_c^n(x, y) = \frac{dx^2 + dy^2}{(cy)^2} = \left(\frac{|dz|}{cy}\right)^2 \tag{5.25}$$

and the basis of its Lie algebra $isom(g)$ of Killing vector fields is:

$$\begin{cases} X_1^n(x, y) = (x^2 - y^2, 2xy) = r^2(\cos 2\theta, \sin 2\theta), \\ X_2^n(x, y) = (x, y) = r(\cos\theta, \sin\theta), \quad X_3^n(x, y) = (1, 0) \end{cases} \tag{5.26}$$

where n means negative. The corresponding holomorphic functions are:

$$f_1^n(z) = z^2, \quad f_2^n(z) = z, \quad f_3^n(z) = 1. \tag{5.27}$$

For above metrics the isometry group is $Isom(g_c^p) = O(3)$ respectively $Isom(g_c^n) = SL(2, \mathbb{R}) \simeq SU(1, 1)$. But:

$$[X_1^p, X_2^p] = X_3^p, \quad [X_2^p, X_3^p] = X_1^p, \quad [X_3^p, X_1^p] = X_2^p, \quad [X_1^n, X_2^n] = -X_1^n,$$
$$[X_2^n, X_3^n] = -X_3^n, \quad [X_3^n, X_1^n] = 2X_2^n$$

and hence $\{X_1^n, X_2^n, X_3^n\}$ is not a Milnor frame for $SL(2, \mathbb{R})$, [4, p. 52].

It is known that the Cayley-Möbius map $\varphi : \Delta^1 \to \mathbb{C}_+$:

$$\varphi(z) = i\frac{1 + z}{1 - z}$$

is a bijection. Hence we have the"Killing" functions on the unit disk $\varphi_j := f_j^n \circ \varphi : \Delta^1 \to \mathbb{C}$, $j = 1, 2, 3$ and the remarkable is:

$$\varphi_1(z) = -\left(\frac{1 + z}{1 - z}\right)^2 = -\left(1 + 4\sum_{n \geq 1} nz^n\right), \quad \varphi_2(z) = \varphi(z), \quad \varphi_3(z) = 1.$$

In fact, $\varphi_1 = -(4k+1)$ with $k(z) = \frac{z}{(1-z)^2}$ the Koebe function; hence the image of φ_1 is $\mathbb{C} \setminus (0, +\infty)$.

Also, it is known that the map $\psi : \mathbb{C}_+ \to \mathbb{C}_+$, $\psi(z) = \frac{-\bar{z}}{|z|^2} = -z^{-1}$, preserves the hyperbolic geodesics of $(\mathbb{C}_+, g_1)$. Hence we have the functions $\psi_j := F_j^n \circ \psi : \mathbb{C}_+ \to \mathbb{C}$, $j = 1, 2, 3$ and the remarkable is:

$$\psi_1(z) = \frac{\bar{z}^2}{|z|^4} = z^{-2}.$$

The geodesic polar coordinates $(\rho, \varphi) \in (0, +\infty) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on the Poincaré upper half-plan $(\mathbb{C}_+, g_1^n)$ are given by $\rho := dist_{g_1^n}(i, z)$, $\varphi := \arctan \frac{x^2+y^2-1}{2x}$ and then:

$$x = \frac{\sinh \rho \cos \varphi}{\cosh \rho - \sinh \rho \sin \varphi}, \quad y = \frac{1}{\cosh \rho - \sinh \rho \sin \varphi}.$$

The function f_2^n becomes:

$$f_2^n(z) = \frac{1}{(\cosh \rho - \sinh \rho \sin \varphi)^2} \left[(\sinh^2 \rho \cos^2 \varphi - 1) + (2 \sinh \rho \cos \varphi) i\right].$$

We can connect the above polar coordinate φ to the Hopf invariant of a matrix. In [26, pp.153–154] the manifold $\mathbb{C}_+$ is identified with the homogeneous space $\mathcal{SP}_2 = SO(2) \setminus SL(2, \mathbb{R})$ of positive definite binary quadratic forms of determinant 1 via the map: $z \in \mathbb{C}_+ \leftrightarrow W_z \in \mathcal{SP}_2$:

$$W_z := \begin{pmatrix} y & 0 \\ 0 & \frac{1}{y} \end{pmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix}, \quad P \cdot [A] = A^t \cdot P \cdot A \to W_z = \frac{1}{y} \begin{pmatrix} x^2 + y^2 & -x \\ -x & 1 \end{pmatrix}.$$

Hence $\varphi(z)$ is exactly the argument of the Hopf-invariant $h(W_z)$. If the trace $Tr(W_z) = \frac{x^2+y^2+1}{y} > 2$ then the matrix W_z is a *hyperbolic* one with real eigenvalues λ, $\frac{1}{\lambda}$. Let $e(z)$ be the eigenvalue which is larger than 1 and the norm of W_z is, by definition, $N(W_z) = e(z)^2$. We obtain:

$$N(W_z) = \frac{1}{4} \left[\frac{|z|^2 + 1}{y} + \sqrt{\left(\frac{|z|^2 + 1}{y} \right)^2 - 4} \right]^2.$$

On the vertical axis $Oy : x = 0$ the hyperbolic condition means $y \neq 1$ and then:

$$N(W_{iy}) = \left(\frac{y^2 + 1 + |1 - y^2|}{2y} \right)^2.$$

Recall also the fundamental region F of the modular group $PSL(2, \mathbb{Z})$:

$$F = \{z \in \mathbb{C}_+; |z| \geq 1, |\Re z| \leq \frac{1}{2}\}$$

and its right corner $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{\pi}{3}} \in F$ which is an elliptic point; then $N(W_\omega) = 3$. Its polar coordinates are provided by $\varphi(\omega) = 0$ and $\cosh \rho(\omega) = \frac{2}{\sqrt{3}}$, $\sinh \rho(\omega) = \frac{1}{\sqrt{3}}$. Another remarkable element of F is *the complex Golden ratio* $\phi_c = \frac{1}{2} + i\frac{\sqrt{5}}{2}$ from [9, p. 1231]. Some of its properties are related to the classical Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$:

$$\begin{cases} N(W_{\phi_c}) = \phi^2, d_{g_1^n}(\omega, \phi_c) = \frac{1}{2}\ln\frac{5}{3}, \cos\varphi(\phi_c) = \frac{2}{\sqrt{5}}, \sin\varphi(\phi_c) = \frac{1}{\sqrt{5}}, \\ \sinh \rho(\phi_c) = \frac{1}{2}, \cosh \rho(\phi_c) = \frac{\sqrt{5}}{2}. \end{cases}$$

The angles of the hyperbolic triangle $\Delta\omega\phi_c i$ and their sum are:

$$\measuredangle(\omega) = \frac{\pi}{3} = 60°, \measuredangle(\phi_c) = 90°, \measuredangle(i) = \arccos\frac{2}{\sqrt{5}} = 26.57°,$$

$sum = 176.57° < 180°$ and then its aria is $S(\Delta\omega\phi i) = \pi - sum = 3.03 < 60 = \frac{\pi}{3} = S(F)$. The hyperbolic distance between ω and ϕ_c is given by: $\cosh[d_{g_1^n}(\omega, \phi_c)] = \frac{4}{\sqrt{15}}$. Since $\ln\frac{5}{3} = 0.51 > 0.50 = \sqrt{5} - \sqrt{3}$ it follows that the hyperbolic distance between ω and ϕ_c is great than the corresponding Euclidean distance. The picture of the Golden hyperbolic triangle $\Delta\omega\phi_c i$ (Fig. 1) was made by my colleague Marian Ioan Munteanu:

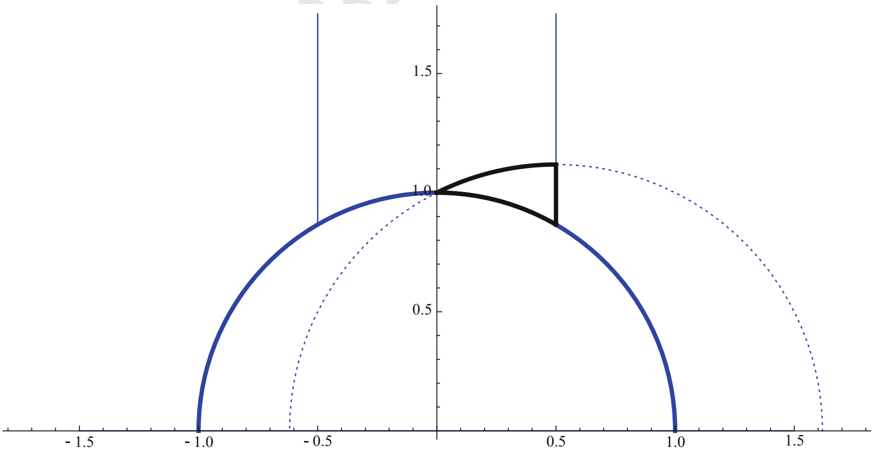

Fig. 1 The Golden hyperbolic triangle

(5.6.3) For $K = 0$ the Euclidean metric $g_{can} = dx^2 + dy^2 = |dz|^2 = dr^2 + r^2 d\theta^2$ the basis of its Lie algebra $isom(g_{can})$ of Killing vector fields is:

$$\begin{cases} X_1^e(x, y) = X_3^n(x, y) = (1, 0) = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial\theta}, \\ X_2^e(x, y) = (0, 1) = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial\theta}, \\ X_3^e(x, y) = X_3^p(x, y) = (-y, x) = \frac{\partial}{\partial\theta} \end{cases} \tag{5.28}$$

with corresponding holomorphic functions and Lie brackets:

$$\begin{cases} f_1^e(z) = 1, \quad f_2^e(z) = i, \quad f_3^e(z) = iz, \\ [X_1^e, X_2^e] = 0, \quad [X_2^e, X_3^e] = -X_1^e, \quad [X_3^e, X_1^e] = -X_2^e. \end{cases} \tag{5.29}$$

The isometry group $Isom(g_{can})$ is the affine orthogonal group $AO^n := \mathbb{R}^n \rtimes O(n)$. In polar coordinates the Wirtinger vector field is $\frac{\partial}{\partial z} = \frac{1}{2e^{\theta}} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial\theta} \right)$. The function $f_3^p = f_3^e$ is the skew-symmetric linear transformation with the matrix:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \nabla^{g_{can}} X_3^e. \tag{5.30}$$

which is the trigonometrical rotation of the plane. The skew-symmetric endomorphisms associated to the above Killing vector fields are:

$$\begin{cases} \nabla^{g_1^p} X_1^p = \frac{-2x}{1+x^2+y^2} J, \quad \nabla^{g_1^p} X_2^p = \frac{2y}{1+x^2+y^2} J, \quad \nabla^{g_1^p} X_3^p = \frac{1-x^2-y^2}{1+x^2+y^2} J \\ \nabla^{g_1^n} X_1^n = \frac{x^2+y^2}{y} J, \quad \nabla^{g_1^n} X_2^n = \frac{x}{y} J, \quad \nabla^{g_1^n} X_2^n = \frac{x}{y} J. \end{cases}$$

The square of their norm is:

$$\begin{cases} \|X^p\|^2 = \frac{(y^2-x^2+1)^2+4(xy)^2}{c^2(1+x^2+y^2)^2}, \|X_2^p\|^2 = \frac{(x^2-y^2+1)^2+4(xy)^2}{c^2(1+x^2+y^2)^2}, \\ \|X_3^p\|^2 = \frac{4(x^2+y^2)}{c^2(1+x^2+y^2)^2}, \quad \|X_1^e\| = \|X_2^e\| = 1, \quad \|X_e^3\|^2 = x^2 + y^2. \end{cases}$$

With the curvature formula of Example 3.5 we have K_f for all above f. The only non-zero are:

$$K_{f_1^p}(z) = K_{f_2^p}(z) = \frac{-1}{(1 + |z|^2)^2}, \quad K_{f_1^n}(z) = \frac{-4}{(1 + 4|z|^2)^2}.$$

Fix $\lambda \in C^{\infty}(\mathbb{R}^2)$ and let X_λ be conformal Killing vector field with λ as factor: $\mathcal{L}_{X_\lambda} g = 2\lambda g$. Then, for a constant $\lambda \in \mathbb{R}$ the set of X_λ depends on a real smooth function f through:

$$X_\lambda(x, y) = (\lambda x + f(y), \lambda y - f(x)), \quad f_\lambda^e(z) = \lambda z - i[f(x) + f(y)i]. \tag{5.31}$$

Remark that the metrics $g_{c>0}$, g_{cigar} and g_{can} of $\mathbb{R}^2$ are all conformally equivalent: $g_c = \frac{4}{c^2(1+x^2+y^2)} g_{cigar}$. These three metrics share the Killing rotation $X_3^p = X_3^e$ with the isothermal metric of the Enneper surface.

Remark that the punctured plane $\mathbb{R}^2 \setminus \{O(0,0)\}$ with the pseudo-Riemannian metric:

$$g = \frac{1}{x^2+y^2}[dx \otimes dy + dy \otimes dx] \tag{5.32}$$

has *the radial* Killing vector field $X^r(x,y) = (x,y) = X_2^n(x,y) = r\frac{\partial}{\partial r}$ with the holomorphic function $f_{X^r}(z) = z$. In fact, X^r is parallel with respect to g. □

This correspondence between Killing vector fields and holomorphic functions is already known, see for example [25, p. 143], while for the conformal Killing vector fields this relationship is discussed in [11, p. 88]. More generally, inspired by almost Ricci solitons we consider given the triple (g, X, f) on M and its associated symmetric covariant tensor field:

$$(g, X, f)^{soliton} := \mathcal{L}_X g + 2Ric_g + 2fg \tag{5.33}$$

whose vanishing characterizes an almost Ricci soliton; here Ric_g is the Ricci tensor of the metric g. Hence, in dimension two the vector field X is a conformal Killing field:

$$\mathcal{L}_X g = -2(K+f)g \tag{5.34}$$

and we can apply the last part of Theorem 5.5:

Proposition 5.7 *If (g, X, f) is an isothermal almost Ricci soliton on M^2 then f_X is a holomorphic function.*

Example 5.8 The cigar soliton is a steady ($f = 0$) gradient Ricci soliton with $X^{cigar} = \nabla_{g_{cigar}} f^{cigar}$ with:

$$f^{cigar}(x,y) = -\ln(1+x^2+y^2), \quad X^{cigar}(x,y) = -2(x,y) = -2X^r,$$

$$f_{X^{cigar}}(z) = -2z. \tag{5.35}$$

X^{cigar} is also a homothetic vector field for the Euclidean metric g_{can} since we make $\lambda = -2$ and $f = 0$ in (5.31). □

6 Subgeodesic Correspondence of Metrics and Connections

In [2, p. 173] we introduce a transformation of Riemannian metrics as follows.

Definition 6.1 The metrics g, $\tilde{g}$ are called *in subgeodesic correspondence* if there exists a vector field ξ and a 1-form ω such that their Levi-Civita connections are given by:

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \omega_j + \delta^k_j \omega_i + g_{ij}\xi^k. \tag{6.1}$$

For our two-dimensional setting M^2 a straightforward computation gives the transformation of the Christoffel-Hopf data:

$$\tilde{h}^k = h^k + h^g \xi^k + (\delta^k_1 - i\delta^k_2)(\omega_1 - i\omega_2). \tag{6.2}$$

Example 6.2 (i) If $\xi = 0$ then the given metrics are *in geodesic* or *projective equivalence* and then:

$$\tilde{h}^k = h^k + (\delta^k_1 - i\delta^k_2)(\omega_1 - i\omega_2). \tag{6.3}$$

(ii) If the metrics are conformal equivalent i.e. $\tilde{g} = e^{2u}g$ then we have the subgeodesic correspondence with $\xi = -\nabla_g u$ and $\omega = du$. Then:

$$\tilde{h}^k = h^k - h^g(\nabla_g u)^k + 2(\delta^k_1 - i\delta^k_2)\partial u \tag{6.4}$$

with ∂ given by (2.14).

We search now a projective equivalence such that $\tilde{h}^1 = \tilde{h}^2 = 0$. From (6.3) it results the initial h^1, h^2:

$$h^k = (\delta^k_1 - i\delta^k_2)(\varphi_1 - i\varphi_2), \quad \varphi = -\omega. \tag{6.5}$$

If φ is the Euclidean gradient of the scalar field f i.e. $(\varphi_1 = f_x, \varphi_2 = f_y)$ then:

$$h^k = 2(\delta^k_1 - i\delta^k_2)\partial f \tag{6.6}$$

and let us say that (g, f) is a *projective pair*. A method to obtain projective Euclidean space is given by:

Proposition 6.3 *If (g, f) is a projective pair such that $\Gamma^1_{22} = \Gamma^2_{11} = 0$ then (M^2, g) is a projective Euclidean space.*

Proof The explicit form of (6.6) is:

$$\Gamma^1_{11} - \Gamma^1_{22} = 2f_x = 2\Gamma^2_{12}, \quad \Gamma^2_{11} - \Gamma^2_{22} = -2f_y = -\Gamma^1_{12} \tag{6.7}$$

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and then the hypothesis implies the Eq. (2.18). We note that the vanishing of Γ^1_{22} and Γ^2_{11} means that the lines $x = constant$ and $y = constant$ are the geodesics and that the transformation $\tilde{x} = \alpha(x)$, $\tilde{y} = \beta(y)$ preserves the equations $\Gamma^1_{22} = \Gamma^2_{11} = 0$. □

Example 6.4 (i) With $\Gamma^1_{22} = \Gamma^2_{11} = 0$ and $f = f^{cigar}$ from (6.7) we derive the Christoffel symbols (2.21) of g_{proj}.

(ii) The *projective connection* associated to the linear connection $\nabla = (\Gamma^k_{ij})_{1\leq i,j,k\leq n}$ is, [16, p. 467]:

$$\Pi^k_{ij} := \Gamma^k_{ij} - \frac{\delta^k_i}{n+1}\Gamma_j - \frac{\delta^k_j}{n+1}\Gamma_i, \quad \Gamma_a = \sum_{i=1}^{n} \Gamma^i_{ia}. \tag{6.8}$$

In dimension 2 this means:

$$\begin{cases} \Pi^1_{11} = \Gamma^1_{11} - \frac{2}{3}\Gamma_1, & \Pi^1_{12} = \Gamma^1_{12} - \frac{1}{3}\Gamma_2, & \Pi^1_{22} = \Gamma^1_{22} \\ \Pi^2_{11} = \Gamma^2_{11}, & \Pi^2_{12} = \Gamma^2_{12} - \frac{1}{3}\Gamma_1, & \Pi^2_{22} = \Gamma^2_{22} - \frac{2}{3}\Gamma_2 \end{cases} \tag{6.9}$$

and a straightforward computation gives for the cigar metric:

$$\Pi^1_{11} = -\Pi^2_{12} = 3\Pi^1_{22} = \frac{x}{3(1+x^2+y^2)},$$

$$\Pi^2_{22} = -\Pi^1_{12} = 3\Pi^2_{11} = \frac{y}{3(1+x^2+y^2)}. \tag{6.10}$$

□

As generalization of conditions (6.7) we allow the symmetric linear connection Γ to be a non-metric connection and let a and b two smooth functions such that (2.18) holds:

$$\Gamma^2_{22} = 2\Gamma^1_{12} = 2a, \quad \Gamma^1_{11} = 2\Gamma^2_{12} = 2b, \quad \Gamma^1_{22} = \Gamma^2_{11} = 0. \tag{6.11}$$

Then the curvature of Γ is zero if and only if:

$$\begin{cases} R^1_{121} = a_x - 2b_y + ab = 0 \\ R^1_{122} = -a_y + a^2 = 0 \\ R^2_{121} = b_x - b^2 = 0 \\ R^2_{122} = 2a_x - b_y - ab = 0. \end{cases} \tag{6.12}$$

We have firstly the trivial solution $a = b = 0$ and we get the Euclidean connection $\Gamma = 0$. The non-trivial solution of system (6.9) is parametrised by two real constants α, β:

$$a = \frac{-1}{y + \alpha x + \beta}, b = \frac{-\alpha}{y + \alpha x + \beta}, h^1 = \frac{-\alpha + i}{y + \alpha x + \beta}, h^2 = -ih^1. \tag{6.13}$$

The classical equations of geodesics for this linear connection are:

$$\ddot{x} = \frac{2\dot{x}(\alpha\dot{x} + \dot{y})}{y + \alpha x + \beta}, \quad \ddot{y} = \frac{2\dot{y}(\alpha\dot{x} + \dot{y})}{y + \alpha x + \beta}. \tag{6.14}$$

This differential system integrates to:

$$\dot{x} = C_1(y + \alpha x + \beta)^2, \quad \dot{y} = C_2(y + \alpha x + \beta)^2, \quad C_1, C_2 \in \mathbb{R} \setminus \{0\}. \tag{6.15}$$

For $\beta = 0$ the final solution is given by lines through origin parametrised as:

$$x(t) = \frac{A}{t}, \quad y(t) = \frac{B}{t}, \quad t \in (0, +\infty). \tag{6.16}$$

If $\alpha = 1$ and $\beta = 0$ then the linear connection (6.11) is an example of locally homogeneous linear connection (3.19) from [18, p. 182].

7 Conclusion

As main benefit of this chapter we point out the richness of informations provided by the Hopf-Levi-Civita data in three remarkable 2D differential geometries, namely Riemannian, Kähler and Hessian. This approach is open to other interesting topics from two-dimensional geometries and we hope to develop it in future papers.

References

1. Chow, B., Knopf, D.: The Ricci Flow: An Introduction. Mathematical Surveys and Monographs 110. American Mathematical Society, Providence, RI (2004). Zbl 1086.53085
2. Crasmareanu, M.: Killing potentials. An. Ştiinţ. Univ. Al. I. Cuza Iaşi Math. **45**(1), 169–176 (1999). Zbl 1011.53032
3. Crasmareanu, M.: New tools in Finsler geometry: stretch and Ricci solitons. Math. Rep., Buchar. **16**(66)(1), 83–93 (2014). Zbl 1313.53091
4. Crasmareanu, M.: Adapted metrics and Webster curvature on three classes of 3-dimensional geometries. Int. Electron. J. Geom. **7**(2), 37–46 (2014). Zbl 1307.53040
5. Crasmareanu, M.: A complex approach to the gradient-type deformations of conics. Bull. Transilv. Univ. Braşov, Ser. III, Math. Inform. Phys. **10**(59) (2), 59–62 (2017). Zbl 06871080, MR3750075
6. Crasmareanu, M.: A new approach to gradient Ricci solitons and generalizations. Filomat **32**(9), 3337–3346 (2018). MR3898903
7. Crasmareanu, M.: New aspects of two Hessian-Riemannian metrics in plane. Surv. Math. Appl. **15**, 217–223 (2020)

8. Crasmareanu, M., Enache, V.: A note on dynamical systems satisfying the Wünschmann-type condition. Int. J. Phys. Sci. **7**(42), 5654–5663 (2012)
9. Crasmareanu, M., Hreţcanu, C.-E.: Golden differential geometry. Chaos Solitons Fractals **38**(5), 1229–1238 (2008). MR2456523 (2009k:53059)
10. Crasmareanu, M., Pişcoran, L.-I.: Wick-Tzitzeica solitons and their Monge-Ampère equation. Int. J. Geom. Methods Mod. Phys. **15**(5), Article ID 1850082, 9 p. (2018). Zbl 1395.53005
11. Fecko, M.: Differential Geometry and Lie Groups for Physicists. Cambridge University Press, Cambridge (2006). Zbl 1121.53001
12. Hopf, H.: Differential geometry in the large. Seminar lectures New York University 1946 and Stanford University 1956. With a preface by S. S. Chern. 2nd ed., Lecture Notes in Mathematics, 1000, Springer, Berlin (1989). Zbl 0669.53001
13. Jensen, G.R., Musso, E., Nicolodi L.: Surfaces in Classical Geometries. A Treatment by Moving Frames. Universitext. Springer, Cham (2016). Zbl 1347.53001
14. Jerrard, R.: Curvatures of surfaces associated with holomorphic functions. Colloq. Math. **21**, 127–132 (1970). Zbl 0203.23802
15. Kenmotsu, K.: Surfaces with Constant Mean Curvature. Translations of Mathematical Monographs 221. American Mathematical Society, Providence, RI (2003). Zbl 1042.53001
16. Levine, J.: Classification of collineations in projectively and affinely connected spaces of two dimensions. Ann. Math. **52**(2), 465–477 (1950). Zbl 0038.34603
17. Munteanu, G.: Complex spaces in Finsler, Lagrange and Hamilton Geometries. Fundamental Theories of Physics, vol. 141. Kluwer Academic Publishers, Dordrecht (2004). MR2102340 (2005g:53133)
18. Opozda, B.: A classification of locally homogeneous connections on 2-dimensional manifolds. Differ. Geom. Appl. 21(2), 173–198 (2004). Zbl 1063.53024
19. Opozda, B.: A realization problem in statistical and affine geometry. J. Geom. Phys. **131**, 147–159 (2018). Zbl 1392.53029
20. Otsuki, T.: On a differential equation related with differential geometry. Mem. Fac. Sci., Kyushu Univ. Ser. A **47**(2), 245–281 (1993). Zbl 0802.34047
21. Petersen, P.: Riemannian Geometry, Graduate Texts in Mathematics, vol. 171, 3rd edn. Springer, Cham (2016). Zbl 06520113
22. Rogers, C. , Schief, W.K.: Bäcklund and Darboux transformations. Geometry and Modern Applications in Soliton Theory. Cambridge Texts in Applied Mathematics, Cambridge University Press (2002). Zbl 1019.53002
23. Sharpe, R.W.: Differential geometry: Cartan's generalization of Klein's Erlangen Program. Graduate Texts in Mathematics, vol. 166. Springer, Berlin (1997). Zbl 0876.53001
24. Shima, H.: The Geometry of Hessian Structures. World Scientific, Hackensack, NJ (2007). Zbl 1244.53004
25. Szabo, R.J.: Equivariant Cohomology and Localization of Path Integrals. Springer, Berlin (2000). Zbl 0998.81520
26. Terras, A.: Harmonic Analysis on Symmetric Spaces-Euclidean Space, the Sphere, and the Poincaré Upper Half Plane. 2nd updated edn. Springer, New York (2013). Zbl 1279.43001