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SEMI-INVARIANT $\xi^{\perp}$-SUBMANIFOLDS OF GENERALIZED QUASI-SASAKIAN MANIFOLDS

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Dedicated to the memory of Prof. Stere Ianuş (1939-2010).

Abstract. A structure on an almost contact metric manifold is defined as a generalization of well-known cases: Sasakian, quasi-Sasakian, Kenmotsu and cosymplectic. This was suggested by a local formula of Eum [9]. Then we consider a semi-invariant $\xi^{\perp}$-submanifold of a manifold endowed with such a structure and two topics are studied: the integrability of distributions defined by this submanifold and characterizations for the totally umbilical case. In particular we recover results of Kenmotsu [11], Eum [9, 10] and Papaghiuc [16].

1. Preliminaries and Basic Formulae

An interesting topic in the differential geometry is the theory of submanifolds in spaces endowed with additional structures, see [7]. In 1978, A. Bejancu (in [2]) studied CR-submanifolds in Kähler manifolds. Starting from it, several papers have been appeared in this field. Let us mention only few of them: a series of papers of B.Y. Chen (e.g. [6]), of A. Bejancu and N. Papaghiuc (e.g. [3] in which the authors studied semi-invariant submanifolds in Sasakian manifolds). See also [14]. The study was extended also to other ambient spaces, for example A. Bejancu in [4] also studied QR-submanifolds in quaternionic manifolds and M. Barros, B.Y. Chen, F. Urbano in [1] investigated CR-submanifolds in quaternionic manifolds. Several important results on CR-submanifolds are being brought together in [4, 6, 12, 14, 15] and the corresponding references. The purpose of the present paper is to investigate the semi-invariant $\xi^{\perp}$-submanifolds in a generalized Quasi-Sasakian manifold.

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Let $\widetilde{M}$ be a real $(2n+1)$-dimensional smooth manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$:

$$\begin{cases} \phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \eta \circ \phi = 0, \ \phi\xi = 0 \\ \eta(X) = \tilde{g}(X, \xi), \tilde{g}(\phi X, Y) + \tilde{g}(X, \phi Y) = 0 \end{cases}$$

for any vector fields X, Y tangent to $\widetilde{M}$ where I is the identity on sections of the tangent bundle $T\widetilde{M}$, ϕ is a tensor field of type $(1,1)$, η is a 1-form, ξ is a vector field and $\tilde{g}$ is a Riemannian metric on $\widetilde{M}$. Throughout the paper all manifolds and maps are smooth. We denote by $\mathcal{F}(\widetilde{M})$ the algebra of the smooth functions on $\widetilde{M}$ and by $\Gamma(E)$ the $\mathcal{F}(\widetilde{M})$-module of the sections of a vector bundle E over $\widetilde{M}$.

The almost contact manifold $\widetilde{M}(\phi, \xi, \eta)$ is said to be *normal* if

$$N_\phi(X, Y) + 2d\eta(X, Y)\xi = 0$$

where

$$N_\phi(X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \quad X, Y \in \Gamma(T\widetilde{M})$$

is the Nijenhuis tensor field corresponding to the tensor field ϕ.

The *fundamental* 2*-form* Φ on $\widetilde{M}$ is defined by $\Phi(X, Y) = \tilde{g}(X, \phi Y)$.

In [9], the author studied hypersurfaces of an almost contact metric manifold $\widetilde{M}$ whose structure tensor fields satisfy the following relation (expressed only in coordinates)

$$(\widetilde{\nabla}_X \phi)Y = \tilde{g}(\widetilde{\nabla}_{\phi X}\xi, Y)\xi - \eta(Y)\widetilde{\nabla}_{\phi X}\xi \tag{1}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of the metric tensor $\tilde{g}$. See also [10]. For the sake of simplicity we say that a manifold $\widetilde{M}$ endowed with an almost contact metric structure satisfying (1) is a *generalized Quasi-Sasakian manifold*, in short G.Q.S. An example of such manifold is furnished in [5]. Define a $(1,1)$ type tensor field F by

$$FX = -\widetilde{\nabla}_X \xi. \tag{2}$$

Proposition 1. *If $\widetilde{M}$ is a G.Q.S manifold then any integral curve of the structure vector field ξ is a geodesic i.e. $\widetilde{\nabla}_\xi \xi = 0$. Moreover $d\Phi = 0$ if and only if ξ is a Killing vector field.*

Proof. The first assertion follows immediately from (1) with $X = Y = \xi$, and taking into account that $\eta(\widetilde{\nabla}_\xi \xi) = 0$. Next, we deduce

$$\begin{aligned} &3d\Phi(X, Y, Z) \\ =\ &\tilde{g}\big((\widetilde{\nabla}_X \phi)Z, Y\big) + \tilde{g}\big((\widetilde{\nabla}_Z \phi)Y, X\big) + \tilde{g}\big((\widetilde{\nabla}_Y \phi)X, Z\big) \\ &+\eta(X)\Big(\tilde{g}(Y, \widetilde{\nabla}_{\phi Z}\xi) + \tilde{g}(\phi Z, \widetilde{\nabla}_Y \xi)\Big) + \eta(Y)\Big(\tilde{g}(Z, \widetilde{\nabla}_{\phi X}\xi) + \tilde{g}(\phi X, \widetilde{\nabla}_Z \xi)\Big) \\ &+\eta(Z)\Big(\tilde{g}(X, \widetilde{\nabla}_{\phi Y}\xi) + \tilde{g}(\phi Y, \widetilde{\nabla}_X \xi)\Big). \end{aligned}$$

If we suppose that ξ is Killing then, from the last equation, we obtain $d\Phi = 0$.

Conversely, suppose that $d\Phi = 0$. Taking into account the first part of the statement, for $X = \xi$, $\eta(Y) = \eta(Z) = 0$, the last relation implies

$$\tilde{g}\big(Y, \widetilde{\nabla}_{\phi Z}\xi\big) + \tilde{g}\big(\phi Z, \widetilde{\nabla}_{Y}\xi\big) = 0.$$

Finally, by replacing Z with ϕZ and Y by $Y - \eta(Y)\xi$ we deduce that ξ is a Killing vector field.

■

The next result can be obtained by direct computation:

Proposition 2. *A G.Q.S manifold $\widetilde{M}$ is normal and*

$$\phi \circ F = F \circ \phi,\ F\xi = 0,\ \eta \circ F = 0,\ \widetilde{\nabla}_{\xi}\phi = 0. \tag{3}$$

Remark 1. (a) It is easy to see that on such manifold $\widetilde{M}$ the structure vector field ξ is not necessarily a Killing vector field i.e. $\widetilde{M}$ is not necessarily a K-contact manifold. Note that any submanifold of a K-contact manifold, normal to the structure vector field, is anti-invariant (see [19]). **b)** It is also interesting to point out that the following particular situations hold

(1) $FX = -\phi X$ then $\widetilde{M}$ is Sasakian,

(2) $FX = -X + \eta(X)\xi$ then $\widetilde{M}$ is Kenmotsu; this is an example of not K-contact structure,

(3) $FX = 0$ then $\widetilde{M}$ is cosymplectic,

(4) if ξ is a Killing vector field then $\widetilde{M}$ is a quasi-Sasakian manifold,

(5) another example of non K-contact manifold arises when F has not a maximal rank,

(6) finally, trans-Sasakian manifolds are not K-contact (see e.g. [8, 13]).

Now, let $\widetilde{M}$ be a G.Q.S manifold and consider an m-dimensional submanifold M, isometrically immersed in $\widetilde{M}$. Denote by g the induced metric on M and by ∇ its Levi-Civita connection. Let $\nabla^{\perp}$ and h be the normal connection induced by $\widetilde{\nabla}$ on the normal bundle $TM^{\perp}$ and the second fundamental form of M, respectively. Then one has the direct sum decomposition $T\widetilde{M} = TM \oplus TM^{\perp}$. Recall the Gauss and Weingarten formulae

$$\text{(G)}\quad \widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$
$$\text{(W)}\quad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad X, Y \in \Gamma(TM)$$

where A_N is the shape operator with respect to the normal section N and satisfies

$$\tilde{g}(h(X, Y), N) = g(A_N X, Y) \quad X, Y \in \Gamma(TM), \quad N \in \Gamma(TM^{\perp}).$$

The aim of the present paper is to investigate the semi-invariant $\xi^\perp$-submanifolds in a G.Q.S manifold. More precisely, we suppose that the structure vector field ξ is orthogonal to the submanifold M. According to Bejancu [4] we say that M is a *semi-invariant* $\xi^\perp$*-submanifold* if there exist two orthogonal distributions, $\mathcal{D}$ and $\mathcal{D}^\perp$, in TM such that:

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp,\ \phi\mathcal{D} = \mathcal{D},\ \phi\mathcal{D}^\perp \subseteq TM^\perp \tag{4}$$

where $\oplus$ denotes the orthogonal sum. If $\mathcal{D}^\perp = \{0\}$ then M is an *invariant* $\xi^\perp$*-submanifold.* The normal bundle can also be decomposed as $TM^\perp = \phi\mathcal{D}^\perp \oplus \mu$, where $\phi\mu \subseteq \mu$. Hence μ contains ξ.

2. Integrability of Distributions on a Semi-invariant $\xi^\perp$-Submanifold

Let M be a semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\widetilde{M}$. Denote by P and Q the projections of TM on $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively, namely for any $X \in \Gamma(TM)$

$$X = PX + QX. \tag{5}$$

Moreover, for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$ we put

$$\phi X = tX + \omega X \tag{6}$$

$$\phi N = BN + CN \tag{7}$$

with $tX \in \Gamma(\mathcal{D})$, $BN \in \Gamma(TM)$ and $\omega X, CN \in \Gamma(TM^\perp)$. We also consider, for $X \in \Gamma(TM)$, the decomposition

$$FX = \alpha X + \beta X, \quad \alpha X \in \Gamma(\mathcal{D}),\ \beta X \in \Gamma(TM^\perp). \tag{8}$$

The purpose of this section is to study the integrability of both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$. With this scope in mind, we state first the following result.

Proposition 3. *Let M be a semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\widetilde{M}$. Then we have*

$$\begin{aligned} &(a)\ (\nabla_X t)Y = A_{\omega Y}X + Bh(X,Y),\\ &(b)\ (\nabla_X \omega)Y = Ch(X,Y) - h(X,tY) + g(FX,\phi Y)\xi, \quad X,Y \in \Gamma(TM). \end{aligned} \tag{9}$$

Proof.

The statement follows immediately from (6)-(8). ■

Taking into consideration the decomposition of $TM^\perp$, it can be easily proved:

Proposition 4. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$. Then for any $N \in \Gamma(TM^{\perp})$ one has:*
(a) $BN \in \mathcal{D}^{\perp}$,
(b) $CN \in \mu$.

Proposition 5. *If M is a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$ then*

$$A_{\omega Z}W = A_{\omega W}Z \tag{10}$$

for any $Z, W \in \Gamma(\mathcal{D}^{\perp})$.

The following two results give necessary and sufficient conditions for the integrability of the two distributions.

Theorem 1. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$. Then the distribution $\mathcal{D}^{\perp}$ is integrable.*

Proof. Let $Z, W \in \Gamma(\mathcal{D}^{\perp})$. Then from (6), (9) and (10) we deduce that

$$t[Z, W] = A_{\omega Z}W - A_{\omega W}Z = 0.$$

Hence the conclusion. ∎

Theorem 2. *If M is a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$ then the distribution $\mathcal{D}$ is integrable if and only if*

$$h(tX, Y) - h(X, tY) = (\mathcal{L}_{\xi}\tilde{g})(X, \phi Y)\ \xi, \quad X, Y \in \Gamma(\mathcal{D}). \tag{11}$$

Proof.
The statement yields directly from (3) and (9)

$$\omega([X, Y]) = h(X, tY) - h(tX, Y) + (\mathcal{L}_{\xi}\tilde{g})(X, \phi Y)\ \xi.$$ ∎

Notice that the two results above are analogue those obtained in the Kenmotsu case in [16] and for the cosymplectic case in [18]. See also [14] when the submanifold is tangent to the structure vector field of the Sasakian manifold.

Moreover, from (8) we deduce

Proposition 6. *Let M be a $\xi^{\perp}$-semi-invariant submanifold of a G.Q.S manifold $\widetilde{M}$. Then*

$$A_{\xi}X = \alpha X, \quad \nabla^{\perp}_{X}\xi = -\beta X, \quad X \in \Gamma(TM). \tag{12}$$

Let now $\{e_i, \phi e_i, e_{2p+j}\}$, $i \in \{1, ..., p\}$, $j \in \{1, ..., q\}$ be an adapted orthonormal local frame on M, where $q = \dim \mathcal{D}^{\perp}$ and $2p = \dim \mathcal{D}$. One can state the following

Theorem 3. *If M is a $\xi^{\perp}$-semi-invariant submanifold of a G.Q.S manifold $\widetilde{M}$ one has*

$$\eta(H) = \frac{1}{m}\ \mathrm{trace}(A_\xi), \quad m = 2p + q.$$

Proof.

Using a general formula for the mean curvature, e.g. $H = \frac{1}{m}\sum_{a=1}^{s}\mathrm{trace}\big(A_{\xi_a}\big)$ ξ_a, where $\{\xi_1, \ldots, \xi_s\}$ is an orthonormal basis in $TM^{\perp}$, the conclusion holds by straightforward computations. ∎

In the case when the ambient space is a Kenmotsu manifold we retrieve the known result from [16, p. 614].

Corollary 1. *There does not exist a minimal semi-invariant $\xi^{\perp}$-submanifold of a Kenmotsu manifold.*

Also it is not difficult to prove:

Theorem 4. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$. Then*

(1) *the distribution $\mathcal{D}$ is integrable and its leaves are totally geodesic in M if and only if $h(X,Y) \in \Gamma(\mu)$, where $X, Y \in \mathcal{D}$;*

(2) *any leaf of the integrable distribution $\mathcal{D}^{\perp}$ is totally geodesic in M if and only if $h(X,Z) \in \Gamma(\mu)$ if $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$.*

Proof.

Let us prove only the first statement. For any $Z \in \mathcal{D}^{\perp}$ we have

$$\begin{aligned}\tilde{g}\big(h(X,Y),\phi Z\big) &= \tilde{g}\big(\widetilde{\nabla}_X Y, \phi Z\big) = -\tilde{g}\big(Y, \widetilde{\nabla}_X(\phi Z)\big) \\ &= -\tilde{g}\big(Y, (\widetilde{\nabla}_X\phi)Z\big) - \tilde{g}\big(\phi Y, \widetilde{\nabla}_X Z\big) = g\big(\nabla_X(\phi Y), Z\big).\end{aligned}$$

Let M^* be a leaf of the integrable distribution $\mathcal{D}$ and h^* the second fundamental form of M^* in M.

For any $Z \in \Gamma(\mathcal{D}^{\perp})$ we have:

$$g(h^*(X,Y),Z) = \tilde{g}(\tilde{\nabla}_X tY, Z) = \tilde{g}((\tilde{\nabla}_X\varphi)Y + \varphi(\tilde{\nabla}_X Y), Z) = -\tilde{g}(h(X,Y),\varphi Z)$$

which proves that the leaf M^* of the integrable distribution $\mathcal{D}$ is totally geodesic in M if and only if $h(X,Y) \in \Gamma(\mu)$. ∎

Notice that the part (2) of the previous Theorem was obtained in the Kenmotsu case by Papaghiuc in [17, p. 115].

We conclude this section with the following

Corollary 2. *If the leaves of the integrable distribution $\mathcal{D}$ are totally geodesic in M then the structure vector field ξ is $\mathcal{D}$-Killing, that is $(\mathcal{L}_\xi g)(X,Y)=0$, $X,Y\in\Gamma(\mathcal{D})$.*

3. Totally umbilical semi-invariant $\xi^\perp$-submanifolds

The main purpose of this section is to obtain a complete characterization of a totally umbilical semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\widetilde{M}$. Recall that for a totally umbilical submanifold we have

$$h(X,Y)=g(X,Y)H,\quad X,Y\in\Gamma(TM).$$

First we state:

Theorem 5. *An invariant $\xi^\perp$-submanifold M of a G.Q.S manifold is totally umbilical if and only if*

$$h(X,Y)=\frac{1}{m}\,g(X,Y)\mathrm{trace}\big(A_\xi\big)\xi. \tag{13}$$

Proof. If M is an invariant $\xi^\perp$-submanifold then for any $X,Y\in\Gamma(TM)$ we have $h(X,\phi Y)=\phi h(X,Y)-g(A_\xi\phi X,Y)\xi$. Let us consider an orthonormal frame $\{e_i,e_{p+i}\}$, $i=1,\dots,p$ on M; from the above relation one obtains that $\phi H=0$. Again, since M is an invariant submanifold:

$$H=g(H,\xi)\xi=\frac{1}{m}\sum_{i=1}^{m}g(h(e_i,e_i),\xi)\xi=\frac{1}{m}\ \mathrm{trace}\big(A_\xi\big)\xi \tag{14}$$

and the proof is complete. ∎

Corollary 3. *A semi-invariant $\xi^\perp$-submanifold of a quasi-Sasakian manifold is minimal.*

The case of a semi-invariant $\xi^\perp$-submanifold in a G.Q.S manifold $\widetilde{M}$ is solved in the next Theorem.

Theorem 6. *Let M be a semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\widetilde{M}$ with $\dim\mathcal{D}^\perp>1$. Then M is totally umbilical if and only if* (13) *holds.*

Proof. Suppose that M is totally umbilical. Let $X\in\Gamma(\mathcal{D})$ be a unit vector field and $N\in\Gamma(\mu)\setminus\mathrm{span}\{\xi\}$. By direct calculation it follows that:

$$g(H,N)=g(h(X,X),N)=g(\widetilde{\nabla}_X\phi X-(\widetilde{\nabla}_X\phi)X,\phi N)=g(h(X,\phi X),\phi N)=0$$

which proves that $H\in\phi\mathcal{D}^\perp\oplus\mathrm{span}\{\xi\}$.

For $Z,W\in\Gamma(\mathcal{D}^\perp)$, from (9) we derive $QA_{\phi Z}W=-g(Z,W)\phi H$ i.e.

$$g(Z,\phi H)g(W,\phi H)=g(Z,W)g(\phi H,\phi H). \tag{15}$$

If we take $Z = W$ orthogonal to ϕH, since $\dim \mathcal{D}^\perp > 1$, from the above relation we infer $\phi H = 0 \Rightarrow H \in \mathrm{span}\{\xi\}$. At this point the conclusion is straightforward.

Conversely, if (13) is supposed to be true, then we get (14) which together with (13) we deduce that M is totally umbilical. ■

Let us remark that when $\widetilde{M}$ is a Kenmotsu manifold the result of the Theorem 6 was proved in [16].

Corollary 4. *Every $\xi^\perp$-hypersurface of a G.Q.S manifold $\widetilde{M}$ is totally umbilical.*

Proof. If M is a hypersurface then $TM^\perp = \mathrm{span}\{\xi\}$ that is $h(X,Y) \in \mathrm{span}\{\xi\}$. Next, from (14) it follows (13). ■

In the particular case of a Kenmotsu manifold this result was obtained by Papaghiuc in [16, p. 617].

As a consequence of Theorem 6, we obtain

Theorem 7. *If M is a totally umbilical semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\widetilde{M}$ with $\dim \mathcal{D}^\perp > 1$, then M is a semi-invariant product.*

Here, by a semi-invariant product we mean a semi-invariant $\xi^\perp$-submanifold of $\widetilde{M}$ which can be locally written as a Riemannian product of a ϕ-invariant submanifold and a ϕ-anti-invariant submanifold of $\widetilde{M}$, both of them orthogonal to ξ.

Proof. From the definition of a totally umbilical submanifold we have $h(X,Z) = 0$ for any $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, so that, by b) of Theorem 4, the leaves of $\mathcal{D}^\perp$ are totally geodesic submanifolds of M. By Theorem 6, we know that $h(X,Y) \in \mathrm{span}\{\xi\} \subset \mu$ for any $X, Y \in \mathcal{D}$. By virtue of a) of Theorem 1, this implies that the invariant distribution $\mathcal{D}$ is integrable and its integral manifolds are totally geodesic submanifolds of M. Therefore, we conclude that M is a semi-invariant product. ■

Without any restriction on the dimension of $\mathcal{D}^\perp$, we have the following

Theorem 8. *Let M be a totally umbilical semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\widetilde{M}$. If $\mathcal{D}$ is integrable, then each leaf of $\mathcal{D}$ is a totally geodesic submanifold of M.*

Proof. By using b) of Proposition 3, for any $X \in \Gamma(\mathcal{D})$, we have

$$\omega(\nabla_X X) = -g(X,X)CH - g(FX, \phi Y)\xi.$$

Since $CH \in \mu$ by b) of Lemma 4 and $\omega U \in \phi\mathcal{D}^\perp$ for any $U \in \Gamma(TM)$, from the above equation we deduce that $\omega(\nabla_X X) = 0$, or equivalently

$$\nabla_X X \in \mathcal{D}, \quad \forall X \in \Gamma(\mathcal{D}).$$

Replacing X by $X+Y$, we get $\nabla_X Y + \nabla_Y X \in \Gamma(\mathcal{D})$ for all $X, Y \in \Gamma(\mathcal{D})$. This condition, together with the integrability of $\mathcal{D}$, implies

$$\nabla_X Y \in \mathcal{D}, \quad \forall X, Y \in \Gamma(\mathcal{D}). \tag{16}$$

As $\mathcal{D}$ is integrable, Frobenius theorem ensures that M is foliated by leaves of $\mathcal{D}$. Combining this fact with (16), we conclude that the leaves of $\mathcal{D}$ are totally geodesic submanifolds of M. ∎

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