Projections on complete, finite-dimensional Riemannian manifolds

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Abstract

We study the best approximation problem on complete, finite-dimensional Riemannian manifolds.

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Introduction

Let Z be a smooth finite-dimensional, complet and connected Riemannian manifold. We denote by d the Riemannian distance on Z and for $p \in Z$ and $r > 0$ let:

$$B(O_p, r) = \{h \in T_pZ; \|h\|_p < r\} \quad \overline{B}(O_p, r) = \{h \in T_pZ; \|h\|_p \leq r\}$$
$$B(p, r) = \{x \in Z; d(p, x) < r\} \quad \overline{B}(p, r) = \{x \in Z; d(p, x) \leq r\}$$

and $\exp_p$ be the exponential map in p.

We recall the Hopf-Rinow theorem([2, p.360]):

The following assertions are equivalent:

(i) Z is complete

(ii) exp_p is defined for all $h \in T_pZ$

(iii) the closed and bounded(related at d) subsets of Z are compacts .

In addition these properties imply:

(iv) $B(p,r) = \exp_p(B(O_p,r))$ $\overline{B}(p,r) = \exp_p(\overline{B}(O_p,r))$ $\forall r > 0$

(v) $\forall q \in Z$ there exists $h \in T_pZ$ (not generally unique) such that $c : [0,1] \to Z$ $c(t) = \exp_p(th)$ is minimal geodesic from p to q with $L(c) = d(p,q) = \|h\|_p$.

In the sections 1-4 of this paper we study the notion which concretises in the Riemannian manifolds context the Dudek and Holly's notion([3]).

Namely, given a nonempty subset M of Z we define *the projection onto* M to be a relation $P \subset Z \times M$ the domain of which is given by:

$$\operatorname{dom} P = \{z \in Z;\ \text{there exists a unique } a \in M \text{ such that } d(z,a) = d(z,M)\}.$$

So we obtain a map $P : \operatorname{dom} P \to M, P(z) = a$. Let Ω be the interior of $\operatorname{dom} P$.

In the last section we discuss the general best approximation problem relative at two, disjoint subsets of Z and generalise a result of Motreanu([7]). According to ([3]) we consider the following set:
$M(cl) = \{z \in Z; \exists r > d(z,M) \text{ such that } M \cap \overline{B}(z,r) \text{ is closed in } Z\}$
(in [3] this set is denoted by D).

1 The domain of P

Firstly, we proves three results which Dudek and Holly point out in [3]:

Proposition 1. *Let $I : Z \to Z$ be an isometry.Then $I \circ P \circ I^{-1}$ is the projection onto $I(M)$.*

Proof. Denote by $\overline{P}$ the map $I \circ P \circ I^{-1}$ and $P_{I(M)}$ the projection onto $I(M)$. It must be proved that $\operatorname{dom} \overline{P} = \operatorname{dom} P_{I(M)}$ and $\forall z \in \operatorname{dom} \overline{P} = \operatorname{dom} P_{I(M)}$ we have $\overline{P}(z) = P_{I(M)}(z)$.
i) Consider $z \in \operatorname{dom} \overline{P}$. Then:

$$(*)\ \ d(z, I \circ P \circ I^{-1}(z)) = d(I^{-1}(z), P(I^{-1}(z))) = d(I^{-1}(z), M) = d(z, I(M)).$$

If $a \in I(M)$ with $d(z,a) = d(z, I(M))$ there exists $p \in M$ such that $I(p) = a$. It follows:

$$d(I^{-1}(z), M) = d(z, I(M)) = d(z,a) = d(z, I(p)) = d(I^{-1}(z), p)$$

But $z \in \operatorname{dom} \overline{P}$ i.e.$I^{-1}(z) \in \operatorname{dom} P$ and the last relation yields $P(I^{-1}(z)) = p$ i.e. $\overline{P}(z) = I(p) = a$. By relation $(*)$ we have $z \in \operatorname{dom} P_{I(M)}$ i.e. $\operatorname{dom} \overline{P} \subseteq \operatorname{dom} P_{I(M)}$ and $\overline{P} = P_{I(M)}|_{\operatorname{dom} \overline{P}}$.
ii) Consider $z \in \operatorname{dom} P_{I(M)}$. Then there exists a unique $a \in M$ such that $P_{I(M)}(z) = I(a)$.We get:
$d(z, I(M)) = d(I^{-1}(z), M) = d(z, I(a)) = d(I^{-1}(z), a)$
i.e. $d(I^{-1}(z), M) = d(I^{-1}(z), a)$.
If $b \in M$ is such that $d(I^{-1}(z), b) = d(I^{-1}(z), M)$ then $d(z, I(M)) = d(z, I(b))$. From this relation it follows that $P_{I(M)}(z) = I(b)$.But $P_{I(M)}(z) = I(a)$. Therefore $I(a) = I(b)$ i.e. $a = b$. Obvious,$I^{-1}(z) \in \operatorname{dom} P$ and $P(I^{-1}(z)) = a$ i.e. $\operatorname{dom} P_{I(M)} \subseteq \operatorname{dom} \overline{P}$.From $P(z) = I(a) = P_{I(M)}(z)$ we have $P_{I(M)} = \overline{P}|_{\operatorname{dom} P_{I(M)}}$. □

Proposition 2. *Let M,U be subsets of Z and $I : Z \to Z$ be an isometry such that $I(M) = M$. Then $I(U \setminus \operatorname{dom} P) = I(U) \setminus \operatorname{dom} P$ where P is the projection onto M.*

Proof. Firstly we prove that $I(U \setminus \operatorname{dom} P) \subseteq I(U) \setminus \operatorname{dom} P$. Fix $u \in U \setminus \operatorname{dom} P$. It must be proved that $I(u) \notin \operatorname{dom} P$. Assume, contrary to our claim, that $I(u) \in \operatorname{dom} P$. Then there exists $P \circ I(u) = v$ with $v \in M$. Since $I(M) = M$ there exists $a \in M$ with $I(a) = v$ and then $P(I(u)) = I(a)$. We have:

$$d(I(u), I(a)) = d(u, a) = d(I(u), M) = d(I(u), I(M)) = d(u, M)$$

Let $b \in M$ such that $d(u, b) = d(u, M)$ (by last relation exists $b = a$ with this property). Then $d(I(u), I(b)) = d(I(u), I(M)) = d(I(u), M)$. It follows that $P(I(u)) = I(b)$; then $I(a) = I(b)$ i.e. $a = b$. This means that a is the unique point of M suct that $d(u, a) = d(u, M)$ contrary to $u \notin \operatorname{dom} P$. There remains to prove that: $I(U) \setminus \operatorname{dom} P \subseteq I(U \setminus \operatorname{dom} P)$. Fix $u \in U$ such that $I(u) \notin \operatorname{dom} P$. It must be proved that $u \notin \operatorname{dom} P$. If $u \in \operatorname{dom} P$ then exists $a \in M$ such that $P(u) = a$. Since $d(u, M) = d(I(u), I(M)) = d(I(u), I(a))$ and a similar argument we have $I(u) \in \operatorname{dom} P$ with $P(I(u)) = I(a)$ contrary to $I(u) \notin \operatorname{dom} P$. □

Proposition 3.

(i) *If M is closed then $M(cl)$=Z*

(ii) *If $z \in M(cl)$ there exists at least one element of M which realises the distance $d(z,M)$.*

Proof. i) Fix $z \in Z$ and $r > d(z,M)$. M and $\overline{B}(z,r)$ are closed. Then $M \cap \overline{B}(z,r)$ is closed i.e. $z \in M(cl)$.
ii) By definition exists $r > d(z,M)$ such that $M \cap \overline{B}(z,r)$ is closed. If $x \in M \cap (Z \setminus \overline{B}(z,r))$ then $d(z,x) > r > d(z,M)$. This means that $d(z,M) = x \in M \cap \overline{B}(z,r) \inf d(z,x)$. By Hopf-Rinow iii) $M \cap \overline{B}(z,r)$ is compact. Hence there exists $x \in M \cap \overline{B}(z,r)$ such that $d(z,x) = a \in M \cap \overline{B}(z,r) \inf d(z,a) = d(z,M)$. $\square$

A consequence of proposition 1 and 2 is the following:

Corollary 1. *Let $I : Z \to Z$ be an isometry such that $I(M) = M$.Then:*

(i) $I \circ P = P \circ I$

(ii) $I(\operatorname{dom} P) = \operatorname{dom} P$.

Proof. i) By proposition 1 there results that $I \circ P \circ I^{-1} = P_{I(M)} = P$ because $I(M) = M$ i.e. $I \circ P = P \circ I$.
ii) By proposition 2 where $U = Z$ we have $I(Z \setminus \operatorname{dom} P) = Z \setminus \operatorname{dom} P$. Since I is bijection it follows ii). $\square$

In the Riemannian manifolds context we obtain the following result which generalises a Dudek and Holly's theorem (theorem 1.5 i) of [3]):

Theorem 1. *Let $z \in Z$ and $a \in M$ with d(z,a)=d(z,M). Let c(z,a) be a minimal geodesic from z to a. Then $c(z,a) \setminus \{z\} \subset \operatorname{dom} P$ and $\forall x \in c(z,a) \setminus \{z\}$ we have P(x)=a.*

Proof. Fix $x \in c(z,a) \setminus \{z\}$; then $d(z,a) = d(z,x) + d(x,a)$. It must be proved that $\forall b \in M \setminus \{a\} \quad d(x,a) < d(x,b)$. Suppose, contrary to our claim, that $\exists b \in M \setminus \{a\}$ with $d(x,b) \leq d(x,a)$. We have two situations:
i) $b \in c(x,a)$. Since $b \neq a$ there results that $d(x,b) < d(x,a)$ and

$$d(z,b) = d(z.x) + d(x,b) < d(z,x) + d(x,a) = d(z,a)$$

contrary to $d(z,a) = d(z,M)$.
ii) $b \notin c(x,a)$. Then $c(x,a)$, $c(x,b)$ are distincts i.e. the tangent versors to minimal geodesics $c(z,x)$, $c(x,b)$ are distincts. Now, we uses the following result([2, p.359]): let a, b, c be distincts points of Z, C_1 a minimal geodesic from a to b and C_2 a minimal geodesic from b to c. If tangent versors to C_1

and C_2 to b are distincts then $d(a,c) < d(a,b) + d(b,c)$.
From this there follows:

$$d(z.b) < d(z,x) + d(x,b) \leq d(z,x) + d(x,a) = d(z,a)$$

again false. □

Corollary 2.

(i) $M(cl) \subset \overline{\operatorname{dom} P}$

(ii) *If M is closed in Z then* $\overline{\operatorname{dom} P} = Z$.

Proof. i) For $z \in M(cl)$ by proposition 3 ii) exists $a \in M$ with $d(z,a) = d(z,M)$. By theorem 1 $c(z,a) \setminus \{z\} \subset \operatorname{dom} P$. Since $z \in \overline{c(z,a) \setminus \{z\}}$ we have i).
ii) By i) and proposition 3 i). □

Now we give another properties of dom P:

Lemma 1. *If $z \in \operatorname{dom} P \setminus M$ then $P(z) \in M \setminus intM$.*

Remark These lemma is a partial answer to exercise 15, [6, p.59].

Proof. Suppose that $P(z) \in intM$. Then exists $r > 0$ such that $\overline{B}(P(z), r) \subset M$. If $d(z, P(z)) \leq r$ then $z \in \overline{B}(P(z), r) \subset M$ false. Suppose $d(z, P(z)) > r$ and let $c(z, P(z))$ be a minimal geodesic from z to $P(z)$. Then exists $u \in c(z, P(z)) \cap FrB(P(z), r)$. It follows $d(z, P(z)) = d(z,u) + d(u, P(z))$ and $d(u, P(z)) = r > 0$. Therefore $d(z,u) < d(z, P(z))$ and $u \in M$ contrary to $u \neq P(z)$. □

Theorem 2. *In the following situations we have* $\operatorname{dom} P = M$*:*

(i) *M is open*

(ii) *M is dense in Z.*

Proof. Since $M \subseteq \operatorname{dom} P$ is suffice to establish that $\operatorname{dom} P \subseteq M$.
i) Suppose that $\exists z \in \operatorname{dom} P \setminus M$. By lemma 1 we have $z \in M \setminus intM = \phi$ false.
ii) Fix $z \in \operatorname{dom} P$. Since M is dense $d(z,M) = 0 = d(z, P(z))$ i.e. $z = P(z) \in M$. □

Theorem 3. $\operatorname{dom} P$ *has'n isolated points.*

Proof. Suppose, contrary to our claim, that exists an isolated point $z \in \operatorname{dom} P$ i.e. there is V a neighbourhood of z such that $(*)\ V \cap \operatorname{dom} P = \{z\}$. From $z \in \operatorname{dom} P$ and theorem 1 $c(z, P(z)) \subset \operatorname{dom} P$. Consider a sequence $(x_n)_n \in c(z, P(z)) \setminus \{z\}$ with $x_n \to z$. Exists n_0 such that $n \geq n_0$ imply $x_n \in V$. Then, for $n \geq n_0 \quad x_n \in V \cap \operatorname{dom} P$ and $x_n \neq z$ contrary to $(*)$. □

2 Continuity of projections

The next lemma extend lemma 1'.1 of [3]:

Lemma 2. *If* $z \in \Omega$ *and* V *is a neighbourhood of* $P(z)$ *in* M *then* $d(z, M) < d(z, M \setminus V)$.

Proof. If the inequality were inverse then $d(z, M) = d(z, M \setminus V)$. There is $(a_n)_n \in M \setminus V$ for which $d(z, a_n) \to d(z, M \setminus V) = d(z, M)$. By Hopf-Rinow v) $\forall n$ there is $h_n \in T_z Z$ such that $a_n = exp_z(h_n)$ and $d(z, a_n) = \|h_n\|_z \to d(z, M)$ i.e. the sequence $(h_n)_n$ is bounded. By Cesaro's lemma we can assume that $h_n \to h \in T_z Z$. Consider $b = exp_z(h)$ and the minimal geodesic $c(z, b)(t) = exp_z(th_n)$, $t \in [0, 1]$ from z to b. Because $(a_n)_n \in M \setminus V$ and $(a_n)_n \to b$ we have $b \neq P(z) \in V$ and $d(z, M) = d(z, b)$. This means $(*)\ b \notin M$. Since $z \in \Omega$ there is $u \in (c(z, b) \setminus \{z\}) \cap \operatorname{dom} P$. By Hopf-Rinow v) there is $\bar{h} \in T_z Z$ with $P(u) = exp_z(\bar{h})$. We get:

$$d(z, b) = d(z, M) \leq d(z, P(u)) \leq d(z, u) + d(u, P(u)) \leq d(z, u) + d(u, b) = d(z, b)$$

(we have $d(u, P(u)) \leq d(u, b)$ because $d(u, P(u)) = d(u, M) \leq d(u, a_n) \to d(u, b)$). Therefore in the last relation is equality between all terms. Then $d(z, P(u)) = d(z, u) + d(u, P(u))$ i.e. $u \in c(z, P(u)) \setminus \{z\}$. Hence $u \in c(z, b) \setminus \{z\}$ and $u \in c(z, P(u)) \setminus \{z\}$. Then $u = exp_z(t_0 h) = exp_z(t_0 \bar{h})$ with $t_0 \in (0, 1)$ ($t_0 = 1 \Rightarrow u = P(u) = b$ false by $(*)$). We can made t_0 small so that $\max\{t_0 \|h\|_z, t_0 \|\bar{h}\|_z\} < r$ and $exp_z|_{B(O_z, r)}$ is difeomorphism. Hence $t_0 h = t_0 \bar{h}$ i.e. $h = \bar{h}$. Then $b = P(u) \in M$ contrary to $(*)$. □

Theorem 4. $P|_{\Omega}$ *is continuous.*

Proof. The argument is inspired from the proof of theorem 1.3 of [3]. Suppose that P is not continous at $z \in \Omega$.There is a neighbourhood V of $P(z)$

in M and a sequence $(x_n)_n \in \operatorname{dom} P$ with $(x_n)_n \to z$ and $P(x_n) \notin V \quad \forall n$. Then:

$$d(z, M \setminus V) \leq d(z, P(x_n)) \leq d(z, x_n) + d(x_n, P(x_n)) = d(z, x_n) + d(x_n, M)$$

At limit we obtain $d(z, M \setminus V) \leq d(z, M)$ contrary to lemma 2. □

Remark For case Z an euclidian space (e.g. R^n) and M a closed subset theorem 3 of [6, p. 119] said that P is continuous on all $\operatorname{dom} P$.

3 Projection onto a submanifold

Now let M be a submanifold of Z. By theorem 3 of [1, p. 151] we have:

Theorem 5. *Let $M_i, i = 1, 2$ be submanifolds of Z and c a geodesic from $m_1 \in M_1$ to $m_2 \in M_2$ such that $L(c) = d(M_1, M_2)$. Then c is normal to M_i at m_i $i = 1, 2$.*

For $M_1 = M$ and $M_2 = \{z\}$ we obtain:

Corollary 3. *Let M be a submanifold of Z and $z \in \operatorname{dom} P$. Then $c(z, P(z))$ is normal to M at $P(z)$. That means $z = exp_{P(z)}(h)$ with $h \in (T_{P(z)}M)^{\perp}$.*

4 The case Z euclidian space

We begin with a more simple proof of lemma 1'. 2 of [3]:

Proposition 4. *Let Z be an euclidian space, $t \in (0, 1]$ and $f_t : \operatorname{dom} P \to Z, f_t(z) = P(z) + t(z - P(z))$. Then f_t is an injection*

Proof. The case $t = 1$ is easy. Let $t \in (0, 1)$ and $a, b \in \operatorname{dom} P$ with $f_t(a) = f_t(b) = u$. By proposition 1 we can assume that $u = 0$ without loss of generality. So for $s = \frac{t}{1-t}$ we have $P(a) = sa$ and $P(b) = sb$. By theorem 1 $P(a) = P(b) = P(0)$ because $u = 0 \in c(a, P(a)) = (a, P(a))$ and $u = 0 \in c(b, P(b)) = (b, P(B))$ (we denote by $c(a, P(a))$ the open segment joining a and $P(a)$ who in Z euclidian space is the unique minimal geodesic from a to $P(a)$)). Then $sa = sb$ i.e. $a = b$. □

We define a new set:

$$M(conv) = \{z \in Z; \exists r > d(z, M) \text{ such that } M \cap \overline{B}(z, r) \text{ is convex } \}$$

Lemma 3. $M(cl) \cap M(conv) \subset \operatorname{dom} P$.

Proof. Fix $z \in M(cl) \cap M(conv)$; there exists $r_1, r_2 > d(z, M)$ such that $M \cap \overline{B}(z, r_1)$ is closed in Z and $M \cap \overline{B}(z, r_2)$ is convex. Let $r = \min(r_1, r_2)$. $M \cap \overline{B}(z, r) \subset M \cap \overline{B}(z, r_1) \Rightarrow cl(M \cap \overline{B}(z, r)) \subseteq cl(M \cap \overline{B}(z, r_1)) = M \cap \overline{B}(z, r_1) \subset M$. Therefore $cl(M \cap \overline{B}(z, r)) \subset M \cap \overline{B}(z, r)$ i.e. $M \cap \overline{B}(z, r)$ is closed in Z. $M \cap \overline{B}(z, r) \subset M \cap \overline{B}(z, r_2) \Rightarrow conv(M \cap \overline{B}(z, r)) \subseteq conv(M \cap \overline{B}(z, r_2)) = M \cap \overline{B}(z, r_2)) \subset M$. Then $conv(M \cap \overline{B}(z, r)) \subseteq M \cap \overline{B}(z, r)$ i.e. $M \cap \overline{B}(z, r)$ is convex. Consequently $M \cap \overline{B}(z, r)$ is closed and convex in an euclidian space and then there exists a unique $a \in M \cap \overline{B}(z, r)$ such that $d(z, a) = d(z, M \cap \overline{B}(z, r)) = d(z, M)$. □

Corollary 4. *Suppose that Z is an euclidian space:*

(i) *If M is convex then $M(cl) \subset \Omega$*

(ii) *If M is convex and locally compact then $M(cl) = \Omega$.*

Proof. If M is convex then $M(conv) = M$;therefore
$M(cl) \cap M(conv) = M(cl)$
i) result from last theorem and fact 2.3 of [3] who said that $M(cl)$ is open
ii) result from i) and proposition 2.6 of [3]. □

5 The general best approximation problem

Return to case Z a complete, finite-dimensional Riemannian manifold. Let S_0 and S_1 be two disjoint subsets of Z. In the following we shall be concerned with the problem of the existence of points $x_i \in S_i$ $i = 0, 1$ such that $d(S_0, S_1) = d(x_0, x_1)$. Denote by R the distance $d(S_0, S_1)$ and for $r > 0$ $B(S_0, r) = \{z \in Z; d(z, S_0) < r\}, A(r) = S_1 \cap \overline{B}(S_0, R + r)$.
Remark. For every $r > 0$ we have:

i) S_1 closed $\Rightarrow A(r)$ closed

ii) $A(r) \neq \phi$.

Theorem 6. *In the following situations there exists* $x_i \in S_i$ $i = 0, 1$ *such that* $d(S_0, S_1) = d(x_0, x_1)$*:*

(i) $\exists\ \varepsilon > 0$ *such that* $A(\varepsilon)$ *is compact and* $A(\varepsilon) \subset S_0(cl)$

(ii) S_1 *is closed ,* S_0 *is bounded and* $\exists\ \varepsilon > 0$ *such that* $A(\varepsilon) \subset S_0(cl)$

(iii) S_1 *is compact and* $S_1 \subset S_0(cl)$

(iv) S_1 *is compact and* S_0 *is closed.*

Proof. i) $\exists\ (x_n^i)_n \in S_i$ $i = 0, 1$ with $d(x_n^0, x_n^1) \to R$. We have:
$R \leq d(x_n^1, S_0) \leq d(x_n^1, x_n^0) \to R$; it follows that $d(x_n^1, S_0) \to R$. Then we can assume that $x_n^1 \in A(\varepsilon)$ $\forall n$. But $A(\varepsilon)$ is compact and then $x_n^1 \to x_1 \in A(\varepsilon)$. By $x_1 \in A(\varepsilon) \subset S_0(cl)$ and proposition 3 ii) there exists $x_0 \in S_0$ such that $d(x_1, S_0) = d(x_0, x_1)$. Therefore

$$d(x_0, x_1) = d(x_1, S_0) = \lim_{n\to\infty} d(x_1^n, S_0) = R = d(S_0, S_1)$$

ii) S_0 bounded and S_1 closed $\Rightarrow A(\varepsilon)$ is closed and bounded. By Hopf-Rinow
iii) $A(\varepsilon)$ is compact and apply i)
iii) the same argument like i) with $A(\varepsilon)$ replaced by S_1
iv) by iii) and proposition 3 i). □

Remark.

i) $d(S_0, S_1) = d(S_0, A(\varepsilon))$

ii) the case iv) of last theorem is theorem 1 (i) of [7].

Final remark. What are possible extensions of these results? The main tool of our study is Hopf-Rinow theorem and which is:

i) false for infinite-dimensional Riemannian manifolds ([4])

ii) true for finite-dimensional Finsler manifolds ([5])

Then most part of these results can be reformulated for complete, finite-dimensional Finsler manifolds.

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