Projections on complete, finite-dimensional Riemannian manifolds

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Abstract

We study the best approximation problem on complete, finite-dimensional Riemannian manifolds.

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Introduction

Let $Z$ be a smooth finite-dimensional, complete and connected Riemannian manifold. We denote by $d$ the Riemannian distance on $Z$ and for $p \in Z$ and $r > 0$ let:

\[ B(O_p, r) = \{ h \in T_pZ; \| h\|_p < r \} \]
\[ B(p, r) = \{ x \in Z; d(p, x) < r \} \]
\[ \overline{B}(O_p, r) = \{ h \in T_pZ; \| h\|_p \leq r \} \]
\[ \overline{B}(p, r) = \{ x \in Z; d(p, x) \leq r \} \]

and $\exp_p$ be the exponential map in $p$.

We recall the Hopf-Rinow theorem([2, p.360]):

The following assertions are equivalent:

(i) $Z$ is complete
(ii) $exp_p$ is defined for all $h \in T_pZ$
(iii) the closed and bounded(related at $d$) subsets of $Z$ are compacts .

In addition these properties imply:
(iv) $B(p, r) = \exp_p(B(O_p, r)) \quad \mathcal{B}(p, r) = \exp_p(\mathcal{B}(O_p, r)) \quad \forall r > 0$

(v) $\forall q \in Z$ there exists $h \in T_p Z$ (not generally unique) such that $c : [0, 1] \rightarrow Z \quad c(t) = \exp_p(th)$ is minimal geodesic from $p$ to $q$ with $L(c) = d(p, q) = \|h\|_p$.

In the sections 1-4 of this paper we study the notion which concretises in the Riemannian manifolds context the Dudek and Holly’s notion([3]).

Namely, given a nonempty subset $M$ of $Z$ we define the projection onto $M$ to be a relation $P \subseteq Z \times M$ the domain of which is given by:

$$\text{dom} P = \{z \in Z; \text{ there exists a unique } a \in M \text{ such that } d(z, a) = d(z, M)\}.$$ 

So we obtain a map $P : \text{dom} P \rightarrow M, P(z) = a$. Let $\Omega$ be the interior of $\text{dom} P$.

In the last section we discuss the general best approximation problem relative at two, disjoint subsets of $Z$ and generalise a result of Motreanu([7]). According to ([3]) we consider the following set:

$M(\overline{c}) = \{z \in Z; \exists r > d(z, M) \text{ such that } M \cap \mathcal{B}(z, r) \text{ is closed in } Z\}$

(in [3] this set is denoted by $D$).

1 The domain of $P$

Firstly, we prove three results which Dudek and Holly point out in [3]:

**Proposition 1.** Let $I : Z \rightarrow Z$ be an isometry. Then $I \circ P \circ I^{-1}$ is the projection onto $I(M)$.

**Proof.** Denote by $\overline{P}$ the map $I \circ P \circ I^{-1}$ and $P_{I(M)}$ the projection onto $I(M)$. It must be proved that $\text{dom} \overline{P} = \text{dom} P_{I(M)}$ and $\forall z \in \text{dom} \overline{P}$ we have $\overline{P}(z) = P_{I(M)}(z)$.

i) Consider $z \in \text{dom} \overline{P}$. Then:

\[
(*) \quad d(z, I \circ P \circ I^{-1}(z)) = d(I^{-1}(z), P(I^{-1}(z))) = d(I^{-1}(z), M) = d(z, I(M)).
\]

If $a \in I(M)$ with $d(z, a) = d(z, I(M))$ there exists $p \in M$ such that $I(p) = a$. It follows:

\[
d(I^{-1}(z), M) = d(z, I(M)) = d(z, a) = d(z, I(p)) = d(I^{-1}(z), p)
\]

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But $z \in \text{dom} \overline{P}$ i.e. $I^{-1}(z) \in \text{dom} P$ and the last relation yields $P(I^{-1}(z)) = p$ i.e. $\overline{P}(z) = I(p) = a$. By relation (*) we have $z \in \text{dom} P_I(M)$ i.e. $\text{dom} \overline{P} \subseteq \text{dom} P_I(M)$ and $\overline{P} = P_I(M)|_{\text{dom} \overline{P}}$.

ii) Consider $z \in \text{dom} P_I(M)$. Then there exists a unique $a \in M$ such that $P_I(M)(z) = I(a)$. We get:

$$d(z, I(M)) = d(I^{-1}(z), M) = d(z, I(a)) = d(I^{-1}(z), a)$$

i.e. $d(I^{-1}(z), M) = d(I^{-1}(z), a)$.

If $b \in M$ is such that $d(I^{-1}(z), b) = d(I^{-1}(z), M)$ then $d(z, I(M)) = d(z, I(b))$.

From this relation it follows that $P_I(M)(z) = I(b)$. But $P_I(M)(z) = I(a)$. Therefore $I(a) = I(b)$ i.e. $a = b$. Obvious, $I^{-1}(z) \in \text{dom} P$ and $P(I^{-1}(z)) = a$ i.e. $\text{dom} P_I(M) \subseteq \text{dom} \overline{P}$. From $P(z) = I(a) = P_I(M)(z)$ we have $P_I(M) = \overline{P}|_{\text{dom} P_I(M)}$. □

**Proposition 2.** Let $M, U$ be subsets of $Z$ and $I : Z \rightarrow Z$ be an isometry such that $I(M) = M$. Then $I(U \setminus \text{dom} P) = I(U) \setminus \text{dom} P$ where $P$ is the projection onto $M$.

**Proof.** Firstly we prove that $I(U \setminus \text{dom} P) \subseteq I(U) \setminus \text{dom} P$. Fix $u \in U \setminus \text{dom} P$. It must be proved that $I(u) \notin \text{dom} P$. Assume, contrary to our claim, that $I(u) \in \text{dom} P$. Then there exists $P \circ I(u) = v$ with $v \in M$. Since $I(M) = M$ there exists $a \in M$ with $I(a) = v$ and then $P(I(u)) = I(a)$. We have:

$$d(I(u), I(a)) = d(u, a) = d(I(u), M) = d(I(u), I(M)) = d(u, M)$$

Let $b \in M$ such that $d(u, b) = d(u, M)$ (by last relation exists $b = a$ with this property). Then $d(I(u), I(b)) = d(I(u), I(M)) = d(I(u), M)$. It follows that $P(I(u)) = I(b)$; then $I(a) = I(b)$ i.e. $a = b$. This means that $a$ is the unique point of $M$ such that $d(u, a) = d(u, M)$ contrary to $u \notin \text{dom} P$. There remains to prove that: $I(U) \setminus \text{dom} P \subseteq I(U) \setminus \text{dom} P$. Fix $u \in U$ such that $I(u) \notin \text{dom} P$. It must be proved that $u \notin \text{dom} P$. If $u \in \text{dom} P$ then exists $a \in M$ such that $P(u) = a$. Since $d(u, M) = d(I(u), I(M)) = d(I(u), I(a))$ and a similar argument we have $I(u) \in \text{dom} P$ with $P(I(u)) = I(a)$ contrary to $I(u) \notin \text{dom} P$. □

**Proposition 3.**

(i) If $M$ is closed then $M(cl) = Z$

(ii) If $z \in M(cl)$ there exists at least one element of $M$ which realises the distance $d(z, M)$.  

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Proof. i) Fix \( z \in Z \) and \( r > d(z, M) \). \( M \) and \( \overline{B}(z, r) \) are closed. Then \( M \cap \overline{B}(z, r) \) is closed i.e. \( z \in M(\text{cl}) \).

ii) By definition exists \( r > d(z, M) \) such that \( M \cap \overline{B}(z, r) \) is closed. If \( x \in M \cap (Z \setminus B(z, r)) \) then \( d(z, x) > r > d(z, M) \). This means that \( d(z, M) = x \in M \cap \overline{B}(z, r) \inf d(z, x) \). By Hopf-Rinow iii) \( M \cap \overline{B}(z, r) \) is compact. Hence there exists \( x \in M \cap \overline{B}(z, r) \) such that \( d(z, x) = a \in M \cap \overline{B}(z, r) \inf d(z, a) = d(z, M) \).

A consequence of proposition 1 and 2 is the following:

Corollary 1. Let \( I : Z \rightarrow Z \) be an isometry such that \( I(M) = M \). Then:

(i) \( I \circ P = P \circ I \)

(ii) \( I(\text{dom } P) = \text{dom } P \).

Proof. i) By proposition 1 there results that \( I \circ P \circ I^{-1} = P_{I(M)} = P \) because \( I(M) = M \) i.e. \( I \circ P = P \circ I \).

ii) By proposition 2 where \( U = Z \) we have \( I(Z \setminus \text{dom } P) = Z \setminus \text{dom } P \). Since \( I \) is bijection it follows ii).

In the Riemannian manifolds context we obtain the following result which generalises a Dudek and Holly’s theorem (theorem 1.5 i) of [3]):

Theorem 1. Let \( z \in Z \) and \( a \in M \) with \( d(z, a) = d(z, M) \). Let \( c(z, a) \) be a minimal geodesic from \( z \) to \( a \). Then \( c(z, a) \setminus \{z\} \subset \text{dom } P \) and \( \forall x \in c(z, a) \setminus \{z\} \) we have \( P(x) = a \).

Proof. Fix \( x \in c(z, a) \setminus \{z\} \); then \( d(z, a) = d(z, x) + d(x, a) \). It must be proved that \( \forall b \in M \setminus \{a\} \) \( d(x, a) < d(x, b) \). Suppose, contrary to our claim, that \( \exists b \in M \setminus \{a\} \) with \( d(x, b) \leq d(x, a) \). We have two situations:

i) \( b \in c(x, a) \). Since \( b \neq a \) there results that \( d(x, b) < d(x, a) \) and

\[
d(z, b) = d(z, x) + d(x, b) < d(z, x) + d(x, a) = d(z, a)
\]

contrary to \( d(z, a) = d(z, M) \).

ii) \( b \notin c(x, a) \). Then \( c(x, a), c(x, b) \) are distincts i.e. the tangent versors to minimal geodesics \( c(z, x), c(x, b) \) are distincts. Now, we uses the following result([2, p.359]): let \( a, b, c \) be distincts points of \( Z \), \( C_1 \) a minimal geodesic from \( a \) to \( b \) and \( C_2 \) a minimal geodesic from \( b \) to \( c \). If tangent versors to \( C_1 \)
and $C_2$ to $b$ are distincts then $d(a, c) < d(a, b) + d(b, c)$.

From this there follows:

$$d(z, b) < d(z, x) + d(x, b) \leq d(z, x) + d(x, a) = d(z, a)$$

again false. □

**Corollary 2.**

(i) $M(cl) \subseteq \overline{\text{dom}} \ P$

(ii) If $M$ is closed in $Z$ then $\overline{\text{dom}} \ P = Z$.

**Proof.** i) For $z \in M(cl)$ by proposition 3 ii) exists $a \in M$ with $d(z, a) = d(z, M)$. By theorem 1 $c(z, a) \setminus \{z\} \subset \text{dom} \ P$. Since $z \in c(z, a) \setminus \{z\}$ we have i).

ii) By i) and proposition 3 i). □

Now we give another properties of dom $P$:

**Lemma 1.** If $z \in \text{dom} \ P \setminus M$ then $P(z) \in M \setminus \text{int} \ M$.

**Remark** These lemma is a partial answer to exercise 15, [6, p.59].

**Proof.** Suppose that $P(z) \in \text{int} \ M$. Then exists $r > 0$ such that $\overline{B}(P(z), r) \subset M$. If $d(z, P(z)) \leq r$ then $z \in \overline{B}(P(z), r) \subset M$ false. Suppose $d(z, P(z)) > r$ and let $c(z, P(z))$ be a minimal geodesic from $z$ to $P(z)$. Then exists $u \in c(z, P(z)) \cap FrB(P(z), r)$. It follows $d(z, P(z)) = d(z, u) + d(u, P(z))$ and $d(u, P(z)) = r > 0$. Therefore $d(z, u) < d(z, P(z))$ and $u \in M$ contrary to $u \neq P(z)$. □

**Theorem 2.** In the following situations we have dom $P = M$:

(i) $M$ is open

(ii) $M$ is dense in $Z$.

**Proof.** Since $M \subseteq \text{dom} \ P$ is suffice to establish that dom $P \subseteq M$.

i) Suppose that $\exists z \in \text{dom} \ P \setminus M$. By lemma 1 we have $z \in M \setminus \text{int} \ M = \phi$ false.

ii) Fix $z \in \text{dom} \ P$. Since $M$ is dense $d(z, M) = 0 = d(z, P(z))$ i.e. $z = P(z) \in M$. □
Theorem 3. dom $P$ has at least one isolated point.

Proof. Suppose, contrary to our claim, that exists an isolated point $z \in \text{dom } P$ i.e. there is $V$ a neighbourhood of $z$ such that $(\ast) V \cap \text{dom } P = \{z\}$. From $z \in \text{dom } P$ and theorem 1 $c(z, P(z)) \subset \text{dom } P$. Consider a sequence $(x_n)_n \in c(z, P(z)) \setminus \{z\}$ with $x_n \rightarrow z$. Exists $n_0$ such that $n \geq n_0$ imply $x_n \in V$. Then, for $n \geq n_0$ $x_n \in V \cap \text{dom } P$ and $x_n \neq z$ contrary to $(\ast)$. □

2 Continuity of projections

The next lemma extend lemma 1.1 of [3]:

Lemma 2. If $z \in \Omega$ and $V$ is a neighbourhood of $P(z)$ in $M$ then 
\[ d(z, M) < d(z, M \setminus V). \]

Proof. If the inequality were inverse then $d(z, M) = d(z, M \setminus V)$. There is $(a_n)_n \in M \setminus V$ for which $d(z, a_n) \rightarrow d(z, M \setminus V) = d(z, M)$. By Hopf-Rinow v) \forall n there is $h_n \in T_z Z$ such that $a_n = \exp_z(h_n)$ and $d(z, a_n) = \|h_n\| \rightarrow d(z, M)$ i.e. the sequence $(h_n)_n$ is bounded. By Cesaro’s lemma we can assume that $h_n \rightarrow h \in T_z Z$. Consider $b = \exp_z(h)$ and the minimal geodesic $c(z, b)(t) = \exp_z(th_n), t \in [0, 1]$ from $z$ to $b$. Because $(a_n)_n \in M \setminus V$ and $(a_n)_n \rightarrow b$ we have $b \notin P(z) \in V$ and $d(z, M) = d(z, b)$. This means $(\ast) b \notin M$. Since $z \in \Omega$ there is $u \in (c(z, b) \setminus \{z\}) \cap \text{dom } P$. By Hopf-Rinow v) there is $\tilde{h} \in T_z Z$ with $P(u) = \exp_z(\tilde{h})$. We get:
\[ d(z, b) = d(z, M) \leq d(z, P(u)) \leq d(z, u) + d(u, P(u)) \leq d(z, u) + d(u, b) = d(z, b) \]

(we have $d(u, P(u)) \leq d(u, b)$ because $d(u, P(u)) = d(u, M) \leq d(u, a_n) \rightarrow d(u, b)$). Therefore in the last relation is equality between all terms. Then $d(z, P(u)) = d(z, u) + d(u, P(u))$ i.e. $u \in c(z, P(u)) \setminus \{z\}$. Hence $u \in c(z, b) \setminus \{z\}$ and $u \in c(z, P(u)) \setminus \{z\}$. Then $u = \exp_z(t_0 h) = \exp_z(t_0 \tilde{h})$ with $t_0 \in (0, 1)$ $(t_0 = 1 \Rightarrow u = P(u) = b$ false by $(\ast)$). We can made $t_0$ small so that $\max\{t_0 \|h\|, t_0 \|\tilde{h}\|\} < r$ and $\exp_z|_{B(O_r, r)}$ is diffeomorphism. Hence $t_0 h = t_0 \tilde{h}$ i.e. $h = \tilde{h}$. Then $b = P(u) \in M$ contrary to $(\ast)$. □

Theorem 4. $P|_\Omega$ is continuous.

Proof. The argument is inspired from the proof of theorem 1.3 of [3]. Suppose that $P$ is not continous at $z \in \Omega$. There is a neighbourhood $V$ of $P(z)$
in $M$ and a sequence $(x_n)_n \in \text{dom } P$ with $(x_n)_n \to z$ and $P(x_n) \notin V \ \forall n$. Then:

$$d(z, M \setminus V) \leq d(z, P(x_n)) \leq d(z, x_n) + d(x_n, P(x_n)) = d(z, x_n) + d(x_n, M)$$

At limit we obtain $d(z, M \setminus V) \leq d(z, M)$ contrary to lemma 2. □

**Remark** For case $Z$ an euclidian space (e.g. $R^n$) and $M$ a closed subset theorem 3 of [6, p. 119] said that $P$ is continuous on all $\text{dom } P$.

3 **Projection onto a submanifold**

Now let $M$ be a submanifold of $Z$. By theorem 3 of [1, p. 151] we have:

**Theorem 5.** Let $M_i, i = 1, 2$ be submanifolds of $Z$ and $c$ a geodesic from $m_1 \in M_1$ to $m_2 \in M_2$ such that $L(c) = d(M_1, M_2)$. Then $c$ is normal to $M_i$ at $m_i, i = 1, 2$.

For $M_1 = M$ and $M_2 = \{z\}$ we obtain:

**Corollary 3.** Let $M$ be a submanifold of $Z$ and $z \in \text{dom } P$. Then $c(z, P(z))$ is normal to $M$ at $P(z)$. That means $z = \exp_{P(z)}(h)$ with $h \in (T_{P(z)}M)^\perp$.

4 **The case $Z$ euclidian space**

We begin with a more simple proof of lemma 1’. 2 of [3]:

**Proposition 4.** Let $Z$ be an euclidian space, $t \in (0, 1]$ and $f_t : \text{dom } P \to Z, f_t(z) = P(z) + t(z - P(z))$. Then $f_t$ is an injection

**Proof.** The case $t = 1$ is easy. Let $t \in (0, 1)$ and $a, b \in \text{dom } P$ with $f_t(a) = f_t(b) = u$. By proposition 1 we can assume that $u = 0$ without loss of generality. So for $s = \frac{t}{1-t}$ we have $P(a) = sa$ and $P(b) = sb$. By theorem 1 $P(a) = P(b) = P(0)$ because $u = 0 \in c(a, P(a)) = (a, P(a))$ and $u = 0 \in c(b, P(b)) = (b, P(B))$ (we denote by $c(a, P(a))$ the open segment joining $a$ and $P(a)$ who in $Z$ euclidian space is the unique minimal geodesic from $a$ to $P(a)$). Then $sa = sb$ i.e. $a = b$. □
We define a new set:

\[ M(\text{conv}) = \{ z \in Z; \exists r > d(z, M) \text{ such that } M \cap \overline{B}(z, r) \text{ is convex} \} \]

**Lemma 3.** \( M(cl) \cap M(\text{conv}) \subset \text{dom} \ P \).

**Proof.** Fix \( z \in M(cl) \cap M(\text{conv}) \); there exists \( r_1, r_2 > d(z, M) \) such that \( M \cap \overline{B}(z, r_1) \) is closed in \( Z \) and \( M \cap \overline{B}(z, r_2) \) is convex. Let \( r = \min(r_1, r_2) \).

\[ M \cap \overline{B}(z, r) \subset M \cap \overline{B}(z, r_1) \Rightarrow cl(M \cap \overline{B}(z, r)) \subset cl(M \cap \overline{B}(z, r_1)) = M \cap \overline{B}(z, r_1) \subset M. \]

Therefore \( cl(M \cap \overline{B}(z, r)) \subset M \cap \overline{B}(z, r) \) i.e. \( M \cap \overline{B}(z, r) \) is closed in \( Z \). \( M \cap \overline{B}(z, r) \subset M \cap \overline{B}(z, r_2) \Rightarrow \text{conv}(M \cap \overline{B}(z, r)) \subset \text{conv}(M \cap \overline{B}(z, r_2)) = M \cap \overline{B}(z, r_2) \subset M \). Then \( \text{conv}(M \cap \overline{B}(z, r)) \subset M \cap \overline{B}(z, r) \) i.e. \( M \cap \overline{B}(z, r) \) is convex. Consequently \( M \cap \overline{B}(z, r) \) is closed and convex in an euclidian space and then there exists a unique \( a \in M \cap \overline{B}(z, r) \) such that \( d(z, a) = d(z, M \cap \overline{B}(z, r)) = d(z, M) \). \( \square \)

**Corollary 4.** Suppose that \( Z \) is an euclidian space:

(i) If \( M \) is convex then \( M(cl) \subset \Omega \)

(ii) If \( M \) is convex and locally compact then \( M(cl) = \Omega \).

**Proof.** If \( M \) is convex then \( M(\text{conv}) = M \); therefore \( M(cl) \cap M(\text{conv}) = M(cl) \)

i) result from last theorem and fact 2.3 of \([3]\) who said that \( M(cl) \) is open

ii) result from i) and proposition 2.6 of \([3]\). \( \square \)

## 5 The general best approximation problem

Return to case \( Z \) a complete, finite-dimensional Riemannian manifold. Let \( S_0 \) and \( S_1 \) be two disjoint subsets of \( Z \). In the following we shall be concerned with the problem of the existence of points \( x_i \in S_i \ i = 0, 1 \) such that \( d(S_0, S_1) = d(x_0, x_1) \). Denote by \( R \) the distance \( d(S_0, S_1) \) and for \( r > 0 \)

\[ B(S_0, r) = \{ z \in Z; d(z, S_0) < r \}, A(r) = S_1 \cap \overline{B}(S_0, R + r). \]

**Remark.** For every \( r > 0 \) we have:

i) \( S_1 \) closed \( \Rightarrow A(r) \) closed

ii) \( A(r) \neq \emptyset \).
Theorem 6. In the following situations there exists \( x_i \in S_i \) \( i = 0, 1 \) such that \( d(S_0, S_1) = d(x_0, x_1) \):

(i) \( \exists \varepsilon > 0 \) such that \( A(\varepsilon) \) is compact and \( A(\varepsilon) \subset S_0(\text{cl}) \)

(ii) \( S_1 \) is closed, \( S_0 \) is bounded and \( \exists \varepsilon > 0 \) such that \( A(\varepsilon) \subset S_0(\text{cl}) \)

(iii) \( S_1 \) is compact and \( S_1 \subset S_0(\text{cl}) \)

(iv) \( S_1 \) is compact and \( S_0 \) is closed.

Proof. i) \( \exists (x_n^0, x_n^1) \in S_i \) \( i = 0, 1 \) with \( d(x_n^0, x_n^1) \to R \). We have:

\[ R \leq d(x_n^1, S_0) \leq d(x_n^1, x_n^0) \to R \]

it follows that \( d(x_n^1, S_0) \to R \). Then we can assume that \( x_n^1 \in A(\varepsilon) \) \( \forall n \). But \( A(\varepsilon) \) is compact and then \( x_n^1 \to x_1 \in A(\varepsilon) \).

By \( x_1 \in A(\varepsilon) \subset S_0(\text{cl}) \) and proposition 3 ii) there exists \( x_0 \in S_0 \) such that

\[ d(x_1, S_0) = d(x_0, x_1) \]

Therefore

\[ d(x_0, x_1) = d(x_1, S_0) = \lim_{n \to \infty} d(x_n^1, S_0) = R = d(S_0, S_1) \]

ii) \( S_0 \) bounded and \( S_1 \) closed \( \Rightarrow A(\varepsilon) \) is closed and bounded. By Hopf-Rinow

iii) \( A(\varepsilon) \) is compact and apply i)

iv) by iii) and proposition 3 i). □

Remark.

i) \( d(S_0, S_1) = d(S_0, A(\varepsilon)) \)

ii) the case iv) of last theorem is theorem 1 (i) of [7].

Final remark. What are possible extensions of these results? The main tool of our study is Hopf-Rinow theorem and which is:

i) false for infinite-dimensional Riemannian manifolds ([4])

ii) true for finite-dimensional Finsler manifolds ([5])

Then most part of these results can be reformulated for complete, finite-dimensional Finsler manifolds.
References


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