A Noetherian conservation law for 2D spinning particle

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Intern. J. of Non-Linear Mechanics, 35(2000), no. 5, 947-951.
MR 2001g:70024

Abstract

Till now, for a 3-dimensional spinning particle only non-Noetherian conservation laws are known([3]). In this paper we use a Noether type theorem due to R. Miron([4]) in order to obtain a Noetherian conservation law for the 2-dimensional spinning particle.

Subject Classification: 58F05, 58F35, 58G35, 70H33, 70H35.
Key words and phrases: Lagrangian of order k, k-jet bundle, symmetry transformation, Noether theorem, spinning particle.

1 A Noether type theorem for autonomous higher order Lagrangians

Every geometrical approach of Analytical Mechanics has to specify an appropriate space. If the mechanical system under consideration has n degree of freedom then the framework is a fibred manifold $\pi : E \longrightarrow I\!R$ with standard fibre a n-dimensional manifold M. If we wish to discuss Lagrangians involving accelerations of order k then the main arena should be the bundle $\pi_k : J^k\pi \longrightarrow E$ of k-jets of sections of π.

Given a trivialization $E \cong I\!R \times M$ one may identify $J^k\pi$ with $I\!R \times T^kM$ where T^kM is the k-tangent bundle over M. Fibered coordinates in E will be denoted by (t, x^i) where $(x^i)_{1\leq i\leq n}$ are the local coordinates in the fibre

M. The fibred manifold $\pi : E \longrightarrow I\!R$ is *the configuration manifold* and the k-jet prologation $J^k\pi$ is *the generalized phase space*. One introduce the local coordinates $(t, x^i, x^i_1, \ldots, x^i_k)$ in $J^k\pi$ with:

$$x^i_\alpha\left(j^k c\right) = \frac{d^\alpha\left(x^i \circ c\right)}{dt^\alpha},\ 1 \leq \alpha \leq k \tag{1.1}$$

for c a section of π and $j^k c \in J^k\pi$ the associated k-jet.

Definition 1 A *Lagrangian of order k* is a smooth mapping $L : J^k\pi \longrightarrow I\!R$. If the Lagrangian does not depend of t, that is $L = L\left(x^i, x^i_1, \ldots, x^i_k\right)$ i.e. L live on T^kM, then we call it *autonomous Lagrangian.*

If $c : [0, 1] \longrightarrow M$ is a regular curve on M, i. e. a section of π, then the *integral of action* of L on c is given by:

$$I\left(c\right) = \int_0^1 L\left(t, x\left(t\right), \frac{dx}{dt}\left(t\right), \frac{d^2x}{dt^2}\left(t\right) \ldots, \frac{d^kx}{dt^k}\left(t\right)\right) dt. \tag{1.2}$$

Applying the usual variational principle to $I\left(c\right)$ we get:

Proposition 2 *If c is an extremal curve of the functional $I\left(c\right)$ then the folowing Euler-Lagrange equations hold:*

$$E_i\left(L\right) = 0 \tag{1.3}$$

where:

$$E_i = \frac{\partial}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial}{\partial x^i_1}\right) + \ldots + \left(-1\right)^k \frac{d^k}{dt^k}\left(\frac{\partial}{\partial x^i_k}\right) \tag{1.4}$$

is the Euler-Lagrange operator of order k.

Let us consider on $J^k\pi$ *the Liouville vector fields* $\overset{1}{\Gamma}, \ldots, \overset{k}{\Gamma}$ given by:

$$\overset{\alpha}{\Gamma} = (k - \alpha + 1)!(x^i_1 \frac{\partial}{\partial x^i_{k-\alpha+1}} + \ldots + \alpha x^i_\alpha \frac{\partial}{\partial x^i_k}) \tag{1.5}$$

and *the higher order energies* of L:

$$\begin{cases} \mathcal{E}^k(L) = \overset{k}{\Gamma}(L) - \frac{1}{2!}\frac{d}{dt}\overset{k-1}{\Gamma}(L) + \ldots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\overset{1}{\Gamma}(L) - L, \\ \mathcal{E}^{k-1}(L) = -\frac{1}{2!}\overset{k-1}{\Gamma}(L) + \frac{d}{dt}\overset{k-2}{\Gamma}(L) + \ldots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-2}}{dt^{k-2}}\overset{1}{\Gamma}(L), \\ \mathcal{E}^{k-2}(L) = \frac{1}{3!}\overset{k-2}{\Gamma}(L) - \overset{k-3}{\Gamma}(L) + \ldots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-3}}{dt^{k-3}}\overset{1}{\Gamma}(L), \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \mathcal{E}^1(L) = (-1)^{k-1}\frac{1}{k!}\overset{1}{\Gamma}(L) \end{cases} \tag{1.6}$$

A straightforward computation yields:

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{dx^i}{dt}E_i(L) + \frac{d}{dt}\overset{k}{\Gamma}(L) - \frac{d^2}{dt^2}\overset{k-1}{\Gamma}(L) + \ldots + (-1)^{k-1}\frac{d^k}{dt^k}\overset{1}{\Gamma}(L) \tag{1.7}$$

$$\frac{d\mathcal{E}^k(L)}{dt} = -E_i(L)\frac{dx^i}{dt} - \frac{\partial L}{\partial t} \tag{1.8}$$

and for any $\tau \in C^\infty(I\!R \times M)$:

$$\frac{d\tau}{dt}L - \left[\frac{d\tau}{dt}\overset{k}{\Gamma}(L) + \frac{d^2\tau}{dt^2}\overset{k-1}{\Gamma}(L) + \ldots + \frac{d^k\tau}{dt^k}\overset{1}{\Gamma}(L)\right] = \tau\frac{d\mathcal{E}^k(L)}{dt} +$$
$$+\frac{d}{dt}\left[-\tau\mathcal{E}^k(L) + \frac{d\tau}{dt}\mathcal{E}^{k-1}(L) - \frac{d^2\tau}{dt^2}\mathcal{E}^{k-2}(L) + \ldots + (-1)^k\frac{d^{k-1}\tau}{dt^{k-1}}\mathcal{E}^1(L)\right]. \tag{1.9}$$

From relation (1.8) it follows:

Proposition 3(Conservation of energy for time-independent Lagrangians) *If L is autonomous then $\mathcal{E}^k(L)$ is a conservation law, that is $\mathcal{E}^k(L)$ is conserved along the extremals of L.*

In the following our aim is to state the Noether type theorem concerning the invariance of higher-order time-independent Lagrangians. For the original Noether theorem see [7] and for the time-dependent higher-order case see [2]. This Noether theorem appears for first time in [4], [5], [6]. Our approach differ from the previous with respect to the local coordinates on T^kM. Another version of Noether theorem for higher-order Lagrangians, based on exterior differential calculus applied to the Cartan forms associated to the Lagrangian, appears in [3].

For a vector field V on E with local expression: $V = \tau\frac{\partial}{\partial t} + V^i\frac{\partial}{\partial x^i}$ let us consider the induced point-transformation:

$$\tilde{t} = t + \varepsilon\tau \tag{1.10a}$$

$$\tilde{x}^i = x^i + \varepsilon V^i, \quad 1 \leq i \leq n \tag{1.10b}$$

where ε is a real number with sufficiently small absolute value.

Definition 4 The transformation (1.10) is called *symmetry transformation* for L if there exists $\Phi \in C^\infty\left(T^{k-1}M\right)$, called *gauge*, such that:

$$\tilde{L}d\tilde{t} = \left(L + \varepsilon\frac{d\Phi}{dt}\right)dt + o\left(\varepsilon\right) \tag{1.11}$$

where $\tilde{L}$ is $L\left(\tilde{x}^i, \tilde{x}^i_1, \ldots, \tilde{x}^i_k\right)$.

Retaining the terms of first order in ε we get:

$$\frac{d\tilde{t}}{dt} = 1 + \varepsilon\frac{d\tau}{dt} \tag{1.12a}$$

$$\frac{dt}{d\tilde{t}} = 1 - \varepsilon\frac{d\tau}{dt} \tag{1.12b}$$

and:

$$\begin{cases} \frac{d\tilde{x}^i}{d\tilde{t}} = \frac{dx^i}{dt} + \varepsilon\Phi^i_1, \\ \frac{d^2\tilde{x}^i}{d\tilde{t}^2} = \frac{d^2x^i}{dt^2} + \varepsilon\Phi^i_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d^k\tilde{x}^i}{d\tilde{t}^k} = \frac{d^kx^i}{dt^k} + \varepsilon\Phi^i_k \end{cases} \tag{1.13}$$

where:

$$\Phi^i_{\alpha+1} = \frac{d\Phi^i_\alpha}{dt} - x^i_{\alpha+1}\frac{d\tau}{dt} \tag{1.14}$$

with $\Phi^i_0 = V^i$.

Returning to L it results:

$$\tilde{L} = \left(L + \varepsilon\frac{d\Phi}{dt}\right)\frac{dt}{d\tilde{t}}$$

and therefore:

$$\tilde{L} - L = \varepsilon \left(\frac{d\Phi}{dt} - L\frac{d\tau}{dt} \right). \tag{1.15}$$

Using the Taylor expansion of first order for $\tilde{L}$ and identifying the coefficients of ε we have:

Theorem 5 *The transformation* (1.10) *is a symmetry transformation for L with the gauge Φ if and only if:*

$$\frac{\partial L}{\partial x^i}V^i + \frac{\partial L}{\partial x_1^i}\Phi_1^i + \ldots + \frac{\partial L}{\partial x_k^i} = \frac{d\Phi}{dt} - L\frac{d\tau}{dt}. \tag{1.16}$$

The last relation is equivalent with:

$$E_i(L) + \frac{d}{dt}I_V^k(L) - \frac{d^2}{dt^2}I_V^{k-1}(L) + \ldots + (-1)^{k-1}\frac{d^k}{dt^k}I_V^1(L) +$$
$$+\tau\frac{d}{dt}\mathcal{E}^k(L) +$$
$$+\frac{d}{dt}\left[-\tau\mathcal{E}^k(L) + \frac{d\tau}{dt}\mathcal{E}^{k-1}(L) - \ldots + (-1)^k \frac{d^{k-1}\tau}{dt^{k-1}}\mathcal{E}^1(L) \right] = \frac{d\Phi}{dt} \tag{1.17}$$

where the vector fields $I_V^1, \ldots, I_V^k$ are:

$$\begin{cases} I_V^1 = k!V^i\frac{\partial}{\partial x_k^i}, \\ I_V^2 = (k-1)!V^i\frac{\partial}{\partial x_{k-1}^i} + k!\frac{dV^i}{dt}\frac{\partial}{\partial x_k^i}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ I_V^k = V^i\frac{\partial}{\partial x_1^i} + 2\frac{dV^i}{dt}\frac{\partial}{\partial x_2^i} + \ldots + k\frac{d^{k-1}V^i}{dt^{k-1}}\frac{\partial}{\partial x_k^i}. \end{cases} \tag{1.18}$$

Following the terminology from classical mechanics, that is $k = 1$([1, p. 184]), we call equation (1.16) or equivalenty (1.17) *the generalized Killing equations of L.*

Therefore we obtain:

Theorem 6(Noether-Miron) *If* (1.10) *is symmetry transformation for L with the gauge Φ then the following function $\mathcal{F} = \mathcal{F}(L, V, \Phi)$ is conserved along the extremals of L:*

$$\mathcal{F} = I_V^k(L) - \frac{1}{2!}\frac{d}{dt}I_V^{k-1}(L) + \ldots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}I_V^1(L) -$$

$$-\tau\mathcal{E}^k(L) + \frac{d\tau}{dt}\mathcal{E}^{k-1}(L) - \ldots + (-1)^k \frac{d^{k-1}\tau}{dt^{k-1}}\mathcal{E}^1(L) - \Phi. \tag{1.19}$$

It is to be noted that for $k = 1$ one obtains a well-known result([1, p. 184]):

$$\mathcal{F} = V^i \frac{\partial L}{\partial x_1^i} - \tau\left(\frac{\partial L}{\partial x_1^i} x_1^i - L\right) - \Phi. \tag{1.20}$$

2 2D spinnig particle

In [3] for a 3D spinning particle the following Lagrangian is pointed out:

$$L = \frac{1}{2}\sum_{i=1}^{3}\left(x_1^i\right)^2 - \frac{1}{2}\sum_{i=1}^{3}\left(x_2^i\right)^2 \tag{2.1}$$

and then some non-Noetherian conservation laws are found. Now we shall find a Noetherian first integral for the 2D case.

For simplification, we shall denote by overdots the derivatives and we shall put $x^1 = x$, $x^2 = y$. Then the Lagrangian for 2D spinning particle is:

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}\left(\ddot{x}^2 + \ddot{y}^2\right) \tag{2.2}$$

The Euler-Lagrange equations are:

$$\frac{d^2x}{dt^2} + \frac{d^4x}{dt^4} = 0 \tag{2.3a}$$

$$\frac{d^2y}{dt^2} + \frac{d^4y}{dt^4} = 0 \tag{2.3b}$$

and therefore we have the first integrals:

$$C_x = \frac{dx}{dt} + \frac{d^3x}{dt^3} \tag{2.4a}$$

$$C_y = \frac{dy}{dt} + \frac{d^4y}{dt^3}. \tag{2.4b}$$

Applying the conservation of energy we have another first integral:

$$\mathcal{E}^2(L) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 - \ddot{x}^2 - \ddot{y}^2\right) + \dot{x}\frac{d^3x}{dt^3} + \dot{y}\frac{d^3y}{dt^3} \tag{2.5}$$

or by virtue of (2.4):

$$\mathcal{E}^2(L) = \dot{x}C_x + \dot{y}C_y - \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \ddot{x}^2 + \ddot{y}^2\right). \tag{2.5'}$$

Let be the point-transformation:

$$\tilde{x} = x - \varepsilon y \tag{2.6a}$$

$$\tilde{y} = y + \varepsilon x \tag{2.6b}$$

that is: $V_x = -y$ and $V_y = x$. Then (1.14) reads as follows:

$$\Phi_{1,x} = -\dot{y} \tag{2.7a}$$

$$\Phi_{1,y} = \dot{x} \tag{2.7b}$$

and:

$$\Phi_{2,x} = -\ddot{y} \tag{2.8a}$$

$$\Phi_{2,y} = \ddot{x}. \tag{2.8b}$$

It results that (2.6) is symmetry transformation for (2.2) with the gauge $\Phi = 0$. The associated conservation law is obtained according to (1.19) in the form:

$$\mathcal{F} = x\frac{dy}{dt} - y\frac{dx}{dt} + \frac{dy}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2y}{dt^2} + x\frac{d^3y}{dt^3} - y\frac{d^3x}{dt^3} \tag{2.9}$$

or by virtue of (2.4):

$$\mathcal{F} = xC_y - yC_x + \dot{y}\ddot{x} - \dot{x}\ddot{y} \tag{2.9'}$$

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