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Magnetic Curves in Three-Dimensional Quasi-Para-Sasakian Geometry

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Abstract. We study (non-geodesic) normal magnetic curves of three-dimensional normal almost paracontact manifolds. We compute their curvature and torsion as well as a Lancret invariant (in the non-Legendre case) and the mean curvature vector field. Two 1-parameter families of magnetic curves (first space like and second time like) are obtained in quasi-para-Sasakian manifolds which are not para-Sasakian; these are non-Legendre helices.

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1. Introduction

A *magnetic field* on a manifold M is a closed 2-form $\Omega \in \Omega^2(M)$. If (M, g) is a (pseudo-) Riemannian manifold then we associate *the Lorentz force* F_Ω thought as a $(1, 1)$-tensor field given by:

$$g(F_\Omega X, Y) = \Omega(X, Y) \tag{1.1}$$

for any vector fields $X, Y \in \mathcal{X}(M)$. Then, *the magnetic curves* on *the magnetic manifold* (M, g, Ω) are the solutions $\gamma : I \subseteq \mathbb{R} \to M$ of the Lorentz equation:

$$\nabla_{\gamma'}\gamma' = F_\Omega(\gamma'). \tag{1.2}$$

A first important property for magnetic curves is that their speed is constant: $\frac{d}{dt}g(\gamma', \gamma') = 0$. In particular, a magnetic curve is called *normal* if it has unit energy, i.e., $\|\gamma'\| = 1$.

Recently, the study of magnetic curves in some special three-dimensional geometries knows a considerable interest and has as starting point the paper [2]. So, there are obtained classes of magnetic curves as well as some classifications as follows: for the Euclidean $\mathbb{R}^3$ in [9] and [13], for the Minkowski $\mathbb{R}^3$

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in [10], for the product manifold $S^2 \times \mathbb{R}$ in [14], in a non-flat quasi-Sasakian $\mathbb{R}^5$ in [15].

The aim of this paper is to study magnetic curves in another three-dimensional pseudo-Riemannian geometry, namely almost paracontact geometry. In fact, beside the dimension condition, to work in a helpful framework we impose another one, called normality, which means the integrability of an associated almost paracomplex structure on the cone manifold $M \times \mathbb{R}$. Let us remark that after the initial submission of this paper, another work concerning a special-type magnetic curves, called Killing, in the same three-dimensional framework was published in [3].

Our work is structured as follows: the first section is a very brief review of (normal) almost paracontact geometry and the closedness of the fundamental form implies the quasi-para-Sasakian framework. The next section is focused on the study of non-geodesic magnetic curves in this setting by choosing as Ω above exactly the fundamental 2-form of the manifold. More precisely, we compute the curvature (which is always constant) and torsion of these curves using the fact that the product between the tangent field and the characteristic paracontact field is a constant c, i.e., a magnetic curve is a c-slant one as considered in [5]; for $c = 0$ we have the case of Legendre curves. An important particular case is that of para-Sasakian manifolds where a magnetic curve is a helix, i.e., both curvature and torsion are constants. The last section is devoted to examples and we obtain for a fixed c a 1-parametric family of magnetic curves which are helices and non-Legendre.

For non-Legendre curves we associate a Lancret invariant defined in terms of c. Another important property we studied is regarding the proper mean curvature vector field of magnetic curves.

2. Normal Almost Paracontact Geometry in Dimension 3

Let M be a $(2n+1)$-dimensional smooth manifold, φ a tensor field of $(1,1)$-type called *the structural endomorphism*, ξ a vector field called *the characteristic vector field*, η a 1-form called *the paracontact form* and g a pseudo-Riemannian metric on M of signature $(n+1, n)$. We say that (φ, ξ, η, g) defines an *almost paracontact metric structure* on M if [19, p. 38]:

1. $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$,
2. $\eta(\xi) = 1$, $\varphi^2 = I - \eta \otimes \xi$,
3. φ induces on the $2n$-dimensional distribution $\mathcal{D} := \ker \eta$ an almost paracomplex structure P, i.e., $P^2 = I$ and the eigensubbundles T^+, T^-, corresponding to the eigenvalues $1, -1$ of P respectively, have equal dimension n; hence $\mathcal{D} = T^+ \oplus T^-$,
4. $g(\varphi\cdot, \varphi\cdot) = -g + \eta \otimes \eta$.

For a list of examples of almost paracontact metric structures see [8] and [11, p. 84]. From the definition it follows that η is the g-dual of ξ, i.e., $\eta(X) = g(X, \xi)$ and ξ is a unitary vector field: $g(\xi, \xi) = 1$. Let ∇ be the Levi-Civita connection of g.

The Nijenhuis tensor field with respect to the tensor field φ, denoted by N_φ, is given by:

$$N_\varphi(X,Y) = [\varphi(X),\varphi(Y)] + \varphi^2([X,Y]) - \varphi([\varphi(X),Y]) \\ -\varphi([X,\varphi(Y)]), \quad \forall X,Y \in \Gamma(TM). \tag{2.1}$$

Definition 2.1. The almost paracontact metric manifold $M(\varphi,\xi,\eta,g)$ is said to be *normal* if the almost paracomplex structure J on the manifold $M \times \mathbb{R}$, given by:

$$J\left(X,\lambda\frac{d}{\mathrm{d}t}\right) := \left(\varphi(X)+\lambda\xi,\eta(X)\frac{d}{\mathrm{d}t}\right), \quad X \in \Gamma(TM), \quad t \in \mathbb{R}, \tag{2.2}$$

is integrable, where λ is a real-valued function on $M \times \mathbb{R}$.

The condition (2.2) is equivalent to:

$$N_\varphi - 2d\eta \otimes \xi = 0. \tag{2.3}$$

In the following, we restrict to the dimension 3 for which the normality is equivalent with, [17, p. 379]:

$$\begin{cases} \nabla_X\xi = \alpha(X-\eta(X)\xi) + \beta\varphi(X), \\ (\nabla_X\varphi)Y = \alpha(g(\varphi X,Y)\xi - \eta(Y)\varphi X) + \beta(g(X,Y)\xi - \eta(Y)X). \end{cases} \tag{2.4}$$

where $\alpha = \frac{1}{2}div\xi$ and $\beta = \frac{1}{2}trace(\varphi\nabla\xi)$. An important consequence of the first equation (2.4) is that ξ is *a geodesic vector field*:

$$\nabla_\xi\xi = 0. \tag{2.5}$$

Like in the almost contact geometry, we associate a differential 2-form:

Definition 2.2. The *fundamental form* of $M(\varphi,\xi,\eta,g)$ is:

$$\Omega(X,Y) := g(\varphi X,Y). \tag{2.6}$$

The condition 4 of almost paracontact metric structures yields the skew-symmetry:
$g(\varphi\cdot,\cdot) = -g(\cdot,\varphi\cdot)$ and then Ω is indeed a 2-form. From the formula:

$$3d\Omega(X,Y,Z) = 2\alpha[\Omega(X,Y)\eta(Z) + \Omega(Y,Z)\eta(X) + \Omega(Z,X)\eta(Y)] \tag{2.7}$$

it results that Ω is closed if and only $\alpha = 0$, i.e., ξ is a divergence-free vector field. For $\alpha = 0$ and $\beta \neq 0$ we get the *quasi-para-Sasakian case* of [17, p. 380] for which we have:

$$\nabla_X\xi = \beta\varphi(X) \tag{2.8}$$

and then ξ is a Killing vector field, i.e., the Lie derivative of g with respect to ξ is zero:

$$\mathcal{L}_\xi g = 0. \tag{2.9}$$

3. Magnetic Curves in Three-Dimensional Quasi-Para-Sasakian Geometry

In dimension 3, the metric g becomes a Minkowski–Lorentzian one, having the signature $(2, 1)$. For a Frenet curve $\gamma : I \subseteq \mathbb{R} \to M^3$ we denotes the Frenet frame as usual $(T = \gamma', N, B)$ and the Frenet equations are, [17, p. 381]:

$$\nabla_T T = k\varepsilon_2 N, \ \nabla_T N = -k\varepsilon_1 T + \tau\varepsilon_3 B, \ \nabla_T B = -\tau\varepsilon_2 N, \tag{3.1}$$

where $k \geq 0$ denote the curvature and $\tau \geq 0$ the torsion. Here, the g-norms of the Frenet vectors are as follows: $g(T,T) = \varepsilon_1, g(N,N) = \varepsilon_2, g(B,B) = \varepsilon_3$ with $\varepsilon_i = \pm 1$ for $1 \leq i \leq 3$. With the discussion of [12, p. 35], we have $\varepsilon_3 = -\varepsilon_1\varepsilon_2$.

Definition 3.1. (i) The *structural function* of γ is the map $c_\gamma : I \to \mathbb{R}$ given by:

$$c_\gamma(s) = g(T(s), \xi) = \eta(T(s)) \tag{3.2}$$

and the curve γ is called *a slant curve*, or more precisely *c-slant curve*, if c_γ is a constant function, $c_\gamma = c \in \mathbb{R}$, see also [18]. In the particular case of $c = 0$, the curve γ is called *Legendre curve* [16].

(ii) The Frenet curve γ on the quasi-para-Sasakian manifold $M(\varphi, \xi, \eta, g)$ is called *magnetic* if:

$$\nabla_T T = \varphi(T). \tag{3.3}$$

In the following, we suppose that γ is non-geodesic, i.e., $k > 0$ and from (2.5) we get that γ cannot be an integral curve of ξ. This means $c \neq \pm 1$. A first property of (normal) magnetic curves and an "a-priori" estimate is given by:

Proposition 3.2. *The magnetic curve γ is a c-slant one with $\eta(N) = 0$ with c constrained by:*

$$\varepsilon_2(c^2 - \varepsilon_1) > 0. \tag{3.4}$$

Proof. Let us take the covariant derivative in the relation (3.2) along γ:

$$c'_\gamma(s) = g(\varphi T, \xi) + g(T, \beta\varphi T) = 0.$$

Comparing (3.1) and (3.3) it results:

$$k\varepsilon_2 N = \varphi(T) \tag{3.5}$$

and then: $\eta(N) = 0$. The square of norms in (3.5) is:

$$k^2\varepsilon_2 = -\varepsilon_1 + c^2 \tag{3.6}$$

and we have (3.4). We note that this inequality can be obtained in a second way which provides other main equations. Let us recall after [12, p. 34] that the decomposition of a vector field in the g-orthonormal frame $\{e_1, e_2, e_3 = e_1 \times_M e_2\}$ is:

$$X = \mu_1 g(X, e_1)e_1 + \mu_2 g(X, e_2)e_2 - \mu_1\mu_2 g(X, e_3)e_3 \tag{3.7}$$

where $\times_M$ is the Minkowski vector product (see the cited book) and $\mu_i = g(e_i, e_i) = \pm 1$ for $1 \leq i \leq 2$. Then, the expression of $\xi|_\gamma$ in the Frenet frame is:

$$\xi|_\gamma = \varepsilon_1 cT - \varepsilon_1\varepsilon_2\eta(B)B \tag{3.8}$$

and since ξ is a unitary vector field we get that:

$$1 = \varepsilon_1 c^2 - \varepsilon_1\varepsilon_2\eta(B)^2$$

and then:

$$\eta(B)^2 = \varepsilon_1\varepsilon_2\left(c^2\varepsilon_1 - 1\right). \tag{3.9}$$

From $c \neq \pm 1$ it results $\eta(B)^2 > 0$ which is again (3.4). □

Remarks 3.3. (i) It is important to point out that the condition (3.4) does not depend on β which means that it holds for all quasi-para-Sasakian geometries in the same form. Also, from $\eta(N) = 0$ it results that γ is a *slant helix* in the sense of [1]; see also [7].

(ii) In the above proof, the relations (3.6) and (3.9) yield:

$$\eta(B)^2 = k^2. \tag{3.10}$$

(iii) With condition (3.4), we define the *Lancret invariant* of a c-slant curve γ as:

$$\mathrm{Lancret}(\gamma) = \frac{c}{\sqrt{\varepsilon_2(c^2 - \varepsilon_1)}}. \tag{3.11}$$

A motivation for this choice is that in the space-like case of γ (i.e., $\varepsilon_1 = -\varepsilon_2 = +1$) the above expression is $\frac{c}{\sqrt{1-c^2}}$, similar to the normal almost contact geometry [4]. □

We arrive now at the expression of the Frenet frame:

$$T = \gamma', \quad N = \frac{\varepsilon_2}{k}\varphi(\gamma'), \quad B = \frac{\varepsilon_3(\xi - \varepsilon_1 c\gamma')}{sgn(\eta(B))k} \tag{3.12}$$

because (3.8)+(3.9) means:

$$\xi = \varepsilon_1 cT + \varepsilon_3 sgn(\eta(B))kB. \tag{3.13}$$

Also (2.8) gives:

$$\nabla_{\gamma'}\xi = \varepsilon_2\beta kN. \tag{3.14}$$

We are ready for the second main result of this paper:

Proposition 3.4. *The curvature and torsion of a c-magnetic curve are:*

$$k = \sqrt{\varepsilon_2(c^2 - \varepsilon_1)} = constant, \quad \tau = \varepsilon_2(\varepsilon_1\beta - c)sgn(\eta(B)). \tag{3.15}$$

The associated Lancret invariant is:

$$Lancret(\gamma) = \frac{\varepsilon_1\beta - \varepsilon_2 sgn(\eta(B))\tau}{k}. \tag{3.16}$$

It follows a new constraint $\varepsilon_2(\varepsilon_1\beta - c)sgn(\eta(B)) \geq 0$.

Proof. From (3.6) we have the expression of the curvature. The second Frenet equation is:

$$\begin{cases} \varepsilon_2 k \nabla_T N = \nabla_T \varphi(T) = (\nabla_T \varphi)T + \varphi(\nabla_T T) = \beta(\varepsilon_1 \xi - c\gamma') + \varphi^2(\gamma') \\ \qquad = \beta(\varepsilon_1 \xi - c\gamma') + \gamma' - c\xi \\ \varepsilon_2 k \nabla_T N = \varepsilon_3 k^2 \gamma' + \varepsilon_2 \tau(\xi - \varepsilon_1 c\gamma') sgn(\eta(B)) \end{cases} \tag{3.17}$$

Hence:

$$\tau(\xi - \varepsilon_1 c\gamma') sgn(\eta(B))\varepsilon_2 = -\varepsilon_3 k^2 \gamma' + \beta(\varepsilon_1 \xi - c\gamma') + \gamma' - c\xi \tag{3.18}$$

and we derive the claimed expression for the torsion. □

Remarks 3.5. (i) We have now the expressions:

$$N = \frac{\varepsilon_2}{\sqrt{\varepsilon_2(c^2 - \varepsilon_1)}} \varphi(\gamma'), \quad \nabla_{\gamma'} \xi = \varepsilon_2 \beta \sqrt{\varepsilon_2(c^2 - \varepsilon_1)} N \tag{3.19}$$

and then $\nabla_{\gamma'}\xi$ is a vector field with norm depending only on the restriction of β to γ:

$$\|\nabla_{\gamma'}\xi\| = |\beta| k. \tag{3.20}$$

(ii) The slant curve is a *helix*, i.e., k and τ are constants, if and only if the restriction of β along γ is constant, e.g., β is a constant. In particular, for $\beta = -1$ we have the *para-Sasakian case*:

$$\begin{cases} k = \sqrt{\varepsilon_2(c^2 - \varepsilon_1)}, \quad \tau = -\varepsilon_2(\varepsilon_1 + c) sgn(\eta(B)), \quad \text{Lancret}(\gamma) \\ \qquad = -\frac{\varepsilon_1 + \varepsilon_2 sgn(\eta(B))\tau}{k}, \\ \|\nabla_{\gamma'}\xi\| = \sqrt{\varepsilon_2(c^2 - \varepsilon_1)}. \end{cases} \tag{3.21}$$

Our Lancret invariant (3.21) is the paracontact version of the Sasakian Lancret $\frac{\tau \pm 1}{k}$ from [6, p. 362] which is generalized in [4]. γ is *a Bertrand curve*, i.e., we find two real numbers x, y such that $xk + y\tau = 1$. A helix with $\tau = 0$ is *a circle*; it follows that the magnetic curves in para-Sasakian manifolds are not circles.

(iii) For $c = 0$ we have the case of Legendre curves and from (3.4) we get $\varepsilon_3 = +1$. From $\eta(T) = \eta(N) = 0$ it results that $B = \pm\xi|_{\gamma}$. Our formulae (3.15) reduce to $k = 1, \tau = -\beta sgn(\eta(B))$. Also $\|\nabla_{\gamma'}\xi\| = |\beta|$. In the para-Sasakian case, the condition $\tau > 0$ implies $sgn(\eta(B)) = +1$ and hence $k = \tau = 1$. □

Denote by h the second fundamental form of γ and by H its mean curvature field. We know that:

$$H = \text{trace}(h) = h(T, T) = \nabla_T T. \tag{3.22}$$

Then, γ is called *a curve with proper mean curvature vector field* if there exists $\lambda \in C^\infty(\gamma)$ so that:

$$\Delta H = \lambda H. \tag{3.23}$$

In particular, if $\lambda = 0$ then γ is known as *a curve with harmonic mean curvature vector field*. Here, the Laplace operator Δ acts on the vector-valued function H and it is given by:

$$\Delta H = -\nabla_T \nabla_T \nabla_T T. \tag{3.24}$$

Making use of Frenet equations, we can rewrite (3.24) as:

$$3\varepsilon_3 k' k T + \left(\varepsilon_2 k'' - \varepsilon_1 k^3 - \varepsilon_3 k \tau^2\right) N - \varepsilon_1 (2k'\tau + k\tau') B = \lambda(-\varepsilon_2 k N). \tag{3.25}$$

It follows that both k and τ are constants and the function λ becomes a constant too, namely:

$$\lambda = -\varepsilon_3 k^2 - \varepsilon_1 \tau^2. \tag{3.26}$$

For our framework, we state the following third main result of magnetic curves:

Proposition 3.6. *A non-geodesic magnetic curve γ in a quasi-para-Sasakian M^3 has a proper mean curvature vector field if and only if β is constant along γ. Then, the curve is a helix with:*

$$\lambda = -1 - \varepsilon_1 \beta^2 + 2\beta c. \tag{3.27}$$

In particular, a magnetic Legendre curve has:

$$\lambda_L = -1 - \varepsilon_1 \beta^2. \tag{3.28}$$

For the para-Sasakian case we have:

$$\lambda = -1 - \varepsilon_1 - 2c, \quad \lambda_L = -\varepsilon_1 - 1. \tag{3.29}$$

4. Examples

Let N be an open connected subset of $\mathbb{R}^2$, (a, b) an open interval in $\mathbb{R}$ and let us consider the manifold $M = N \times (a, b)$. Let (x, y) be the coordinates on N induced from the cartesian coordinates on $\mathbb{R}^2$ and let z be the coordinate on (a, b) induced from the cartesian coordinate on $\mathbb{R}$. Thus, (x, y, z) are the coordinates on M. Now, we choose the functions:

$$\omega_1, \omega_2 : N \to \mathbb{R}, \quad F : M \to \mathbb{R}, \tag{4.1}$$

and following the idea from [16] we define a normal almost paracontact metric structure (φ, ξ, η, g) on M as follows:

$$\begin{gathered} \varphi(\tfrac{\partial}{\partial x}) = \tfrac{\partial}{\partial y} - \omega_2 \tfrac{\partial}{\partial z}, \quad \varphi(\tfrac{\partial}{\partial y}) = \tfrac{\partial}{\partial x} - \omega_1 \tfrac{\partial}{\partial z}, \quad \varphi(\xi = \tfrac{\partial}{\partial z}) = 0, \\ \eta = \mathrm{d}z + \omega_1 \mathrm{d}x + \omega_2 \mathrm{d}y, \end{gathered} \tag{4.2}$$

$$g = [g_{ij}] = \begin{bmatrix} \omega_1^2 - F & \omega_1\omega_2 & \omega_1 \\ \omega_1\omega_2 & \omega_2^2 + F & \omega_2 \\ \omega_1 & \omega_2 & 1 \end{bmatrix}. \tag{4.3}$$

It follows that:

$$\alpha = \tfrac{1}{2F} \tfrac{\partial F}{\partial z}, \quad \beta = \tfrac{1}{2F}\left(-\tfrac{\partial \omega_1}{\partial y} + \tfrac{\partial \omega_2}{\partial x}\right). \tag{4.4}$$

(M^3, g) is quasi-para-Sasakian if and only if $F = F(x, y)$ and we work with this expression in the following. For $\omega_1 = \omega_2 = 0$ and $F = 1$, we have the Minkowski space $\mathbb{E}^3_1$ of [10].

If we denote $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ then γ is a c-slant curve if and only if:

$$\begin{cases} \omega_1\gamma_1' + \omega_2\gamma_2' + \gamma_3' = c \\ (\omega_1^2 - F(\gamma_1,\gamma_2))(\gamma_1')^2 + (\omega_2^2 + F(\gamma_1,\gamma_2))(\gamma_2')^2 + (\gamma_3')^2 \\ \qquad + 2\omega_1\omega_2\gamma_1'\gamma_2' + 2\omega_1\gamma_1'\gamma_3' + 2\omega_2\gamma_2'\gamma_3' = \varepsilon_1. \end{cases} \tag{4.5}$$

But $(3.5b)$ becomes:

$$(\omega_1\gamma_1' + \omega_2\gamma_2' + \gamma_3')^2 + F(\gamma_1,\gamma_2)\left[-(\gamma_1')^2 + (\gamma_2')^2\right] = \varepsilon_1 \tag{4.6}$$

and then γ is a c-slant curve if and only if:

$$\begin{cases} \omega_1\gamma_1' + \omega_2\gamma_2' + \gamma_3' = c \\ F(\gamma_1,\gamma_2)\left[-(\gamma_1')^2 + (\gamma_2')^2\right] = \varepsilon_1 - c^2. \end{cases} \tag{4.7}$$

The Example from [17, p. 385] of space-like curve ($\varepsilon_1 = \varepsilon_3 = +1, \varepsilon_2 = -1$) is recovered with $N = \mathbb{R}^2$, $(a,b) = (0,+\infty)$ and:

$$\omega_1 = 0, \quad \omega_2 = 2x, \quad F = x^2 \tag{4.8}$$

which yields:

$$\beta = \frac{1}{x^2} \tag{4.9}$$

and then M is a quasi-para-Sasakian manifold which is not para-Sasakian. Fix the real parameter $\rho \in (0,\infty)$; from (3.4) the parameter c belongs to $(-1,1)$. From (4.7) we get the c-slant curve:

$$\gamma_\rho^c(t) = \left(\frac{1}{\rho}, \sqrt{1-c^2}\rho t, (c - 2\sqrt{1-c^2})t\right) \tag{4.10}$$

which for $c = 0$ is the curve (a) of the cited paper with c replaced by ρ; the interval $I \subseteq \mathbb{R}$ of definition for γ corresponds to the condition $(c - 2\sqrt{1-c^2})t > 0$ for any $t \in I$. The covariant derivative along γ_ρ^c is:

$$\nabla_T X = \sqrt{1-c^2}\rho\nabla_{\frac{\partial}{\partial y}} X + (c - 2\sqrt{1-c^2})\nabla_{\frac{\partial}{\partial z}} X \tag{4.11}$$

With the Levi-Civita connection computed in the cited paper we have:

$$\begin{cases} \nabla_{\frac{\partial}{\partial y}} X = X^1\left[\frac{3}{x}\frac{\partial}{\partial y} - 5\frac{\partial}{\partial z}\right] + X^2\left[\frac{5}{x}\frac{\partial}{\partial x}\right] + X^3\left[\frac{1}{x^2}\frac{\partial}{\partial x}\right] + \frac{\partial X^1}{\partial y}\frac{\partial}{\partial x} + \frac{\partial X^2}{\partial y}\frac{\partial}{\partial y} \\ \qquad + \frac{\partial X^3}{\partial y}\frac{\partial}{\partial z} \\ \nabla_{\frac{\partial}{\partial z}} X = X^1\left[\frac{1}{x^2}\frac{\partial}{\partial y} - \frac{2}{x}\frac{\partial}{\partial z}\right] + X^2\left[\frac{1}{x^2}\frac{\partial}{\partial x}\right] + \frac{\partial X^1}{\partial z}\frac{\partial}{\partial x} + \frac{\partial X^2}{\partial z}\frac{\partial}{\partial y} + \frac{\partial X^3}{\partial z}\frac{\partial}{\partial z}. \end{cases} \tag{4.12}$$

for $X = X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial y} + X^3\frac{\partial}{\partial z}$ and then:

$$\nabla_T T = \sqrt{1-c^2}\rho^3(\sqrt{1-c^2} + 2c)\frac{\partial}{\partial x} \tag{4.13}$$

while:

$$\varphi(T) = \sqrt{1-c^2}\rho\frac{\partial}{\partial x}. \tag{4.14}$$

Taking into account the constraint of Proposition 3.4, we derive the following classes of examples:

Proposition 4.1. *Case I: sign(η(B)) = −1. Let* $c \in \left(-\frac{1}{\sqrt{5}}, \sqrt{\frac{5-\sqrt{5}}{10}}\right)$ *and* $\rho = \frac{1}{\sqrt{\sqrt{1-c^2}+2c}}$*. Then, the space-like curve* γ_ρ^c *of* (4.10) *is a helix and magnetic curve which is not Legendre. Its curvature and torsion are:*

$$k = \sqrt{1-c^2}, \quad \tau = \frac{1}{\sqrt{1-c^2}+2c} - c. \tag{4.15}$$

Its Lancret invariant and λ *are:*

$$\text{Lancret}(\gamma_\rho^c) = \frac{\rho^2 - \tau}{k}, \quad \lambda = \frac{c^2 - 2\sqrt{1-c^2} - 2}{(\sqrt{1-c^2}+2c)^2}. \tag{4.16}$$

Case II: sign(η(B)) = 1. Let $c \in \left(\sqrt{\frac{5-\sqrt{5}}{10}}, 1\right)$ *and* $\rho = \frac{1}{\sqrt{\sqrt{1-c^2}+2c}}$*. Then, the space-like curve* γ_ρ^c *of* (4.10) *is a helix and magnetic curve which is not Legendre. Its curvature and torsion are:*

$$k = \sqrt{1-c^2}, \quad \tau = c - \frac{1}{\sqrt{1-c^2}+2c}. \tag{4.17}$$

Its Lancret invariant and λ *are:*

$$\text{Lancret}(\gamma_\rho^c) = \frac{\rho^2 + \tau}{k}, \quad \lambda = \frac{c^2 - 2\sqrt{1-c^2} - 2}{(\sqrt{1-c^2}+2c)^2}. \tag{4.18}$$

For a second example, we consider the same $N = \mathbb{R}^2$, $(a,b) = (0,+\infty)$, $\omega_1 = 0$ and $\omega_2 = 2x$ but:

$$F = -x^2 \tag{4.19}$$

which yields:

$$\beta = -\frac{1}{x^2} \tag{4.20}$$

and then M is again a quasi-para-Sasakian manifold which is not para-Sasakian. Fix again the real parameter $\rho \in (0,\infty)$. We consider now the c-slant curve:

$$\gamma_\rho^c(t) = \left(\frac{1}{\rho}, \sqrt{c^2+1}\rho t, (c - 2\sqrt{c^2+1})t\right) \tag{4.21}$$

with the interval $I = \mathbb{R}$. It is a time-like curve with: $\varepsilon_1 = -1, \varepsilon_2 = \varepsilon_3 = +1$. The covariant derivative along γ_ρ is:

$$\nabla_T X = \sqrt{c^2+1}\rho \nabla_{\frac{\partial}{\partial y}} X + \left(c - 2\sqrt{c^2+1}\right) \nabla_{\frac{\partial}{\partial z}} X \tag{4.22}$$

The Levi-Civita connection of this new metric is:

$$\begin{cases} \nabla_{\frac{\partial}{\partial y}} X = X^1 \left[-\frac{1}{x}\frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}\right] + X^2[-\frac{3}{x}\frac{\partial}{\partial x}] + X^3[-\frac{1}{x^2}\frac{\partial}{\partial x}] + \frac{\partial X^1}{\partial y}\frac{\partial}{\partial x} \\ \qquad + \frac{\partial X^2}{\partial y}\frac{\partial}{\partial y} + \frac{\partial X^3}{\partial y}\frac{\partial}{\partial z} \\ \nabla_{\frac{\partial}{\partial z}} X = X^1 \left[-\frac{1}{x^2}\frac{\partial}{\partial y} + \frac{2}{x}\frac{\partial}{\partial z}\right] + X^2[-\frac{1}{x^2}\frac{\partial}{\partial x}] + \frac{\partial X^1}{\partial z}\frac{\partial}{\partial x} + \frac{\partial X^2}{\partial z}\frac{\partial}{\partial y} + \frac{\partial X^3}{\partial z}\frac{\partial}{\partial z}. \end{cases} \tag{4.23}$$

and then:

$$\nabla_T T = \sqrt{1+c^2}\rho^3 \left(\sqrt{1+c^2} - 2c\right) \frac{\partial}{\partial x} \tag{4.24}$$

while:

$$\varphi(T) = \sqrt{1+c^2}\rho\frac{\partial}{\partial x}. \tag{4.25}$$

Taking into account the constraint of Proposition 3.4, the case $sign(\eta(B)) = -1$ is impossible and for $sign(\eta(B)) = +1$ we obtain:

Proposition 4.2. *Let $c \in \left(-\infty, \frac{1}{\sqrt{3}}\right)$ and $\rho = \frac{1}{\sqrt{\sqrt{1+c^2}-2c}}$. Then, the time-like curve γ_ρ of (4.21) is a helix and magnetic curve which is not Legendre. Its curvature and torsion are:*

$$k = \sqrt{1+c^2}, \quad \tau = \frac{1}{\sqrt{1+c^2}-2c} - c. \tag{4.26}$$

Its Lancret invariant has the same expression (4.16) and λ is:

$$\lambda = -1 - 2c^2 - \frac{2c\sqrt{1+c^2} - 4c^2 - 1}{(\sqrt{1+c^2}-2c)^2}. \tag{4.27}$$

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