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FUNCTIONAL ANALYSIS

Weighted Mazur-Ulam Spaces

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Abstract—The notion of a Mazur-Ulam space, introduced by C. P. Niculescu in [6] by using the midpoints, is extended here for an arbitrary weight $\lambda \in (0, 1)$. A similar characterization in terms of a class of isometries and their unique fixed point is obtained for the rational case $\lambda = \frac{m}{n}$ and under more complicated conditions than that of in [6] or [7, p. 166].

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The classical result of S. Mazur and S. Ulam [4], stating that every isometry between real normed linear spaces is a linear map up to translation, was recently reconsidered from the point of view of its extensions. Notice that this property is not true in the category complex linear spaces, for example, for the conjugation map in the complex plane. Note that the surjectivity hypothesis is essential and that without this assumption, J. A. Baker proved that every isometry from a normed real space into a strictly convex normed real space is a linear map up to translation.

An interesting framework to deal with the Mazur-Ulam theorem was developed in [6], see also [7, p. 165]. This approach is based on the notion of *midpoint* in a metric space, and therefore the notion of *Mazur-Ulam space* is naturally considered. The aim of this note is to extend this notion to a weighted case, by using an arbitrary, but fixed, intermediate point. On this way, several aspects of Mazur-Ulam spaces are generalized for a weight $\lambda \in (0, 1)$, the previous setting being obtained for $\lambda = \frac{1}{2}$. Also, for this special case $\lambda = \frac{1}{2}$, we derive some new aspects, for instance, we show that the midpoint is an example of metric center, in the sense of [9].

The present paper is divided into two sections. The first section is devoted to the *weighted Mazur-Ulam spaces* and their characterizations in terms of isometries with a unique fixed point satisfying some additional properties, but only for the rational λ. Two types of examples are provided: the real normed spaces and intervals with distances induced by a bijection. In the second class we recover the points provided by the geometric and harmonic means. The second section deals with the convexity between weighted Mazur-Ulam spaces, and the geometric convexity, discussed in [2].

Let (M, d) be a metric space and let $Isom(M, d)$ denote the group of isometries of (M, d). Recall that the points $x, y, z \in (M, d)$ are called *collinear* (in this order) if $d(x, y) + d(y, z) = d(x, z)$.

Definition 1. Let $\lambda \in (0, 1)$ be a fixed real number. A λ-*Mazur-Ulam space* (λMU-space for short) is defined to be a triple $(M, d, \sharp)$ with (M, d) a metric space and $\sharp : M \times M \to M$ satisfying for all $x, y \in M$ the following conditions:

A) (the idempotent property) $x\sharp x = x$;

B) (the λ-commutative property) $d(x\sharp y, y\sharp x) = |2\lambda - 1|d(x, y)$;

C) (the weighted property) the points x, $x\sharp y$, y are collinear and $d(x, x\sharp y) = \lambda d(x, y)$;

D) (the transformation property) $T(x\sharp y) = T(x)\sharp T(y)$ for all $T \in Isom(M, d)$.

The point $x\sharp y$ is called *the λ-intermediate point* between x and y.

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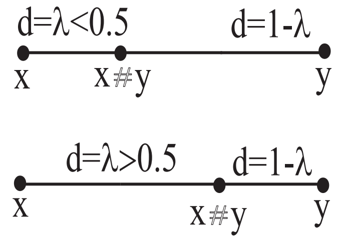
Remark 1. i) For $\lambda = \frac{1}{2}$ we recover the notion of *Mazur-Ulam space* considered in [7, p. 166]. The point $x\sharp y$ is called *midpoint* by Niculescu, but it appears also with different names, e.g., *metric midpoint* (see [8] and also [1, p. 18]). If we ask for the uniqueness of midpoints, we obtain the class of metric spaces with UMP property for which the survey [3] is available.
ii) From condition C) we derive: $d(y, x\sharp y) = (1-\lambda)d(x, y)$. The existence of $x\sharp y$ for every $\lambda \in (0, 1)$ in [1, p. 18] is called *metric convexity* of (M, d), while the uniqueness of the λ-intermediate point is called *strict metric convexity*.
iii) The essence of Mazur-Ulam theorem is contained in condition D).

Example 1. Let (M, d) be a real normed space with distance: $d(x, y) = \|x - y\|$. With $x\sharp_\lambda y = (1-\lambda)x + \lambda y$ we obtain a λMU-space. A remarkable example of $\frac{1}{2}$MU-space, treated in [6] and [7, p. 167-168], is that of $(Sym^{++}(n, \mathbb{R}), d_{trace})$, where $Sym^{++}(n, \mathbb{R})$ denotes the set of all $n \times n$ dimensional positive definite matrices with the trace metric $d_{trace}(A, B) = \left(\sum_{k=1}^{n} \log^2 \lambda_k\right)^{1/2}$, where $\lambda_1, ..., \lambda_n$ are the eigenvalues of AB^{-1}. The analogue of $(1-\lambda)x + \lambda y$ in this cone is the matrix $A\sharp_\lambda B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\lambda} A^{1/2}$.

Example 2. A large class of nonlinear examples can be obtained in the case where $M \subseteq \mathbb{R}$ is a real interval and $f : M \to f(M)$ is a bijection. Then $d_f : M \times M \to \mathbb{R}_+$ given by $d_f(x, y) = |f(x) - f(y)|$ is a distance on M, and with $x\sharp_f y = f^{-1}((1-\lambda)f(x) + \lambda f(y))$ we get a λMU-space.

The following two examples are of interest.

Example 2a. Let $M = \mathbb{R}_+^* = (0, +\infty)$ and $f(x) = \log x$. Then $f(M) = \mathbb{R}$ and $d_f(x, y) = |\log \frac{y}{x}|$. Hence $x\sharp_f y = x^{1-\lambda}y^{\lambda}$, and for $\lambda = \frac{1}{2}$ we have the Mazur-Ulam space (see [7, p. 166]), provided that the midpoint is equal to the geometric mean.



The cases $\lambda \neq \frac{1}{2}$.

Example 2b. Let M be as in example 2a), and let $f(x) = \frac{1}{x} = inv(x)$. Then $f(M) = M$ and $d_f(x, y) = |\frac{1}{x} - \frac{1}{y}|$. Therefore $x\sharp_f y = \frac{xy}{\lambda x + (1-\lambda)y}$, and the case $\lambda = \frac{1}{2}$ gives $x\sharp y =$ the harmonic mean M_{-1} of x and y (see [7, p. 1]), that is, $x\sharp y = \frac{2xy}{x+y} = M_{-1}(x, y)$.

Following the ideas of [7, p. 166], a characterization of λMU-spaces can be derived for the rational weights $\lambda = \frac{m}{n}$, where m, n are integers with $n > m \geq 1$ and $gcd(m, n) = 1$. Specifically, we have the following result.

Theorem 1. *Let (M, d) be a metric space. Suppose that for any pair $(a, b) \in M \times M$ there exists $G_{a,b} \in Isom(M, d)$ with the properties*:
(λMU1) $d(G_{a,b}(a), b) = (m-1)d(a, b)$ *and* $d(G_{a,b}(b), a) = (n-m-1)d(a, b)$;
(λMU2) $G_{a,b}$ *admits a unique fixed point* $z^{a,b}$ *and* $d(G_{a,b}(x), x) = nd(x, z^{a,b})$ *for all* $x \in M$;
(λMU3) $d(G_{a,b}(x), G_{b,a}(x)) = |2m - n|d(a, b)$ *for all* $x \in M$;
(λMU4) *the points a,* $z^{a,b}$*, b are collinear and* $nd(a, z^{a,b}) = md(a, b)$;
(λMU5) $G_{Ta,Tb} \circ T = T \circ G_{a,b}$ *for all* $T \in Isom(M, d)$.
Then $(M, d, \sharp)$ with $a\sharp b = z^{a,b}$ is a λMU-space.

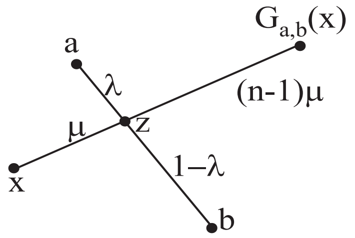
Proof. To prove the theorem we have to verify conditions A)–D) of Definition 1.1. For A), observe that by condition (λMU2) we have $nd(a, z^{a,b}) = d(G_{a,b}(a), a)$, while the condition ($\lambda$MU1) gives $d(G_{a,a}(a), a) = 0$.
B) From condition (λMU2) we have $nd(a\sharp b, b\sharp a) = d(G_{b,a}(a\sharp b), a\sharp b)$, and the fixed point property gives

$d(G_{a,b}(a\sharp b), G_{b,a}(a\sharp b)) = |2m-n|d(a,b)$. Hence $d(a\sharp b, b\sharp a) = |2\lambda - 1|d(a,b)$.
C) the condition (λMU4) is exactly the claimed property.
D) From the fixed point property and the condition (λMU5) we have $G_{Ta,Tb}(T(\sharp b)) = T(a\sharp b)$ which yields $T(a\sharp b) = T(a)\sharp T(b)$.

Remark 2. i) In the Niculescu's works [6, 7] the function $G_{a,b}$ is an isometry, but this property was not used in the above proof of Theorem 1.

ii) The condition (λMU3) implies that the distance between $G_{a,b}(x)$ and $G_{b,a}(x)$ does not depend on $x \in M$.

iii) We have $nd(b, z^{a,b}) = (n-m)d(a,b)$ and also the collinearity of all four points $a, z^{a,b}, z^{b,a}, b$.



The collinearity of points.

iv) A comparison of Theorem 3.13.2 of [7, p. 166] and our Theorem 1 reveals the complexity of the arbitrary rational case of λ and the special case $\lambda = \frac{1}{2}$. More precisely, our $\frac{1}{2}$MU1-2 are exactly the conditions MU 1-2 of the above cited theorem from [7], while the condition $\frac{1}{2}$MU3 means that $G_{a,b} = G_{b,a}$.

v) Very close notion appears in [9, p. 100]: for $a, b \in (M, d)$ a point $z \in M$ is called a *metric center of* a and b if there exists a surjective isometry $\psi : M \to M$ such that $\psi(a) = b$, $\psi(b) = a$, and for every $S \subseteq M$ with $0 < \sup_{x\in S} d(x,z) < +\infty$ we have

$$\sup_{x\in S} d(x,z) < \sup_{x\in S} d(\psi(x), x). \tag{1}$$

Lemma 4 of [9] shows that z is the unique metric center of a and b, and also, it is the unique fixed point of ψ. For $\frac{1}{2}$MU-spaces provided by Theorem 3.13.2 of [7, p. 166], we have that the point $a\sharp b$ is the metric center of a and b since $G_{a,b}(a) = b$, $G_{a,b}(b) = a$, and the right-hand side of (1) is the double of the left-hand side. Also, assuming that (M, d) is bounded we have that $a\sharp b$ is a *dissimilarity center of* M (see [9, p. 99] for the definition), and applying Lemma 1 of [9] we conclude that a bounded $\frac{1}{2}$MU-space provided by the isometries $G_{\cdot,\cdot}$ has a "universal" midpoint z, that is, for all $a, b, c, d \in M$ we have $a\sharp b = c\sharp d = z$ and z is the unique fixed point of all elements of $Isom(M, d)$.

Example 3. Returning to the Example 2 we consider
1) $G^{\lambda}_{a,b}(x) = (n-m)a + mb - (n-1)x$. Then we have $G^{\lambda}_{a,b}(a) = (1-m)a + mb$ and $G^{\lambda}_{a,b}(b) = (n-m)a + (m+1-n)b$.
2) $G^{f}_{a,b}(x) = f^{-1}\left((n-m)f(a) + mf(b) - (n-1)f(x)\right)$. Then for $f(x) = \log x$ we have $G^{\log}_{a,b}(x) = \frac{a^{n-m}b^m}{x^{n-1}}$, while for $f(x) = \frac{1}{x}$ we have $G^{inv}_{a,b}(x) = \frac{abx}{(n-m)bx+max-(n-1)ab}$. Hence $G^{\log}_{a,b}(a) = a^{1-m}b^m$ and $G^{\log}_{a,b}(b) = a^{n-m}b^{m+1-n}$, and respectively, $G^{inv}_{a,b}(a) = \frac{ab}{ma+(1-m)b}$ and $G^{inv}_{a,b}(b) = \frac{ab}{(m+1-n)a+(n-m)b}$.
3) Let (M, d) be a *metric absolute plane* according to [5, p. 236], that is, a set satisfying the axioms of incidence (in the plane), those of ordering, those of congruence and those of continuity. Then, in view of Theorem 1.4 of [5, p. 237], we conclude that an isometry with a unique fixed point is a product of two axial symmetries.

Now we show that the weighted Mazur-Ulam spaces constitute a natural framework to deal with convexity.

Definition 2. Let $(M, d, \sharp)$ and $(M', d', \sharp')$ be two weighted Mazur-Ulam spaces with M' being a subinterval in the real line $\mathbb{R}$. A continuous function $f : M \to M'$ is called $(\sharp, \sharp')$*-convex* if for all $x, y \in M$ we have

$$f(x\sharp y) \le f(x)\sharp' f(y), \tag{2}$$

and it is called $(\sharp, \sharp')$-*concave* if the opposite inequality holds. In the equality case:

$$f(x\sharp y) = f(x)\sharp' f(y) \tag{3}$$

we call it $(\sharp, \sharp')$-*affine*.

Observe that for $(M, d, \sharp) = (M', d', \sharp')$ given in Example 2a, that is, in the case of real normed spaces, the relation (3.1) is the classical convexity of real functions, and (3.2) is the usual affinity property. Since the weights of M and M' can be different, we obtain a very large setting for the notions of convexity, concavity and affinity.

Example 4. A function $g : (0, +\infty) \to (0, +\infty)$ is called *geometrically convex* if for all $\lambda \in (0, 1)$ and all $x, y \in (0, +\infty)$ we have (see [2, p. 154]:

$$g(x^\lambda y^{1-\lambda}) \leq g(x)^\lambda g(y)^{1-\lambda}. \tag{4}$$

Hence g is geometrically convex if and only if its exponential conjugate $\log \circ g \circ \exp$ is convex on the real line $\mathbb{R}$. It follows from (4) that $g(x\sharp^\lambda_{log} y) \leq g(x)\sharp^\lambda_{\log} g(y)$, and hence

$$g : (M = (0, \infty), d_{\log}, \sharp^\bullet_{\log}) \to (M = (0, \infty), d_{\log}, \sharp^\bullet_{\log})$$

is geometrically convex if and only if it is $(\sharp^\lambda_{\log}, \sharp^\lambda_{\log})$-convex for all $\lambda \in (0, 1)$.

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