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ArticleTitle	Weak Para-CR Structures on Vertical Bundles	
Article Sub-Title		
Article CopyRight	Springer International Publishing (This will be the copyright line in the final PDF)	
Journal Name	Advances in Applied Clifford Algebras	

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	Received	24 September 2015
Schedule	Revised	
	Accepted	29 February 2016

Abstract	We consider the paracomplex geometry of the vertical bundle for a given manifold in relationship with paracomplex CR-structures by following the complex case studied by Bejancu (Tensor 46:361–364, 1987). Adding a neutral metric, the corresponding structures on the vertical bundle of submanifolds, particularly hypersurfaces, are also studied through their invariant and anti-invariant distributions.

Mathematics Subject Classification (separated by '-')	53C15 - 53C12 - 53C17 - 53C40
Keywords (separated by '-')	(Weak almost) para-CR structure - Para-CRV manifold - (Almost) Paracomplex CR-structure
Footnote Information	

Adv. Appl. Clifford Algebras

DOI 10.1007/s00006-016-0652-3

Advances in Applied Clifford Algebras

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Weak Para-CR Structures on Vertical Bundles

Mircea Crasmareanu* and Laurian-Ioan Pişcoran

Abstract. We consider the paracomplex geometry of the vertical bundle for a given manifold in relationship with paracomplex CR-structures by following the complex case studied by Bejancu (Tensor 46:361–364, 1987). Adding a neutral metric, the corresponding structures on the vertical bundle of submanifolds, particularly hypersurfaces, are also studied through their invariant and anti-invariant distributions.

Mathematics Subject Classification. 53C15, 53C12, 53C17, 53C40.

Keywords. (Weak almost) para-CR structure, Para-CRV manifold, (Almost) Paracomplex CR-structure.

1. Introduction

The rôle of the vertical bundle (being an integrable distribution) in some remarkable geometries and gauge theories was pointed out recently in the book [6]. The purpose of this paper is to develop the paracomplex geometry of these bundles having as model their complex geometry considered by Bejancu in [5]. A strong motivation for our study is the fact that paracomplex geometry becomes recently the suitable mathematical tool in important physical theories (e.g. supersymmetric field theory) as was pointed out in [8,9] and subsequent works, as well as in a Hermitian Clifford analysis conform [20]. From a geometrical point of view, the para-CR structures are naturally associated with the geometry of the tangent bundle in Riemann–Finsler geometry in the recent paper of the authors [12] and with the paracontact geometry in [22]. Under the name of *almost D-structure*, these structures are studied in [21] by using mainly the algebraic tools of differential graded Lie algebras.

We continue here to study the interplay between paracomplex geometry and the geometry of some natural vector bundle by considering the notion of (*weak almost*) *para-CRV manifold* as one having a (weak almost) para-CR structure on its vertical bundle. Since for the complex case an important tool

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Journal: 6 Article No.: 652 TYPESET DISK LE CP Disp.:2016/3/7 Pages: 10

in characterizations is played by CR-structures we introduce also the paracomplex analog as compatible subbundles of the paracomplexified tangent bundle. The fundamental example for this approach is exactly the paracomplex (or dual) algebra $\mathbb{A}^n = \mathbb{R}^n[x]/(x^2 - 1)$ with its natural paracomplex structure. The vertical bundle considered as sub-bundle of the tangent bundle is an integrable distribution and we point out that in [19] there are studied distributions endowed with remarkable endomorphisms as almost complex and almost product structures.

We add also a semi-Riemannian metric g of neutral type in order to extend our framework to para-Hermitian geometry; the neutral space-time in dimension four is used in [18, p. 123]. So, we consider the restrictions of the above tensor fields to submanifolds, particularly hypersurfaces. Let us remark that in the latter case, due to the pseudo-Riemannian nature of g in order to construct an invariant distribution there are necessary several technical conditions, called by us *regularity*, *vertical regularity* and finally *strongly vertical regularity*. In the former case, we study the integrability of both invariant and anti-invariant distributions (denoted by D and $D^{\perp}$ respectively) defined by a para-CRV submanifold by using the Nijenhuis tensor of an intrinsic tensor field of $(1,1)$-type φ. For example, if the ambient manifold is weak para-CRV and φ is integrable then both D and $D^{\perp}$ are integrable. A case which provides the integrability of φ is given by the parallelism of φ with respect to the Levi-Civita connection of g.

2. Weak Para-CR Structures on Vertical Bundles

Let N be a smooth $(2n+k)$-dimensional manifold with $n, k \geq 1$ and fix D a distribution on N of rank $2n$. Considering D as a vector bundle over N let $\Gamma(D)$ be the $C^{\infty}(N)$-module of its sections. Suppose D is endowed with a endomorphism $K : D \to D$ of vector bundles satisfying $K^2 = I$ and $K \neq \pm I$ where $I = (\delta_{ij})$ is the identity (Kronecker) morphism on D; then the pair (D, K) is called *a weak almost para-CR structure*. The first main notion is given by [1, p. 3], see also [2]:

Definition 2.1. If for all $X, Y \in \Gamma(D)$ we have:

$$\begin{cases} [KX, KY] + [X, Y] \in \Gamma(D) \\ N_K(X, Y) := [KX, KY] + [X, Y] - K([X, KY] + [KX, Y]) = 0 \end{cases} \tag{2.1}$$

then (D, K) is a *weak para-CR structure* on N. If, in addition, the subdistributions of D corresponding to the eigenvalues ± 1 have the same rank n then the pair (D, K) is a *para-CR structure*.

Fix now $\tilde{M}$ a smooth m-dimensional manifold with $\tilde{\pi} : T\tilde{M} \to \tilde{M}$ its tangent bundle. The *vertical bundle* of $\tilde{M}$ is $VT\tilde{M} = \ker d\tilde{\pi}$; recall that $VT\tilde{M}$ is an integrable distribution of rank m. Also, there exists a natural bundle isomorphism between the pullback $\tilde{\pi}^{-1}T\tilde{M}$ by $\tilde{\pi}$ of $T\tilde{M}$ and $VT\tilde{M}$ called *the vertical lift*; it is detailed in [14, p. 3]. The framework of this paper is provided by:

Definition 2.2. The pair $(\tilde{M}, K)$ with a linear endomorphism $K : \Gamma(VT\tilde{M}) \to \Gamma(VT\tilde{M})$ is called a (*weak almost*) *para-CRV manifold* if $(VT\tilde{M}, K)$ is a (weak almost) para-CR structure on $T\tilde{M}$. It follows $m = 2n$.

Further, let N be a real manifold and $\mathbb{A}TN$ its *paracomplexified tangent bundle*; its sections are $U = X + jY$ with $X, Y \in \Gamma(TN)$ and $j^2 = 1$. Here, $\mathbb{A}$ is the algebra of paracomplex numbers $z = x + jy$; for details see [13]. Following the complex case we consider *the paraconjugation* $\bar{U} = X - jY$ and the real part function $Re(U) = X$; hence $Y = Re(jU)$.

As in the complex geometry we define:

Definition 2.3. An *almost paracomplex CR-structure* on N is a paracomplex subbundle H of $\mathbb{A}TN$ such that $H \cap \bar{H} = \{0\}$. If, in addition H is involutive, i.e. for every paracomplex vector fields $U, V \in \Gamma(H)$ we have $[U, V] \in \Gamma(H)$, then H is called a *paracomplex CR-structure*. H is called *of injective type* if the real part function is injective.

The first main result is a relationship between the last two notions which is a characterization:

Theorem 2.4. *The pair $(\tilde{M}, K)$ is a weak para-CRV manifold if and only if the tangent bundle $T\tilde{M}$ has a paracomplex CR-structure H of injective type such that $Re(H) = VT\tilde{M}$.*

Proof. Suppose $(\tilde{M}, K)$ is a weak para-CRV manifold and consider $H = \{X + jKX; X \in \Gamma(VT\tilde{M})\}$. It follows directly that $H \cap \bar{H} = \{0\}$, H is of injective type and $Re(K) = VT\tilde{M}$. For $U, V \in \Gamma(H)$ with expression $U = X + jKX$, $V = Y + jKY$ we get:

$$[U, V] = [KX, KY] + [X, Y] + j([X, KY] + [KX, Y]). \tag{2.2}$$

The vanishing of N_K on the pair (X, Y) gives:

$$Re(j[U, V]) = K([KX, KY] + [X, Y]) = K(Re([U, V]))$$

and then we have the integrability of H.

Conversely, suppose $T\tilde{M}$ has the injective type paracomplex CR-structure H with $Re(H) = VT\tilde{M}$. Fix $X \in \Gamma(VT\tilde{M})$ arbitrary and consider $U \in \Gamma(H)$ uniquely defined by $X = Re(U)$. We define $K : \Gamma(VT\tilde{M}) \to \Gamma(VT\tilde{M})$ by:

$$KX = Re(jU). \tag{2.3}$$

A straightforward computation gives $K^2 = I$ but $K \neq \pm I$ since $H \cap \bar{H} = \{0\}$. With the formula (2.2) we get:

$$[KX, KY] + [X, Y] = Re([U, V]), \quad [X, KY] + [KX, Y] = Re(j[U, V]) \tag{2.4}$$

and from the first relation we have $[KX, KY] + [X, Y] \in \Gamma(VT\tilde{M})$. Concerning the Nijenhuis tensor field we have:

$$N_K(X, Y) = Re([U, V]) - K(Re(j[U, V])) = Re([U, V]) - Re(j^2[U, V]) = 0$$

and the proof is complete. □

Remark 2.5. (i) The similar result for the complex case is Theorem 1.1 of [5, p. 362] stated without the injectivity hypothesis.

(ii) Interesting open problems for further studies are to replace the tangent bundle with: (i) the big tangent bundle $T^{big}M := TM \oplus T^*M \to M$ as in [7]; (ii) a tangent manifold (M, J) $(J^2 = 0)$ as in [10]; (iii) a Lie algebroid as in [11].

Example 2.6. (*Fundamental*) Let $\tilde{M} = \mathbb{R}^{2n} = \mathbb{A}^n$ be the flat Euclidean space with the global coordinates $x = (x^i) = (x^1, \ldots, x^{2n})$. Then $T\tilde{M} = \mathbb{R}^{4n}$ has the global coordinates $(x, y) = (x^i, y^i) = (x^1, \ldots, x^{2n}, y^1, \ldots, y^{2n})$ and every $X \in \Gamma(VT\tilde{M})$ has the global expression:

$$X = X^i \frac{\partial}{\partial y^i} = X^a \frac{\partial}{\partial y^a} + X^{n+a} \frac{\partial}{\partial y^{n+a}} \tag{2.5}$$

with $X^i \in C^\infty(\mathbb{R}^{4n})$, $1 \leq i \leq 2n$ and $1 \leq a \leq n$. We define as in [4, p. 133]:

$$KX = X^{n+a} \frac{\partial}{\partial y^a} + X^a \frac{\partial}{\partial y^{n+a}} \tag{2.6}$$

and then $(\mathbb{R}^{2n}, K)$ is a para-CRV manifold with the ± 1-eigenspaces:

$$\begin{cases} V_+T\tilde{M} = \left\{X \in \Gamma(VTM); X = X^a \left(\frac{\partial}{\partial y^a} + \frac{\partial}{\partial y^{n+a}}\right)\right\} \\ V_-T\tilde{M} = \left\{X \in \Gamma(VTM); X = X^a \left(\frac{\partial}{\partial y^a} - \frac{\partial}{\partial y^{a+n}}\right)\right\}. \end{cases} \tag{2.7}$$

The paracomplex CR-structure of Theorem 2.4 is:

$$H = \Big\{(X^a + jX^{n+a}) \frac{\partial}{\partial y^a} + (X^{n+a} + jX^a) \frac{\partial}{\partial y^{n+a}}; X^a, \\ X^{n+a} \in C^\infty(\mathbb{R}^{4n}), 1 \leq a \leq n\Big\}. \tag{2.8}$$

Example 2.7. In dimension 2, the Nijenhuis tensor field of any almost product structure vanishes. Then, if $n = 1$ we have that $(\tilde{M}, K)$ is a para-CRV manifold. □

In order to sketch a partial answer to the open problems of Remark 2.5(ii) we start now with a vector bundle $\xi = (E, \pi, M^n)$ having the fiber $\mathbb{R}^m$. The local coordinates $x = (x^1, \ldots, x^n) = (x^i)_{1 \leq i \leq n}$ on an open domain $U \subset M$ yield on $\pi^{-1}(U)$ the local coordinates $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^m) = (x^i, y^a)^{1 \leq i \leq n}_{1 \leq a \leq m}$. Let $T\pi : TE \to TM$ be the differential of π. Then $V\xi = \ker T\pi$ is a subbundle of $T\xi$ called *the vertical subbundle* of ξ. For every $u \in \pi^{-1}(U)$ the fiber $V_u\xi$ is spanned by $\left\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}\right\}$. The extension of the notions previously defined is as follows:

Definition 2.8. (i) The pair (ξ, K) with a linear endomorphism $K : \Gamma(V\xi) \to \Gamma(V\xi)$ is called a (*weak almost*) *para-CRV bundle* if $(V\xi, K)$ is a (weak almost) para-CR structure on TE. It follows $m = 2r$.

(ii) An *almost paracomplex CR-structure* on ξ is a paracomplex subbundle H of $\mathbb{A}TE \to \mathbb{A}TM$ such that $H \cap \bar{H} = \{0\}$. If, in addition H is involutive, i.e. for every paracomplex sections $U, V \in \Gamma(H)$ we have $[U, V] \in \Gamma(H)$, then H is called a *paracomplex CR-structure*.

Hence, the Theorem 2.4 extends to:

Theorem 2.9. *The pair (ξ, K) is a weak para-CRV bundle if and only if the paracomplex tangent bundle $\mathbb{A}TE \to \mathbb{A}TM$ has an injective paracomplex CR-structure H such that $Re(H) = V\xi$.*

3. Strongly Vertical Regular Hypersurfaces in Weak Para-Hermitian-CRV Geometry

Suppose now that $n \geq 2$ and $(\tilde{M}, K)$ is a weak para-CRV manifold such that $VT\tilde{M}$ is endowed with a semi-Riemannian metric g satisfying the skew-symmetry:

$$g(KX, Y) = -g(X, KY) \tag{3.1}$$

for all $X, Y \in \Gamma(VT\tilde{M})$. Under the above conditions we say that $(\tilde{M}, g, K)$ is a *weak para-Hermitian-CRV manifold*. Hence, g is a neutral metric having the signature (n, n) and $V_+T\tilde{M}$, $V_-T\tilde{M}$ are maximally isotropic with respect to g [4, p. 134]. Moreover, $V_+T\tilde{M}$ and $V_-T\tilde{M}$ are Lagrangian subbundles with respect to the fundamental 2-form $\Omega = g(K\cdot, \cdot)$; for the geometry of a Hamiltonian and a Lagrangian subbundle associated to a differential pseudo-form Ω on a general vector bundle E see [16].

Next, let M be a hypersurface of $\tilde{M}$ given by the immersion $i : M \to \tilde{M}$ and $\pi : TM \to M$ its tangent bundle. Since $d(di)(VTM) \subset VT\tilde{M}$, the metric g restricts to a semi-Riemannian metric on VTM denoted also by g. However, $(M, g, K|_{VTM})$ is not a weak para-Hermitian-CRV manifold due to odd dimension of M namely $2n - 1$ and this yields the following new notion:

Definition 3.1. The submanifold M of $(\tilde{M}, g, K)$ is called *regular* if VTM admits a g-orthogonal complement $VTM^{\perp}$ in $VT\tilde{M}$:

$$VT\tilde{M} = VTM \oplus VTM^{\perp}. \tag{3.2}$$

Now let us define $D^{\perp} = K(VTM^{\perp})$, then the rank of $D^{\perp}$ is 1 and from (3.1) we have $D^{\perp} \perp K(VTM)$. Remark that (3.1) with $Y = X$ yields also $VTM \perp K(VTM)$ but this does not imply the *K-anti-invariance* $D^{\perp} \subset VTM$ due to the signature of metric g. Hence we introduce:

Definition 3.2. A regular submanifold M of $(\tilde{M}, g, K)$ is called *vertical regular* if $D^{\perp} \subset VTM$ and *strongly vertical regular* if $D^{\perp}$ admits a g-orthogonal complement D in VTM:

$$VTM = D \oplus D^{\perp}. \tag{3.3}$$

Combining the last two relations it follows:

$$VT\tilde{M} = D \oplus D^{\perp} \oplus K(D^{\perp}) \tag{3.4}$$

and then D is a K-invariant distribution of rank $2n - 2$: $K(D) = D$. The main result of this Section is the para-Hermitian version of Theorem 2.1 of [5, p. 363]:

Theorem 3.3. *If M is a strongly vertical regular hypersurface of the weak para-Hermitian-CRV manifold $(\tilde{M}, g, K)$ then the tangent bundle TM has an injective paracomplex CR-structure H^* such that $Re(H^*) = D$.*

Proof. Let P and Q be the projectors from (3.3) and fix $X, Y \in \Gamma(D)$. From $KX, KY \in \Gamma(D) \subset \Gamma(VTM)$ and the integrability of VTM it results:

$$[X, KY] + [KX, Y] \in VTM = D \oplus D^{\perp}. \tag{3.5}$$

Then:

$$K([X, KY] + [KX, Y]) \in D \oplus VTM^{\perp} \tag{3.6}$$

while the vanishing of $N_K(X, Y)$ gives:

$$K([X, KY] + [KX, Y]) = [KX, KY] + [X, Y] \in VTM. \tag{3.7}$$

The last two equations yield:

$$K([X, KY] + [KX, Y]) \in \Gamma(D) \tag{3.8}$$

or, equivalently:

$$Q([X, KY] + [KX, Y]) = 0. \tag{3.9}$$

Hence:

$$[KX, KY] + [X, Y] = KP([X, KY] + [KX, Y]). \tag{3.10}$$

Following the pattern of Theorem 2.4 we define:

$$H^* = \{X + jKX; X \in \Gamma(D)\} \tag{3.11}$$

and then it follows directly $H \cap \bar{H} = \{0\}$, $Re(H^*) = D$ and the type of injectivity property of H^*. If $U, V \in \Gamma(H^*)$ have the expressions $U = X + jKX$, $V = Y + jKY$ then:

$$[U, V] = [KX, KY] + [X, Y] + jP([X, KY] + [KX, Y]) \tag{3.12}$$

and from (3.10) it yields that $[U, V] \in \Gamma(H^*)$. □

Example 3.4. Returning to Example 2.6 we add the semi-Riemannian metric g on $VT\mathbb{R}^{2n}$:

$$g = \delta_{ab} dy^a \otimes dy^b - \delta_{ab} dy^{a+n} \otimes dy^{b+n} \tag{3.13}$$

which makes $(\mathbb{R}^{2n}, g, K)$ para-Hermitian-CRV manifold. We consider $M = \mathbb{R}^{2n-1}$ as hypersurface in $\tilde{M} = \mathbb{R}^{2n}$ given by $x^{2n} = 0$. Then $VTM = span\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{2n-1}}\}$ and M is regular with $VTM^{\perp} = span\{\frac{\partial}{\partial y^{2n}}\}$. It results immediately the vertical regularity with $D^{\perp} = span\{\frac{\partial}{\partial y^n}\}$ and a direct verification proves that our setting is also strongly vertical regular hypersurface with $D = span\{\frac{\partial}{\partial y^{\alpha}}, \frac{\partial}{\partial y^{n+\alpha}}; 1 \leq \alpha \leq n-1\}$. Remark that $g|_{VTM}$ has the signature $(n, n-1)$.

Then the paracomplex CR-structure of Theorem 3.3 is:

$$H^* = \left\{(X^{\alpha} + jX^{n+\alpha})\frac{\partial}{\partial y^{\alpha}} + (X^{n+\alpha} + jX^{\alpha})\frac{\partial}{\partial y^{n+\alpha}}; X^{\alpha}, X^{n+\alpha} \in C^{\infty}(\mathbb{R}^{4n}), 1 \leq \alpha \leq n-1\right\}. \tag{3.14}$$

4. Para-CRV Submanifolds in Weak Almost Para-Hermitian CRV Geometry

Suppose now that M is a submanifold of $\tilde{M}$ not necessary hypersurface and not necessary vertical regular. Also, we do not use the conditions (2.1) and then $(\tilde{M}, g, K)$ can be a weak almost CRV-manifold. Inspired by the previous section we introduce:

Definition 4.1. M is a *para-CRV submanifold* of $(\tilde{M}, g, K)$ if there exists two g-orthogonal distributions D, $D^{\perp}$ on TM such that:

(i) $VTM = D \oplus D^{\perp}$,
(ii) D is K-invariant: $K(D) = D$; hence the dimension of D is even,
(iii) $D^{\perp}$ is K-anti-invariant: $K(D^{\perp}) \subset VTM^{\perp}$.

We call D and $D^{\perp}$ the *invariant distribution* and *anti-invariant distribution* respectively on TM.

Denote by P and Q the projection morphisms of VTM on D and $D^{\perp}$ respectively. We consider also two morphisms of vector bundles $\varphi : VTM \to VTM$ and $\omega : VTM \to VTM^{\perp}$ given by $\varphi = K \circ P$ and $\omega = K \circ Q$ respectively; hence $KX = \varphi X + \omega X$ for all $X \in \Gamma(VTM)$.

The integrability of the given distributions is provided by the following result, a counter-part of Theorem 3.1 from [5, p. 364]:

Theorem 4.2. (i) *The invariant distribution D is integrable if and only if for any $X, Y \in \Gamma(D)$:*

$$N_K(X, Y)^{VTM} = N_{\varphi}(X, Y) \tag{4.1}$$

where X^{VTM} means the part of $X \in \Gamma(VT\tilde{M})$ belonging to $\Gamma(VTM)$ with respect to an orthogonal decomposition similar to (3.2). In particular, if $(\tilde{M}, g, K)$ is a weak CRV-manifold then D is integrable if and only if N_{φ} vanishes on D.

(ii) *The anti-invariant distribution $D^{\perp}$ is integrable if and only if for any $X, Y \in \Gamma(D^{\perp})$:*

$$N_{\varphi}(X, Y) = 0. \tag{4.2}$$

Thus, in the weak para-CRV case the integrability of φ yields the integrability of both D and $D^{\perp}$. In particular, if φ is parallel with respect to the Levi-Civita connection ∇ of g then φ is integrable.

Proof. (i) By using the Nijenhuis tensor field of φ:

$$N_{\varphi}(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi([X, \varphi Y] + [\varphi X, Y]) \tag{4.3}$$

we get for all $X, Y \in \Gamma(VTM)$:

$$\begin{aligned}(N_K - N_{\varphi})(X, Y) = {} & -\omega([X, \varphi Y] + [\varphi X, Y]) + (I - \varphi^2)([X, Y]) + [\varphi X, \omega Y] \\ & + [\omega X, \varphi Y] + [\omega X, \omega Y] - K([X, \omega Y] + [\omega X, Y]).\end{aligned} \tag{4.4}$$

In particular, if $X, Y \in \Gamma(D)$ then $QX = QY = 0$ which yields $\omega X = \omega Y = 0$ and then:

$$N_K(X, Y) = N_{\varphi}(X, Y) - \omega([X, \varphi Y] + [\varphi X, Y]) + (I - \varphi^2)([X, Y]). \tag{4.5}$$

But $K \circ \varphi = P$ and $K \circ \omega = Q$ implies $\varphi^2 = (K - \omega) \circ \varphi = P - \omega \circ \varphi = P$. It results that:

$$N_K(X, Y) = N_\varphi(X, Y) - \omega([X, \varphi Y] + [\varphi X, Y]) + Q([X, Y]). \tag{4.6}$$

The left hand-side is from $\Gamma(VT\tilde{M})$ while in the right hand-side the first and third terms are from $\Gamma(VTM)$ and the second term belongs to $\Gamma(VTM^\perp)$. Therefore projecting this relation on VTM we get:

$$N_K(X, Y)^{VTM} = N_\varphi(X, Y) + Q([X, Y]). \tag{4.7}$$

and the last term is zero if and only if (4.1) holds.
(ii) If $X, Y \in \Gamma(D^\perp)$ then:

$$N_\varphi(X, Y) = \varphi^2([X, Y]) = P([X, Y]) \tag{4.8}$$

and (4.2) is directly. The last statement is a straightforward consequence of the relation:

$$N_\varphi(X, Y) = (\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X + \varphi[(\nabla_Y \varphi)X - (\nabla_X \varphi)Y] \tag{4.9}$$

which holds for any X and Y. □

Remark 4.3. (i) From $\varphi^2 = P$ it results that $\varphi^3 = P \circ K \circ P = K \circ P = \varphi$ which means that φ is a $f(3, -1)$ structure. An almost product structure on VTM is $P - Q$.

(ii) Let $X \in \Gamma(TM)$ and $Y, Z \in \Gamma(D)$. The covariant derivative of φ is:

$$g((\nabla_X \varphi)Y, Z) = X(g(KY, Z)) + g(P(\nabla_X Y), KZ) - g(KY, P(\nabla_X Z)). \tag{4.10}$$

Example 4.4. A large class of examples for $n = 2$ can be provided by following the table of page 284 from [15] as well as following [3]. Another type of examples, for any n, can be constructed from g-isometric actions of Lie groups G on $VT\tilde{M}$ with a suitable moment map having $0 \in Lie(G)$(=the Lie algebra of G) as regular value. This construction appears in [18].

Acknowledgments

The authors are extremely indebted to two anonymous referees for their useful remarks and improvements.

References

[1] Alekseevsky, D.V., Medori, C., Tomassini, A.: Maximally homogeneous para-CR manifolds. Ann. Glob. Anal. Geom. **30**(1), 1–27 (2006)

[2] Alekseevsky, D.V., Medori, C., Tomassini, A.: Maximally homogeneous para-CR manifolds of semisimple type. In: Handbook of Pseudo-Riemannian Geometry and Supersymmetry, pp. 559–577. IRMA Lectures in Mathematics and Theoretical Physics, vol. 16. European Mathematical Society, Zürich (2010)

[3] Bejan, C.-L.: CR-submanifolds of hyperbolic almost Hermitian manifolds. Demonstr. Math. **23**(2), 335–343 (1990)

[4] Bejan, C.-L.: The existence problem of hyperbolic structures on vector bundles. Publ. Inst. Math. Beogr. **53**(67), 133–138 (1993)

[5] Bejancu, A.: Complex structures on vertical bundle and CR-structures. Tensor **46**, 361–364 (1987)

[6] Bejancu, A., Farran, H.R.: Foliations and Geometric Structures. Mathematics and its Applications, vol. 580. Springer, Dordrecht (2006)

[7] Blaga, A.M., Crasmareanu, M.: A class of almost tangent structures in generalized geometry. Balk. J. Geom. Appl. **19**(2), 23–35 (2014)

[8] Cortés, V.: The special geometry of Euclidian supersymmetry: a survey. Rev. Un. Mat. Argent. **47**(1), 29–34 (2007)

[9] Cortés, V., Mayer, C., Mohaupt, T., Saueressig, F.: Special geometry of Euclidean supersymmetry. I. Vector multiplets. J. High Energy Phys. **3**, 028 (2004)

[10] Crasmareanu, M.: Nonlinear connections and semisprays on tangent manifolds. Novi. Sad J. Math. **33**(2), 11–22 (2003)

[11] Crasmareanu, M., Ida, C.: Almost analyticity on almost (para) complex Lie algebroids. Results Math. **67**(3–4), 495–519 (2015)

[12] Crasmareanu, M., Piscoran, L.-I.: Para-CR structures of codimension 2 on tangent bundles in Riemann–Finsler geometry. Acta Math. Sin. (Engl. Ser.) **30**(11), 1877–1884 (2014)

[13] Cruceanu, V., Fortuny, P., Gadea, P.M.: A survey on paracomplex geometry. Rocky Mt. J. Math. **26**(1), 83–115 (1996)

[14] Dragomir, S., Perrone, D.: Harmonic Vector Fields. Variational Principles and Differential Geometry. Elsevier, Inc., Amsterdam (2012)

[15] Etayo, F., Fioravanti, M., Trías, U.R.: On the submanifolds of an almost para-Hermitian manifold. Acta Math. Hung. **85**(4), 277–286 (1999)

[16] Gusiew-Czudzak, M., Oziewicz, Z.: Poincaré-Cartan subbundles. Adv. Appl. Clifford Algebras **4**(Suppl. 1), 325–339 (1994)

[17] Ławrynowicz, L., Suzuki, O.: Hurwitz duality theorems for Fueter and Dirac equations. Adv. Appl. Clifford Algebras **7**(2), 113–132 (1997)

[18] Ornea, L.: CR-submanifolds. A class of examples. Rev. Roumaine Math. Pures Appl. **51**(1), 77–85 (2006)

[19] Özdemir, F., Crasmareanu, M.: Geometrical objects associated to a substructure. Turk. J. Math. **35**(4), 717–728 (2011)

[20] Ren, G., Wang, H., Chen, L.: Paracomplex Hermitean Clifford analysis. Complex Anal. Oper. Theory **8**(6), 1367–1382 (2014)

[21] Rossi, F.A.: On deformations of D-manifolds and CR D-manifolds. J. Geom. Phys. **62**(2), 464–478 (2012)

[22] Wełyczko, J.: Para-CR structures on almost paracontact metric manifolds. J. Appl. Anal. **20**(2), 105–117 (2014)

Journal: 6 Article No.: 652 ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:2016/3/7 Pages: 10

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Received: September 24, 2015.
Accepted: February 29, 2016.