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GOLDEN-STATISTICAL STRUCTURES

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Abstract

Statistical structures are introduced in para-golden geometry and the characterization of holomorphic class of such structures is provided. A large class of examples is obtained by using an arbitrary 1-form and deforming the Levi-Civita connection with a mixed projective and dual-projective transformation.

Key words: statistical manifold, para-golden structure, holomorphic golden-statistical structure, (dual) projective equivalence

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Introduction. Information geometry has emerged from investigating the geometrical structure of a class of probability distributions depending on various parameters and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory conform [1]. The main notion of this theory is that of *statistical manifold* which is a triple (M, g, ∇) with g a Riemannian metric on the manifold M and ∇ a (symmetric) linear connection for which $C := \nabla g$ is totally symmetric, i.e. *the Codazzi equation* holds:

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(Z, X) \ (= (\nabla_Z g)(X, Y)). \tag{0.1}$$

In this framework there exists another torsion-free linear connection ∇^* defined by the relation:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{0.2}$$

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for any X, Y, $Z \in \mathfrak{X}(M)$, called the *conjugate* (or the *dual*) connection of ∇ with respect to g. From the direct consequence $\nabla^* g = -\nabla g$ it results that (M, ∇^*, g) is a statistical manifold, too. The conjugate of the conjugate connection of ∇ coincides with ∇, i.e. $(\nabla^*)^* = \nabla$.

Until now, the statistical manifolds have been studied in few particular geometries, e.g. paracontact geometry in [2]. In this paper we shall study the interference of a statistical structure on a golden manifold [3] of special type following the studies above. A motivation of this choice is the flow of recent studies in golden geometry.

More precisely, in the framework of para-golden geometry we introduce various types of statistical structures ∇ according to the behaviour of some remarkable tensors associated to the golden structure, such as the fundamental form, with respect to ∇. Let us remark that in order to introduce a *fundamental* 2-form we need a semi-Riemannian metric of signature (n, n) on an even dimensional (i.e. $2n$) manifold; sometimes such metrics are called *neutral*, [4]. This means that we work in the setting of para-Hermitian geometry [5] and motivates the prefix "para" used throughout the paper.

A large class of examples are produced via a differential 1-form λ by deforming the Levi-Civita connection ∇^g of the para-golden structure according to a combination of projective and dual-projective transformations. Since these transformations are in relationship with the study of geodesics an important particular case is provided by *a geodesic field* $\xi = \lambda^{\sharp_g}$ which means $\nabla^g_\xi \xi = 0$.

1. Golden-statistical manifolds. The setting of this work is provided by:

Definition 1.1. The triple (M^{2n}, g, J) consisting of a semi-Riemannian manifold of signature (n, n) endowed with an endomorphism $J \in \mathcal{T}^1_1(M)$ is called *para-golden Riemannian manifold* if:

$$J^2 - J - I = 0, \quad g(JX, Y) + g(X, JY) = 0 \tag{1.1}$$

for all vector fields X, Y, where I is the Kronecker endomorphism.

If (M^{2n}, P, g) is an almost para-Hermitian manifold, i.e. P is an almost product structure ([6]) on M satisfying the second relation (1.1), then with the approach of [3] we obtain two associated para-golden manifolds. As in the almost para-Hermitian geometry, we consider the tensor field:

$$\omega(X, Y) := g(X, JY) \tag{1.2}$$

which is skew-symmetric and called *the fundamental form.*

We introduce now the main notion of this paper:

Definition 1.2. i) The data (M, ∇, J, g) is a *golden-statistical manifold* if (M, ∇, g) is a statistical manifold and (M, g, J) is a para-golden Riemannian manifold.

ii) If J is covariant constant with respect to ∇, i.e. $\nabla J = 0$, then ∇ is called *J-connection.*

Concerning the conjugate connection of a golden-statistical manifold its existence is not necessary since the metric is semi-Riemannian and not a Riemannian one. Then a golden-statistical manifold with conjugate connection will be called *strong* in the following. We state now its properties:

Proposition 1.3. *Let (M, ∇, J, g) be a strong golden-statistical manifold and ∇^* the conjugate connection of ∇. Then*:

(i) *(M, ∇^*, J, g) is a golden-statistical manifold, too,*

(ii) *∇ is a J-connection if and only if so is ∇^* as well as $\nabla^\alpha := (1-\alpha)\nabla + \alpha\nabla^*$ for any real number $\alpha \neq 0$.*

Proof. i) It results directly from Definition 1.2.

ii) A straightforward calculus gives:

$$g((\nabla_X J)Y, Z) = -g(Y, (\nabla^*_X J)Z) \tag{1.3}$$

for any vector fields X, Y, $Z \in \mathfrak{X}(M)$ which have the required consequence. □

An analogue of the notion of *holomorphic statistical manifold* defined in [7] can be here considered:

Definition 1.4. The golden-statistical manifold (M, ∇, J, g) is *golden-holomorphic* if the fundamental 2-form ω is ∇-parallel.

A necessary and sufficient condition for the existence of the above notion is given in the following:

Proposition 1.5. *The golden-statistical manifold (M, ∇, J, g) is golden-holomorphic if and only if for any $X \in \mathfrak{X}(M)$ we have:*

$$(\nabla_X g) \circ (I \times J) = -g \circ (I \times \nabla_X J). \tag{1.4}$$

In particular, if ∇ is a J-connection, then the golden-statistical manifold M is golden-holomorphic if and only if:

$$(\nabla_X g) \circ (I \times J) = 0 \tag{1.5}$$

holds for any $X \in \mathfrak{X}(M)$.

Proof. We compute $(\nabla_X \omega)(Y, Z) = (\nabla_X g)(Y, JZ) + g(Y, (\nabla_X J)Z)$ for any X, Y, $Z \in \mathfrak{X}(M)$ and from the condition $\nabla\omega = 0$ we obtain the required relation. □

Another relationship between ∇ and ∇^* in the strong case is given by:

$$(\nabla_X \omega + \nabla^*_X \omega)(Y, Z) = g(Y, (\nabla_X J + \nabla^*_X J)Z) \tag{1.6}$$

and then on a strong golden-holomorphic-statistical manifold, if ∇ is a J-connection, we also have $\nabla^* \omega = 0$ due to ii) of Proposition 1.3; in conclusion $\nabla^\alpha \omega = 0$. Let us remark that from the second part of (1.1) we have $g(X, JX) = 0$ and then the left-hand-side of (1.4) applied on a pair (Y, Y) is $-g((\nabla_X J)Y, Y)$.

Example 1.6. Suppose that (M, P, g) is a para-Hermitian geometry which is para-Kähler, i.e. $\nabla^g P = 0$ ([5]), where ∇^g is the Levi-Civita connection of g. Then the associated para-golden geometries have ∇^g as J-connection. Any product connection ([6]) ∇ is J-connection.

2. Projective equivalences in golden-statistical geometry. Geometrically, two torsion-free linear connections are projectively equivalent if they have the same geodesics as unparameterized curves. Thus, they determine a class of equivalence on a given manifold called *projective structure*.

Fix a 1-form η on M. Two linear connections ∇ and ∇^* on M are:

(i) ([8]) *η-projectively equivalent* if

$$\nabla^* - \nabla = \eta \otimes I + I \otimes \eta; \tag{2.1}$$

(ii) ([9]) *η-dual-projectively equivalent* if

$$\nabla^* - \nabla = -g \circ \eta^{\sharp_g}, \tag{2.2}$$

where $\eta^{\sharp_g}$ is the vector field g-dual to η.

Note that if two connections are projectively equivalent or dual-projectively equivalent, then their conjugate connections with respect to a Riemannian metric associated in a statistical structure may not be projectively or dual-projectively equivalent, respectively:

Proposition 2.1. *Let (M, ∇, g) be a statistical manifold of dimension $n \geq 2$ and ∇^* its conjugate connection. Then ∇ and ∇^* are neither η-projectively equivalent nor η-dual projectively equivalent.*

Proof. i) Replacing the expression (2.1) of ∇^* in (0.2) we get:

$$(\nabla_X g)(Y, Z) = \eta(X) g(Y, Z) + \eta(Z) g(X, Y) \tag{2.3}$$

and taking into account the symmetry (0.1) we obtain:

$$\eta(Z) g(X, Y) = \eta(Y) g(Z, X) \tag{2.4}$$

which implies:

$$\eta \otimes I = I \otimes \eta. \tag{2.5}$$

Applying this relation on the pair $(X, \xi = \eta^{\sharp_g})$ with $X \perp \xi$ it results $0 = X$ which is impossible.

ii) Replacing (2.2) in (0.2) it results:

$$(\nabla_X g)(Y, Z) = -\eta(Y)g(X, Z) \tag{2.6}$$

and again the symmetry argument yields (2.5). □

Example 2.2. The computations above inspire us to construct a large class of golden-statistical manifolds. Fix a para-golden manifold (M, J, g) and a 1-form λ. We define the linear connection:

$$\nabla^\lambda = \nabla^g + \lambda \otimes I + I \otimes \lambda + g \otimes \xi, \tag{2.7}$$

where ξ is the g-dual of λ. A straightforward computation gives that the cubic form C^λ from Introduction is:

$$C^\lambda(X, Y, Z) = (\nabla^\lambda_X g)(Y, Z) = -2 \sum_{\text{cyclic}} [\lambda(X)g(Y, Z)] \tag{2.8}$$

and then ∇^λ is a strong statistical structure on M with the dual connection $(\nabla^\lambda)^* = \nabla^{-\lambda}$. Another important feature of ∇^λ is the fact that is torsion-free. Let us remark that a choice with the opposite sign in front of the last term:

$$\tilde{\nabla}^\lambda = \nabla^g + \lambda \otimes I + I \otimes \lambda - g \otimes \xi \tag{2.7$'$}$$

yields a non-statistical structure since:

$$(\tilde{\nabla}^\lambda_X g)(Y, Z) = -2\lambda(X)g(Y, Z). \tag{2.8$'$}$$

In particular, we consider the case of a *geodesic field* ξ which means: $\nabla^g_\xi \xi = 0$. Hence:

$$\nabla^\lambda_\xi \xi = 3\|\xi\|^2_g \xi \tag{2.9}$$

and then, ξ is also a geodesic field for ∇^λ if and only if it is a null vector field, i.e. $\|\xi\|_g = 0$. For example, the eigenvectors of J are null vector fields since we apply the second identity (1.1) and the fact that the eigenvalues of J are different from zero. Moreover, if X is a null vector field, then JX is also a null vector field. □

We have:

$$(\nabla^\lambda_X J)Y = (\nabla^g_X J)Y + (\lambda \circ J)(Y)X - \lambda(Y)JX + g(X, JY)\xi - g(X, Y)J(\xi) \tag{2.10}$$

and whence:

Proposition 2.3. i) *The covariant derivative of the fundamental form is*:

$$(2.11)\quad (\nabla^{\lambda}_{X}\omega)(Y,Z)=g(Y,(\nabla^{g}_{X}J)Z)+\lambda(JY)g(X,Z) \\ -\lambda(JZ)g(X,Y)-2\lambda(X)g(Y,JZ)-\lambda(Y)g(X,JZ)+\lambda(Z)g(X,JY).$$

ii) *The data* (M,λ,J,g) *is a strong golden-statistical manifold with* ∇^{λ} *a* J*-connection if and only if*:

$$(2.12)\qquad (\nabla^{g}_{X}J)Y=g(X,Y)J(\xi)+g(JX,Y)\xi+\lambda(Y)JX-(\lambda\circ J)(Y)X.$$

If, in addition ξ *is a geodesic field, then*:

$$(2.13)\qquad \nabla^{g}_{\xi}J(\xi)=2\|\xi\|^{2}_{g}J(\xi).$$

The condition of golden-holomorphic statistical structure means:

$$(2.14)\quad 2\lambda(X)g(Y,JZ)+\lambda(Y)g(X,JZ)+\lambda(JZ)g(X,Y) \\ =g(Y,(\nabla^{g}_{X}J)Z)+\lambda(Z)g(X,JY)+\lambda(JY)g(X,Z)$$

which yields:

Proposition 2.4. *The strong golden-statistical manifold* (M,λ,J,g) *with* ∇^{λ} *a* J*-connection is golden-holomorphic if and only if:*

(2.15)

$$2\lambda(X)g(Y,JZ)+\lambda(Y)g(X,JZ)+\lambda(JZ)g(X,Y)=\lambda(Z)g(X,JY)+\lambda(JY)g(X,Z).$$

In the particular case $X=Y$ *this means:*

$$(2.16)\qquad 3\lambda(X)g(X,JZ)+\lambda(JZ)\|X\|^{2}_{g}=\lambda(JX)g(X,Z).$$

For a null vector field X *and* $Z=\xi$ *this implies*: $\lambda(X)\lambda(JX)=0$.

Also (2.10) gives:

$$(2.17)\qquad (\nabla^{\lambda}_{X}J)X=(\nabla^{g}_{X}J)X+(\lambda\circ J)(X)X-\lambda(X)JX-\|X\|^{2}_{g}J(\xi)$$

and hence $(\nabla^{\lambda}_{X}-\nabla^{g}_{X})J(X)$ belongs to the kernel of λ for any vector field X.

The relationship between the curvature fields of ∇^{g} and ∇^{λ} is expressed by:

$$(2.18)\quad (R^{\lambda}-R^{g})(X,Y)Z=[g(Y,\nabla^{g}_{X}\xi)-g(X,\nabla^{g}_{Y}\xi)]Z \\ +[\lambda(Y)\lambda(Z)+\|\xi\|^{2}_{g}-g(Z,\nabla^{g}_{Y}\xi)]X-[\lambda(X)\lambda(Z)+\|\xi\|^{2}_{g}-g(Z,\nabla^{g}_{X}\xi)]Y \\ +g(Y,Z)\nabla^{g}_{X}\xi-g(X,Z)\nabla^{g}_{Y}\xi+[\lambda(X)g(Y,Z)-\lambda(Y)g(X,Z)]\xi.$$

In particular, if ξ is g-parallel (and hence geodesic field), i.e. $\nabla^g \xi = 0$, then:

$$(2.19)\quad (R^\lambda - R^g)(X,Y)Z$$
$$= [\lambda(Y)\lambda(Z) + \|\xi\|_g^2]X - [\lambda(X)\lambda(Z) + \|\xi\|_g^2]Y + [\lambda(X)g(Y,Z) - \lambda(Y)g(X,Z)]\xi.$$

More particularly, if the semi-Riemannian metric g admits a null and parallel vector field, then $R^\lambda - R^g$ belongs to the kernel of λ; this is the case of (even dimensional) Walker manifolds as treated in [4].

Recall also that for an arbitrary linear connection ∇ the exterior covariant derivative of J with respect to ∇ is ([6], p. 76):

$$d^\nabla J(X,Y) := (\nabla_X J)Y - (\nabla_Y J)X \tag{2.20}$$

For our ∇^λ we derive:
(2.21)
$$(d^{\nabla^\lambda} - d^{\nabla^g})J(X,Y) = \lambda(JY)X - \lambda(JX)Y + \lambda(X)JY - \lambda(Y)JX + 2g(X,JY)\xi.$$

Also, we have:

$$(\nabla^\lambda_X \lambda)Y = g(Y, \nabla^g_X \xi) - 2\lambda(X)\lambda(Y) - g(X,Y)\|\xi\|_g^2. \tag{2.22}$$

We finish this note with the Laplacian of a smooth function $f \in C^\infty(M)$ with respect to ∇^λ. Let us denote by Δ^λ this Laplacian and Δ^g the Laplacian with respect to g. Then a direct computation yields:

$$\Delta^\lambda(f) = \Delta^g(f) - 2g^{-1}(\lambda, df) - 2n\xi(f), \tag{2.23}$$

where g^{-1} is the variant of g on 1-forms. In particular, if λ is an exact 1-form with $\lambda = df$, then:

$$\Delta^{df}(f) = \Delta^g(f) - 2(n+1)\|df\|^2_{g^{-1}}. \tag{2.24}$$

Hence a g-harmonic function f gives a ∇^{df}-super-harmonic function: $\Delta^{df}(f) \leq 0$.

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