PARALLEL SECOND ORDER TENSORS ON VAISMAN MANIFOLDS

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Abstract. The aim of this paper is to study the class of parallel tensor fields α of $(0,2)$-type in a Vaisman geometry (M,J,g). A sufficient condition for the reduction of such symmetric tensors α to a constant multiple of g is given by the skew-symmetry of α with respect to the complex structure J. As an application of the main result we prove that certain vector fields on a P_0K-manifold turn out to be Killing. Also, we connect our main result with the Weyl connection of conformal geometry as well as with possible Ricci solitons in P_0K manifolds.

Introduction

The Theorem 2 of [11] states that a parallel second order tensor field in a non-flat complex space form is a linear combination (with constant coefficients) of the underlying Kähler metric and Kähler 2-form. The aim of this paper is to extend this result in the non-Kähler setting provided by locally conformal Kähler (lcK) geometry, more precisely Vaisman geometries. These are introduced in [12] under the name of *generalized Hopf manifolds* or PK-manifolds.

Our main result, namely Theorem 2.1, asserts that the above statement holds again in this framework for symmetric and J-skew-symmetric tensor fields of $(0,2)$-type as well as for J-skew-symmetric 2-forms satisfying a special identity with respect to its covariant derivative; here J denotes the complex structure of the given Hermitian geometry. As application, we obtain a reduction result for a special type if holomorphic vector fields in a subclass of Vaisman manifolds, usually denoted by P_0K-manifolds and given by the flatness of the local Kähler metrics of our structure. This reduction result is of the same nature as Theorem 3 from [11, p. 789] and states that a certain holomorphic vector field is in fact a homothetic one. Another reduction result of this type, but for conformal Killing vector fields on a special class of compact Vaisman manifolds, is Theorem 3.2 of [7, p. 99]. Recently, the compact lck manifolds with parallel vector fields are completely classified in [6].

We finish this short paper by pointing out some possible applications of our main result in two other settings: Weyl structures (naturally connected with l.c.k. geometry) and Ricci solitons (extensively studied until now in the Kähler framework).

1. Vaisman manifolds

Let (M^{2n},J,g) be a complex n-dimensional Hermitian manifold and Ω its fundamental 2-form given by $\Omega(X,Y)=g(X,JY)$ for any vector fields $X,Y\in\Gamma(TM)$. Recall from [3, p. 1] that (M,J,g,Ω) is a *locally conformal Kähler manifold* (lcK) if and only if there exists a closed 1-form $\omega\in\Gamma(T_1^0(M))$ such that: $d\Omega=\omega\wedge\Omega$. In particular, M is called *strongly non-Kähler*

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if ω is without singularities i.e. $\omega \neq 0$ everywhere; hence we consider $2c = \|\omega\|$ and $u = \omega/2c$ the corresponding 1-form. Since ω is called *the Lee form* of M the vector field $U = u^\sharp$ will be called *the Lee vector field*. Consider also the unit vector field $V = JU$, *the anti-Lee vector field*, as well as its dual form $v = V^\flat$, so: $u(V) = v(U) = 0, v = -u \circ J, u = v \circ J$.

Our setting is provided by the particular case of strongly non-Kähler l.c.K. manifolds, called *Vaisman manifolds*, and given by the parallelism of ω with respect to the Levi-Civita connection ∇ of g. Hence c is a positive constant and the Lemma 2 of [9] gives the covariant derivative of V with respect to any $X \in \Gamma(TM)$:

$$\nabla_X V = c[u(X)V - v(X)U - JX] \tag{1.1}$$

which yields the dual:

$$(\nabla_X v)Y = c[u(X)v(Y) - u(Y)v(X) + \Omega(X, Y)] \tag{1.2}$$

and the curvature:

$$R(X, Y)V = c^2\{[u(X)v(Y) - u(Y)v(X)]U + v(X)Y - v(Y)X\}. \tag{1.3}$$

Hence:

$$R(X, V)V = c^2[u(X)U + v(X)V - X] \tag{1.4}$$

and for an unitary X, orthogonal to V we derive the sectional curvature:

$$K(X, V) = c^2[u(X)^2 - 1]. \tag{1.5}$$

In particular: $K(U, V) = 0$.

The class of Vaisman manifolds was introduced in [12] and the old notation is that of *PK-manifolds*. A main subclass of Vaisman manifolds, denoted P_0K, is provided by the flatness of the local Kähler metrics generated by g and the local exactness of ω; see details in [12]. For these manifolds it is known the expression of the Ricci tensor of g; with formula (2.10) of [5, p. 125] one obtains:

$$Ric = 2c^2(n-1)[g - u \otimes u] \tag{1.6}$$

which means that the triple (M, g, U) is an *eta-Einstein manifold* which is not Einstein. P_0K-manifolds are locally analytically homothetic to Hopf manifolds (quotients $(\mathbb{C}^n \setminus \{0\})/\Gamma \simeq \mathbb{S}^1 \times \mathbb{S}^{2n-1}$, [8]) and it was shown in [5, p. 125] that P_0K-manifolds are locally symmetric spaces. Hence P_0K-manifolds are locally Ricci symmetric spaces, i.e. $\nabla Ric = 0$, which results also directly from (1.6).

In the last part of this section we recall the notion of Ricci solitons from [2]. On the manifold M, a *Ricci soliton* is a triple (g, ξ, λ) with g a Riemannian metric, ξ a vector field (called *the potential vector field*) and λ a real scalar such that the symmetric tensor field $Ric_{(\xi,\lambda)}$ of $(0, 2)$-type:

$$Ric_{(\xi,\lambda)} := \mathcal{L}_\xi g + 2Ric + 2\lambda g = 0 \tag{1.7}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ and Ric is the Ricci tensor of g. A straightforward consequence of (1.6) is that on a non-Kähler P_0K manifold the potential vector field of a Ricci soliton can not be a conformal Killing vector field: $\mathcal{L}_\xi g = 2\varepsilon g$ since by applying (1.7) on the pair (U, U) we get $\varepsilon + \lambda = 0$ while by applying on the pair (V, V) one gets: $\varepsilon + \lambda = -2c^2(n-1)$.

2. Parallel second order tensors in a Vaisman geometry

The purpose of this Section is to prove the main result of the paper:

Theorem 2.1 *Let (M, J, g, Ω) be a Vaisman manifold.*
i) *Fix a tensor field $\alpha \in \Gamma(T^0_2(M))$ which is symmetric and J-skew-symmetric i.e.:*

$$\alpha(JX, Y) + \alpha(X, JY) = 0 \tag{2.1}$$

for all $X, Y \in \Gamma(TM)$. If α is parallel with respect to ∇ then it is a constant multiple of the metric tensor g.
ii) *Let the 2-form $\beta \in \bigwedge^2(M)$ which is J-skew-symmetric and satisfies:*

$$(\nabla_Z \beta)(X, Y) = c[g(X, Z)\beta(U, Y) - \Omega(X, Z)\beta(V, Y) - v(X)\beta(Y, JZ) + u(X)\beta(Y, Z)] \tag{2.2}$$

for all $X, Y, Z \in \Gamma(TM)$. Then β is a constant multiple of the fundamental form Ω.

Proof i) Applying the Ricci commutation identity [2, p. 14] and $\nabla^2_{X,Y}\alpha(Z, W) - \nabla^2_{X,Y}\alpha(W, Z) = 0$ for all vector fields X, Y, Z, W we obtain the relation (1.1) of [11, p. 787]:

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0 \tag{2.3}$$

which is fundamental to this subject. Replacing $Z = W = V$ and using (1.3) it results, by the symmetry of α:

$$v(X)\alpha(Y, V) = v(Y)\alpha(X, V). \tag{2.4}$$

With $X = V$ we get:

$$\alpha(Y, V) = v(Y)\alpha(V, V). \tag{2.5}$$

From the symmetry and J-symmetry of α we have:

$$\alpha(U, V) = 0 \tag{2.6}$$

and then the parallelism of α and formulae (2.5) – (2.6) imply that $\alpha(V, V)$ is a constant. Applying X to (2.5) and using (1.2) we have:

$$X(\alpha(Y, V)) = \alpha(\nabla_X Y, V) + \alpha(Y, \nabla_X V) = X(v(Y))\alpha(V, V) + 2v(Y)\alpha(\nabla_X V, V)$$

which means that:

$$c\alpha(Y, u(X)V - v(X)U - JX) = (\nabla_X v)(Y)\alpha(V, V) + 2cv(Y)\alpha(u(X)V - JX, V).$$

Due to (2.5) and $v \circ J = u$ the last term above is zero. By using (1.2) and (2.5) again it follows:

$$-v(X)\alpha(U, Y) + \alpha(X, JY) = -u(Y)v(X)\alpha(V, V) + \Omega(X, Y)\alpha(V, V). \tag{2.7}$$

We have a relation similar to (2.5) but in terms of U:

$$\alpha(Y, U) = \alpha(Y, -JV) = \alpha(JY, V) = v(JY)\alpha(V, V) = u(Y)\alpha(V, V) \tag{2.8}$$

and then, returning to (2.7) we get:

$$\alpha(X, JY) = \alpha(V, V)\Omega(X, Y) \tag{2.9}$$

and a transformation $Y \to JY$ gives the conclusion.
ii) Let $\alpha \in \Gamma(T^0_2(M))$ be given by a relation dual to that defining Ω through g:

$$\alpha(X, Y) := \beta(JX, Y). \tag{2.10}$$

Hence: $\alpha(Y, X) = \beta(JY, X) = -\beta(X, JY)$ which by J-skew-symmetry means $\beta(JX, Y)$ and consequently α is symmetric. Also:

$$\alpha(JX, Y) + \alpha(X, JY) = -\beta(X, Y) + \beta(JX, JY) = -\beta(X, Y) - \beta(J^2X, Y) = 0.$$

Finally, (2.2) express the parallelism of α by using the following covariant derivative of J resulting from Proposition 1 of [9, p. 338]:

$$(\nabla_Z J)X = c[\Omega(X,Z)U + g(X,Z)V - u(X)JZ - v(X)Z]. \tag{2.11}$$

Therefore we apply i) for α and (2.9) is exactly the conclusion: $\beta(X,Y) = \alpha(V,V)\Omega(X,Y)$ with $\alpha(V,V) = -\beta(U,V)$. □

Remarks 2.2 i) The reduction of a covariant second order tensor field to a multiple of the metric holds generally under the hypothesis of irreducibility of the holonomy group/algebra, see for example the theorem 57 of [10, p. 254]. Our result above implies weaker conditions for the l.c.K. metric in the Vaisman framework.
ii) The parallel forms of compact connected Vaisman manifolds are completely treated in Theorem 7.7. of [3, p. 78]. The covariant derivative of the fundamental form is:

$$(\nabla_Z \Omega)(X,Y) = c[u(Y)g(JX,Z) - u(X)g(JY,Z) + v(X)g(Y,Z) - v(Y)g(X,Z)]. \tag{2.2F}$$

iii) It is well known that $\bigwedge^2(M)$ admits the decomposition: $\bigwedge^2(M) = \mathbb{R}\Omega \oplus \bigwedge^{1,1}_J(M) \oplus LM$, where $\bigwedge^{1,1}_J(M)$ (LM, respectively) denotes the subbundle of $\bigwedge^2(M)$ of primitive J-invariant (J-skew-invariant, respectively) 2-forms on M. The J-skew-symmetry of $\beta \in \bigwedge^2(M)$ means exactly its J-invariance (Ω is J-invariant but not primitive) and hence the case ii) of Theorem 2.1 gives a necessary condition on $\beta \in \mathbb{R}\Omega \oplus \bigwedge^{1,1}_J(M) \subset \bigwedge^2(M)$ to belongs to the first factor of the decomposition above. □

As application of Theorem 2.1 we have a result similar to theorem 3 of [11]:

Corollary 2.3 *Let ξ be a holomorphic vector field on a Vaisman manifold such that $\mathcal{L}_\xi g$ is parallel. Then ξ is a homothetic vector field. Moreover, if (M,g,J) is a P_0K-manifold then ξ is a Killing vector field.*

Proof For the second order covariant tensor field $\alpha = \mathcal{L}_\xi g$ we can apply the previous theorem if the skew-symmetry (2.1) is satisfied. We have:

$$\alpha(JX,Y) + \alpha(X,JY) = g(\nabla_{JX}\xi - J(\nabla_X \xi), Y) + g(X, \nabla_{JY}\xi - J(\nabla_Y \xi)) \tag{2.12}$$

and the holomorphic hypothesis $\mathcal{L}_\xi J = 0$ yields:

$$\nabla_{JX}\xi - J(\nabla_X \xi) = \nabla_\xi JX - J(\nabla_\xi X)$$

which means that:

$$\begin{aligned} \alpha(JX,Y) + \alpha(X,JY) &= g(\nabla_\xi JX - J(\nabla_\xi X), Y) + g(\nabla_\xi JY - J(\nabla_\xi Y), X) = \\ &= -g(JX, \nabla_\xi Y) - g(J(\nabla_\xi X), Y) - g(JY, \nabla_\xi X) - g(J(\nabla_\xi Y), X) = 0. \end{aligned} \tag{2.13}$$

Hence the claimed skew-symmetry holds and consequently:

$$\mathcal{L}_\xi g = \alpha(V,V)g \tag{2.14}$$

is just the first conclusion regarding ξ. This relation implies $\mathcal{L}_\xi Ric = 0$ and in the P_0K setting the equation (1.6) gives:

$$\mathcal{L}_\xi g = \mathcal{L}_\xi(u \otimes u). \tag{2.15}$$

The right hand side of (2.15) applied to (V,V) gives that $\alpha(V,V) = 0$ and then (2.14) gives the second conclusion. □

Examples 2.4 i) The Lee vector field U is a holomorphic ([3, p. 37]) and Killing one in a Vaisman manifold since it is parallel: $\nabla U = 0$. Then $\alpha := u \otimes u$ is symmetric and parallel while the condition (2.1) means the Kählerian setting $\omega = 0$. Indeed, with $X = Y$,

the equation (2.1) reads $u(X)u(JX) = 0$ for all X i.e. $u = 0$.
ii) Again from [3, p. 37] the anti-Lee vector field V is holomorphic and Killing.
iii)([8]) It is well known that on compact Ricci-flat Kähler manifolds, any holomorphic vector field is parallel. □

Let us finish this note by pointed out some other possible applications of our main result 2.1:

Remarks 2.5 i) The locally conformal Kähler geometry can be studied in terms of Weyl structures and their associated Weyl connections according to Theorem 1.4 of [3, p. 5]. The expression of the Weyl connection of (M, g, J, ω) is formula (2) of [7, p. 94] which for our notation becomes:

$$D = \nabla - c(u \otimes I + I \otimes u - g \otimes U) \tag{2.16}$$

with the Kronecker tensor field I. Hence, a symmetric tensor field $\alpha \in \Gamma(T^0_2(M))$ is ∇-parallel if and only if its Weyl derivative is:

$$D_Z\alpha(X,Y) = c[2u(Z)\alpha(X,Y)+u(X)\alpha(Y,Z)+u(Y)\alpha(X,Z)-g(X,Z)\alpha(U,Y)-g(Y,Z)\alpha(U,X)] \tag{2.17}$$

for all vector fields X, Y, Z. Also, $\beta \in \bigwedge^2(M)$ is ∇-parallel if and only if its Weyl derivative is:

$$D_Z\beta(X,Y) = c[2u(Z)\beta(X,Y)+u(X)\beta(Z,Y)+u(Y)\beta(X,Z)-g(X,Z)\beta(U,Y)-g(Y,Z)\beta(X,U)] \tag{2.18}$$

for all vector fields X, Y, Z. The conversely problem, namely the Weyl-parallelism of forms, is studied in [1].
ii) An interpretation of (2.2) is provided by the so-called *Hermitian* or *first canonical* connection of (M, g, J) defined by ([4, p. 273]):

$$\tilde{\nabla}_X Y := \nabla_X Y - \frac{1}{2}J(\nabla X J)Y = \frac{1}{2}\nabla_X Y - \frac{1}{2}J(\nabla_X JY). \tag{2.19}$$

For a J-invariant 2-form β we have:

$$\tilde{\nabla}_Z\beta(X,Y) = \frac{1}{2}\nabla_Z\beta(X,Y) + \frac{1}{2}\nabla_Z\beta(JX,JY) \tag{2.20}$$

and hence β satisfies (2.2) if and only if β is $\tilde{\nabla}$-parallel. □

If we intend to use the first part of Theorem 2.1 for searching Ricci solitons in P_0K manifolds then it appears as natural to consider the symmetric tensor field:

$$\alpha_\xi := \mathcal{L}_\xi g - 4c^2(n-1)u \otimes u. \tag{2.21}$$

The J-skew-symmetry of this tensor field means:

$$(\mathcal{L}_\xi g)(JX,Y) + (\mathcal{L}_\xi g)(X,JY) = -4c^2(n-1)(u \otimes v + v \otimes u)(X,Y) \tag{2.22}$$

for all vector fields X, Y. Then:
ii1) applying (2.22) on (U, U) we get: $(\mathcal{L}_\xi g)(U, V) = 0$,
ii2) applying (2.22) on (U, V) we get: $(\mathcal{L}_\xi g)(V, V) - (\mathcal{L}_\xi g)(U, U) = -4c^2(n-1)$.
With (1.7) we obtain the scalar λ as: $\lambda = g(\mathcal{L}_\xi U, U) = -g(\nabla_U \xi, U) = U(g(\xi, U))$ which yields:

Proposition 2.6 *Let ξ be a vector field on a compact P_0K-manifold such that α_ξ defined by* (2.21) *is parallel and J-skew-symmetric. Hence, the angle between ξ and U is not constant (hence $\xi \neq U$) and ξ is not a Lie symmetry of U (hence $\xi \neq V$).*

Proof Assuming the contrary of one of the above claims means that the Ricci soliton provided by ξ is steady: $\lambda = 0$. But according to the Proposition 5.20 of [2, p. 117] a steady Ricci soliton on a compact manifold is Einstein which is impossible for P_0K-manifolds. □

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