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Last Multipliers for Riemannian Geometries, Dirichlet Forms and Markov Diffusion Semigroups

Mircea Crasmareanu[1]

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Abstract We start this study with last multipliers and the Liouville equation for a symmetric and non-degenerate tensor field Z of $(0, 2)$-type on a given Riemannian geometry (M, g) as a measure of how far away is Z from being divergence-free (and hence g^C-harmonic) with respect to g. The some topics are studied also for the Riemannian curvature tensor of (M, g) and finally for a general tensor field of $(1, k)$-type. Several examples are provided, some of them in relationship with Ricci solitons. Inspired by the Riemannian setting, we introduce last multipliers in the abstract framework of Dirichlet forms and symmetric Markov diffusion semigroups. For the last framework, we use the Bakry-Emery carré du champ associated to the infinitesimal generator of the semigroup.

Keywords Riemannian manifold · Symmetric covariant 2-tensor field · Last multiplier · Liouville equation · Jacobi form · Modular manifold · Ricci soliton · Dirichlet form · Markov diffusion semigroups

Mathematics Subject Classification 53C21 · 53C25 · 35Q75 · 53C99 · 53B20

Introduction

The method of study dynamical systems through Jacobi last multipliers is well known, and a modern approach can be found in [26]. Recently, we extend in [12] the notion of last multiplier and its associated Liouville equation to vector fields on manifolds

✉ Mircea Crasmareanu
mcrasm@uaic.ro
http://www.math.uaic.ro/˜mcrasm

[1] Faculty of Mathematics, University "Al. I. Cuza", 700506 Iasi, Romania

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endowed with a volume form. Also, these objects are studied in some remarkable settings: for Poisson geometry in [13], for weighted manifolds in [15], for Lie algebroids in [17] and in a complex framework in [18].

All these works concern with vectors and multivectors. This paper starts with the case of covariant tensor fields Z of second order and in a Riemannian geometry (M, g). We impose two conditions of such Z: (1) the symmetry, in order to define the divergence of Z with respect to g; (2) the non-degeneration, in order to deal with the corresponding Liouville equation, see (1.5) below. It follows the birth of a remarkable 1-form, $\omega_Z := Z_{\sharp}^{-1}(divZ)$, called the *Jacobi form* of the triple (M, g, Z). Then the existence of last multipliers for Z with respect to g means that this form is an exact one. Since exactness implies closedness, we arrive at a de Rham cohomology $[\omega_Z] \in H^1(M)$ when Z admits last multipliers.

The motivation for this subject is both geometrical and dynamical. From a geometric point of view, we study several important cases of Z, some of them involved in the Ricci flow theory (see for example [8] for general theory and the particular case of Ricci solitons in [2]): the Ricci tensor of g, the Hessian of a smooth function, the Lie derivative of g with respect to a given vector field. Also, since the divergence-free nature of Z expresses the harmonicity of its $(1, 1)$-version with respect to the complete lift of g (which is a semi-Riemannian metric, [21,22]), we connect our study with the theory of harmonic self-maps of (TM, g^C). From the dynamical point of view, the divergence-free covariant tensors provide physical conservation laws (see the whole of [6, Chap. 5]) and the generic example is the Einstein tensor of g discussed in Sect. 2.

In fact, the main result, namely Theorem 1.3, gives a condition for the existence of last multipliers for a given Z with respect to g and also, their generic expression in terms of a potential $u \in C^{\infty}(M)$ of the Jacobi form. This closedness condition, (1.7) or equivalently (1.8), is expressed in terms of ∇=the Levi-Civita connection of g and the $(1, 1)$-version of Z; so, there exist curvature restrictions generated by g as well as the nature of Z. Also, this condition (1.7) involves the exterior differential d on M, and hence there are de Rham cohomology restrictions. It follows that there exists Z without last multipliers with respect to g.

The paper is organized as follows. The first section introduces the setting and its main result, namely Theorem 1.3, discusses the existence and expression of the last multipliers for a fixed Z. Several remarks are included towards a better picture of this framework; for example the Jacobi form is expressed in an adapted orthonormal co-frame.

The Sect. 2 is devoted to applications and some remarkable 2-tensor fields are discussed: the metric (as the simpler case), the Ricci tensor, Chen-Nagano harmonicity, the Lie derivative of g with respect to a given vector field, the Hessian of a smooth function, the second fundamental of a hypersurface, and 2-tensors obtained from 1-forms. We continue their study in Sect. 3 with concrete examples: rotationally symmetric metrics, quasi-constant curvature manifolds, quasi-Einstein manifolds, Ricci solitons and spheres. For Ricci solitons, we derive the (non-vanishing) Jacobi-Ricci form in the gradient case while for the general (not necessary gradient) case the harmonicity of $(\mathcal{L}_V g)^{\sharp}$ is equivalent with the constancy of the scalar curvature. Also for the gradient case the measure and the diffusion operator (weighted Laplacian) of the canonically

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associated metric measure space are expressed by using the last multiplier instead of the potential function u; in fact this was the initial motivation for this work, namely to derive relationships between (gradient) Ricci solitons and last multipliers.

The fourth section concerns with two types of deformations: (1) conformal deformations and the case of a traceless Z is discussed from the point of preserving the exactness (closedness) character of the triple (M, g, Z); (2) the curvature deformations under the action of curvature operator of g. The following section discusses the case of the Riemannian curvature tensor of g, and we finish with a general tensor field $T \in \mathcal{T}^1_k(M)$ for which we express the Liouville equation. By using the Weitzenböck formula, we express this equation for $T = \nabla \alpha$ with α a k-form in terms of Laplacian and the rho tensor field of α.

Inspired by a formula for last multipliers of vector fields from the Riemannian geometry which involves the Laplacian, we extend this notion firstly for the setting of Dirichlet forms and secondly for symmetric Markov diffusion semigroups in the last section. For the last framework, we use the Bakry-Emery carré du champ Γ associated to the infinitesimal generator L of the semigroup and then an example of last multiplier is put in relationship with the harmonicity with respect to L.

1 Last Multipliers for Symmetric Covariant 2-Tensors

Let (M^n, g) be a smooth, n-dimensional Riemannian manifold and fix an orthonormal frame $\{e_i; 1 \leq i \leq n\} = \{e_1, \ldots, e_n\} \subset \mathcal{X}(M)$. As usual, we denote by $C^\infty(M)$ the algebra of smooth real functions on M, by $\mathcal{X}(M)$ the $C^\infty(M)$-module of vector fields and by $\Omega^k(M)$ the $C^\infty(M)$-module of differential k-forms on M with $1 \leq k \leq n$. We need also $C^\infty_+(M)$ the cone of positive smooth functions on M. Let ∇ be the Levi-Civita connection of g and Tr the trace operator with respect to g.

The main object of our study is a fixed symmetric tensor field of $(0, 2)$-type: $Z \in \mathcal{T}^0_{2,s}(M)$. Its associated $(1, 1)$-tensor field has two variants: 1) $Z^\sharp : \mathcal{X}(M) \to \mathcal{X}(M)$ and 2) $Z_\sharp : \Omega^1(M) \to \Omega^1(M)$, respectively. The *divergence* of Z with respect to g is $\mathrm{div}\, Z \in \Omega^1(M)$ defined by [2, p. 9]:

$$\mathrm{div} Z = Tr(\nabla Z^\sharp) \tag{1.1}$$

which means for $X \in \mathcal{X}(M)$ that [1, p. 334]:

$$\mathrm{div} Z(X) = \sum_{i=1}^{n} (\nabla_{e_i} Z)(X, e_i). \tag{1.2}$$

Sometimes, a local expression is useful. In a local coordinate system $(x^i; 1 \leq i \leq n)$ on M, we have $g = g_{ij} dx^i \otimes dx^j$ and $Z = Z_{ij} dx^i \otimes dx^j$ with $Z_{ij} = Z_{ji}$; hence $Z^\sharp(Z_\sharp) = Z^i_j \frac{\partial}{\partial x^i} \otimes dx^j$ with $Z^i_j = g^{ia} Z_{aj}$. Let (Γ^a_{bc}) be the set of Christoffel symbols of ∇. Then

$$\mathrm{div} Z = Z^k_{j|k} dx^j, \quad Z^k_{j|i} = \frac{\partial Z^k_j}{\partial x^i} + Z^l_j \Gamma^k_{li} - Z^k_l \Gamma^l_{ji}. \tag{1.3}$$

Fix also $f \in C^{\infty}(M)$. A straightforward computation gives

$$\mathrm{div}(fZ) = Z_{\sharp}(\mathrm{d}f) + f\,\mathrm{div}Z. \tag{1.4}$$

The aim of this paper is to study the following notion:

Definition 1.1 The function $f \in C^{\infty}_{+}(M)$ is *a last multiplier for Z with respect to g* if $\mathrm{div}(fZ) = 0$. The corresponding equation

$$Z_{\sharp}(d\ln f) = -\mathrm{div}Z \tag{1.5}$$

is called *the Liouville equation of Z* with respect to g.

In order to solve the Liouville equation, we need an additional hypothesis: Z is non-degenerate i.e., $Z^{\sharp}$ is non-singular operator. Let $\mathcal{T}^1_{1,i}(M)$ be the cone of invertible endomorphisms of the tangent bundle TM; then $Z^{\sharp} \in \mathcal{T}^1_{1,i}(M)$. Hence our setting is described by the following notions:

Definition 1.2 (i) The triple (M, g, Z) with non-degenerate Z is called *exact* (*closed*) *modular manifold* if its *Jacobi form* $\omega_Z := Z^{-1}_{\sharp}(divZ) \in \Omega^1(M)$ is exact (closed).
(ii) In the first case above, the function $u \in C^{\infty}(M)$ is called *potential* if $\omega_Z = du$. In the second case above, the cohomology class $[\omega_Z] \in H^1(M)$ is called *the modular class* of the closed modular manifold (M, g, Z).

We obtain a characterization for the existence of last multipliers:

Theorem 1.3 i. *Let $Z \in \mathcal{T}^0_{2,s}(M)$ be non-degenerate. Then Z admits last multipliers with respect to g if and only if (M, g, Z) is an exact modular manifold; hence if $u \in C^{\infty}(M)$ is a potential of it then the last multipliers of Z have the form:*

$$f = f_C = C\exp(-u) \tag{1.6}$$

for $C > 0$. It results that if f_1 and f_2 are last multipliers then there exists a constant $C > 0$ such that $f_2 = Cf_1$.

ii. *The triple (M, g, Z) is a closed modular manifold if and only if*

$$\mathrm{div}Z \in Ker(d \circ Z^{-1}_{\sharp}) \tag{1.7}$$

equivalently

$$Z^{\sharp} \in Ker(d \circ Z^{-1}_{\sharp} \circ Tr \circ \nabla). \tag{1.8}$$

iii. *In particular, if Z is divergence-free then its last multipliers are the (positive) constant functions.*

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Remark 1.4 (i) Let us denote the operator $LM^g Z := d \circ Z_\sharp^{-1} \circ Tr \circ \nabla$ involved in condition (1.8). It results that $LM^g Z$ has the decomposition:

$$\mathcal{T}^1_{1,i}(M) \overset{\nabla}{\to} \mathcal{T}^1_2(M) \overset{Tr}{\to} \mathcal{T}^0_1(M) = \Omega^1(M) \overset{Z_\sharp^{-1}}{\to} \Omega^1(M) \overset{d}{\to} \Omega^2(M). \quad (1.9)$$

Let us remark that the four operators involved above have different natures: the middle terms $(Tr, Z_\sharp^{-1})$ are algebraic while the extremal terms (∇, d) are differential. Two of them (∇, Tr) depend on g; only one depends of Z, namely $Z_\sharp^{-1}$, and the last, namely d, concerns with the nature of ambient setting M.

(ii) Following the case of weighted divergence for vector fields from [15], we define *the weighted f-divergence* of Z as

$$\mathrm{div}_f Z := \frac{1}{f}\mathrm{div}(fZ). \quad (1.10)$$

Then the Liouville equation is $\mathrm{div}\ _f Z = 0$ and the set of last multipliers is a ”measure of how far away” is Z from being f-divergence-free with respect to g.

(iii) In [21, p. 26] or [22, p. 127], it is remarked that the divergence-free character of Z is equivalent with the harmonicity of the map $Z^\sharp : (TM, g^C) \to (TM, g^C)$ where g^C is the complete lift of g to the tangent bundle of M. Hence, if f is a last multiplier we can say that $Z^\sharp$ is *f-harmonic with respect to g^C*.

(iv) We can consider a Frolicher-Nijenhuis type approach. For a 1-form ω and a $(1, 1)$-tensor field F, we can define the *F-differential* of ω by

$$d_F\omega(\cdot, \cdot) = d\omega(F\cdot, \cdot) - d(F_\sharp(\omega)). \quad (1.11)$$

The condition (1.7) means $d(Z_\sharp^{-1}(\mathrm{div}Z)) = 0$ and hence the Liouville equation means in terms of 2-forms:

$$(d_{Z_\sharp^{-1}}\mathrm{div}Z)(\cdot, \cdot) = d(\mathrm{div}Z)(Z^\sharp\cdot, \cdot). \quad (1.12)$$

(V) The Liouville equation can be completely integrated in the 1-dimensional case: $g = g(x) > 0$, $Z = Z(x)dx \otimes dx$. Since $Z^\sharp(Z_\sharp) = \frac{Z}{g}\frac{\partial}{\partial x} \otimes dx$, the non-degeneration of Z means $Z \neq 0$. The divergence of Z is $divZ = \frac{1}{g}(\frac{Z}{g})'dx$ where we use the derivative with respect to variable x of M. The operator involved in (1.7) is $Z_\sharp^{-1} : \omega \in \Omega^1(M) \to \frac{g}{Z}\omega \in \Omega^1(M)$ and hence the formal equation

$$f(x) = \exp\left(-\int Z_\sharp^{-1}(\mathrm{div}Z)dx\right) \quad (1.13)$$

is expressed as

$$f(x) = \exp\left(-\int \frac{1}{Z}\left(\frac{Z}{g}\right)' dx\right). \quad (1.14)$$

For example, if $Z = g^k$ then a straightforward computation yields: $f(x) = C\exp(\frac{k-1}{g(x)})$ with the constant $C > 0$.

(vi) There exists an orthonormal frame adapted to our setting. Indeed, since $Z^\sharp$ is symmetric i.e., g-self-adjoint

$$g(Z^\sharp X, Y) = g(X, Z^\sharp Y), \tag{1.15}$$

there exists such an orthonormal frame and there exists $\{\lambda_1, \ldots, \lambda_n\} \subset C^\infty(M)$ such that e_i is unit of the eigenvector corresponding to the eigenvalue λ_i:

$$Z^\sharp e_i = \lambda_i e_i. \tag{1.16}$$

Let $\{e^1, \ldots, e^n\} \subset \Omega^1(M)$ be the dual frame. Then we express the divergence of Z as

$$\mathrm{div} Z = A_j e^j, \quad A_j = \mathrm{div} Z(e_j). \tag{1.17}$$

In order to express the coefficient A_j, we introduce the connection coefficients $\{C^k_{ij}\} \subset C^\infty(M)$ of ∇ with respect to the adapted orthonormal frame:

$$\nabla_{e_i} e_j = C^k_{ij} e_k. \tag{1.18}$$

Hence, a long but straightforward computation gives

$$A_j = e_j(\lambda_j) - \sum_{i=1}^{n} \left(C^i_{ij}\lambda_i - C^j_{ii}\lambda j \right). \tag{1.19}$$

It follows an expression of the Jacobi form. Since

$$Z_\sharp : \omega_k e^k \in \Omega^1(M) \to \omega_k \lambda_k e^k \in \Omega^1(M), \tag{1.20}$$

we obtain that Z is non-degenerate if and only if all its eigenvalues λ_i are different to zero and the inverse:

$$Z_\sharp^{-1} : \omega_k e^k \in \Omega^1(M) \to \frac{\omega_k}{\lambda_k} e^k \in \Omega^1(M). \tag{1.21}$$

In conclusion, the Jacobi form of (M, g, Z) expressed in the adapted dual frame is

$$\omega_Z = \frac{A_j}{\lambda_j} e^j. \tag{1.22}$$

Its differential is

$$d\omega_Z = d\left(\frac{A_j}{\lambda_j}\right) \wedge e^j + \frac{A_j}{\lambda_j} de^j = e_k\left(\frac{A_j}{\lambda_j}\right) e^k \wedge e^j - \frac{A_j}{\lambda_j}\theta^j_k \wedge e^k \tag{1.23}$$

with θ_k^j the connection 1-forms of g, [2, p. 2]. But

$$\theta_k^j = C_{ik}^j e^i \tag{1.24}$$

and then

$$d\omega_Z = \left[e_i \left(\frac{A_k}{\lambda_k} \right) - C_{ik}^j \frac{A_j}{\lambda_j} \right] e^i \wedge e^k. \tag{1.25}$$

(vii) The 2-tensor field Z being symmetric and non-degenerate can be considered as another Riemannian metric on M. To the pair of Riemannian metrics (g, Z) and the application $\varphi : M \to M$ in [9, p. 337], it is associated as a *map-Laplacian*

$$\Delta_{g,Z}\varphi := Tr_g(\nabla^{g\boxtimes_\varphi Z} d\varphi) \tag{1.26}$$

with $g \boxtimes_\varphi Z := g^{-1} \otimes \varphi^* Z$ the natural bundle metric on $T^*M \otimes \varphi^{-1}(TM)$ and $\nabla^{g\boxtimes_\varphi Z} d\varphi$ the associated *map-Hessian* of $d\varphi : TM \to TM$. Hence, with the computation of the cited book on page 338, we get that the divergence of Z can be computed in another way from

$$\mathrm{div}_g Z = (\Delta_{g,Z} 1_M)^g_\sharp + \frac{1}{2} d(Tr_g Z) \tag{1.27}$$

with 1_M the identity map of M and $Tr_g Z$ the trace of Z with respect to g. The term $\Delta_{g,Z} 1_M$ is a vector field along the map 1_M and hence is a section in the pull-back bundle $1_M^{-1} TM = TM$ i.e., an usual vector field on M; the notation from (1.27) gives its dual 1-form with respect to g. Then f is a last multiplier of Z if and only if:

$$(\Delta_{g,fZ} 1_M)^g_\sharp + \frac{1}{2} d(f Tr_g Z) = 0. \tag{1.28}$$

Hence we introduce a new type of multiplier:

Definition 1.5 Let M be endowed with the Riemannian metrics g, Z, and $f \in C^\infty_+(M)$. We call f as being *a conformal harmonic multiplier for Z with respect to g* if $1_M : (M, g) \to (M, fZ)$ is a harmonic map.

It follows that a conformal harmonic multiplier f is also a last multiplier for Z with respect to g if and only if it has the expression $\frac{C}{Tr_g Z}$ supposing that $Tr_g Z \neq 0$.

2 Applications to Some Remarkable 2-Tensor Fields

In this section, we provide several examples of above settings.

(I) $Z = g$. Let $I \in \mathcal{T}^1_{1,i}(M)$ be the Kronecker endomorphism given locally by δ^i_j. Since $\nabla g^\sharp = \nabla I = 0$ we have two results: (1) a well-known one: g is divergence-free;

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(2) the triple (M, g, g) is exact modular manifold with zero Jacobi form. Hence we have the case iii of Theorem 1.3.

Moreover, if Z is a conformal deformation of g, i.e., $Z = ug$ with $u \in C^\infty_+(M)$, then the triple (M, g, ug) is an exact modular manifold since $Z_\sharp^{-1} = \frac{1}{u} I$ and its Jacobi form is $\omega_Z = d \ln u$; its modular class is zero. The Liouville equation yields the last multipliers $f = f_C = \frac{C}{u}$ with $C > 0$.

(II) $Z = \mathrm{Ric}$ the Ricci tensor field of g. Let us denote $Q = \mathrm{Ric}^\sharp$, respectively, $S = \mathrm{Ric}_\sharp$ and suppose that Ric is non-degenerate. Denote by R the scalar curvature of g. The divergence of Ric is given by [27, p. 39]

$$\mathrm{div}\mathrm{Ric} = \frac{1}{2} dR \tag{2.1}$$

and then we introduce the following:

Definition 2.1 If the Ricci tensor is non-degenerate then the *Jacobi-Ricci form* of (M, g) is $\omega_{\mathrm{Ric}} := S^{-1}(dR) \in \Omega^1(M)$. The Riemannian manifold (M, g) is called *Ricci-exact* (*Ricci-closed*) *modular manifold* if ω_{Ric} is exact (closed). In the second case, the de Rham cohomology class $[\omega_{\mathrm{Ric}}] \in H^1(M)$ is called the *Ricci-modular* class of (M, g).

Hence Ric admits last multipliers with respect to g if and only if (M, g) is a Ricci-exact modular manifold and if u is a potential for it, i.e. $\omega_{\mathrm{Ric}} = du$, then the last multipliers of Ric have the form

$$f = f_C = C \exp\left(-\frac{u}{2}\right) \tag{2.2}$$

with $C > 0$. The Riemannian manifold (M, g) is a Ricci-closed modular manifold if and only if

$$R \in Ker(d \circ S^{-1} \circ d) \tag{2.3}$$

and the Liouville equation for Ric is

$$d \ln f = -\frac{1}{2} \omega_{\mathrm{Ric}}. \tag{2.4}$$

In particular, if R is constant then Ric is divergence-free and the last multipliers of Ric with respect to g are again the constant functions. Two related tensors are
(a) *the Einstein tensor of* g, [27, p. 106]: $Einstein(g) := \mathrm{Ric} - \frac{R}{2} g$ which is again divergence-free and we have a variant of Proposition 3.1 from [21, p. 26]:

Proposition 2.2 *For any Riemannian geometry* (M, g)*, the map* $Q - \frac{R}{2} g : (TM, g^C) \to (TM, g^C)$ *is harmonic and in particular, the scalar curvature of* g *is constant if and only if* $Q : (TM, g^C) \to (TM, g^C)$ *is a harmonic map. Moreover, if* Ric *is non-degenerate then the map* $1_M : (M, g) \to (M, Ric)$ *is harmonic.*

(b) *the Schouten tensor* of g, [27, p. 109], for $n > 2$: $P = \frac{1}{n-2}(2\text{Ric} - \frac{R}{n-1}g)$. Its divergence is div $P = \frac{1}{n-1}dR$.

Let us point out that in [24,25] is considered a tensor field of type $Z = \text{Ric} + \varphi g$ with $\varphi \in C^\infty(M)$ and its physical importance. [3]

(III) (Chen–Nagano harmonicity) A common generalization of the cases I and II is provided by the harmonicity in the Chen–Nagano (CN) sense. Recall, after [7], that the metric Z is *CN-harmonic with respect to* g if the identity map $1_M : (M, g) \to (M, Z)$ is harmonic. With the discussion of [21, p. 26], this is equivalent with the divergence-free character of the tensor field: $Z - \frac{TrZ}{2}g$. We derive:

Proposition 2.3 *Suppose that Z is CH-harmonic with respect to g. Then (M, g, Z) is a closed modular manifold if and only if*

$$TrZ \in Ker(d \circ Z_\sharp^{-1} \circ d). \tag{2.5}$$

In particular, if TrZ is constant (for example Z is traceless) then the last multipliers of Z are the (positive) constant functions.

A more general case is when TrZ is an eigenvalue of $Z_\sharp$: $Z_\sharp(TrZ) = \lambda TrZ$ with $\lambda \neq 0$. Then the last multipliers of Z have the expression: $f = C\exp(-\frac{TrZ}{2\lambda})$ with $C > 0$. The case of traceless operators is discussed in the section 4.

(IV) Fix $V \in \mathcal{X}(M)$ and consider $Z = \mathcal{L}_V g$ where $\mathcal{L}_V$ denotes the Lie derivative with respect to g. Its local expression is

$$Z_{ij} = V_{i|j} + V_{j|i}, \quad V_{a|b} = \frac{\partial V_a}{\partial x^b} - V_l\Gamma^l_{ab}, \quad Z^k_j = V^k_{|j} + g^{ka}V_{j|a}. \tag{2.6}$$

The non-degeneration of this Z excludes the case of a Killing V. Let $V^\flat$ be the 1-form dual of V with respect to g and Δ the Laplacian of g. The divergence of this Z is expressed in Lemma 1.10 of [8, p. 6] as

$$\text{div}Z = (\Delta + S)(V^\flat) + d(\text{div}V). \tag{2.7}$$

The operator $\Delta + S$ can be considered as a "Schrödinger" one on 1-forms and hence:

Proposition 2.4 *The triple $(M, g, \mathcal{L}_V g)$ is a closed modular manifold if and only if*

$$d \circ (\mathcal{L}_V g)_\sharp^{-1}\left[((\Delta + S)(V^\flat) + d(\text{div}V)\right] = 0. \tag{2.8}$$

If $V^\flat$ is a solution of the "Schrödinger-Ricci" equation i.e.,

$$(\Delta + S)(V^\flat) = -d(\text{div}V) \tag{2.9}$$

then $(\mathcal{L}_V g)^\sharp : (TM, g^C) \to (TM, g^C)$ is a harmonic map.

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In a local coordinate system the Schrödinger-Ricci equation is

$$\Delta V_j + R_{jk}V^k = -\frac{\partial}{\partial x^j}\left(\sum_{i=1}^{n} V^i_{|i}\right), \tag{2.9loc}$$

and if V is divergence-free then it means that $V^\flat$ belongs to the kernel of $\Delta + S$.

(V) Fix $u \in C^\infty(M)$ and consider $Z = H(u)$ the Hessian of u with respect to g. Its local components are

$$H(u)_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial u}{\partial x^k}. \tag{2.10}$$

The class of smooth functions with vanishing Hessian are called *linear* in [27, p. 283] and *Killing potentials* in [11] since their gradient are Killing vector fields. From (2.7) we obtain

$$\mathrm{div} H(u) = \frac{1}{2}\left[(\Delta + S)(du) + d(\Delta u)\right] \tag{2.11}$$

and then we get, with $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ the co-differential induced by g:

Proposition 2.5 *For a nonlinear function u, the triple $(M, g, H(u))$ is a closed modular manifold if and only if*

$$d \circ (H(u))^{-1}_{\sharp}\left[(2d\delta + S)(du)\right] = 0. \tag{2.12}$$

If u is a solution of the "exact Schrödinger-Ricci" equation i.e.,

$$(2d\delta + S)(du) = 0 \tag{2.13}$$

then $H(u)^\sharp : (TM, g^C) \to (TM, g^C)$ is a harmonic map.

A combination of this application and II) consists in the *Bakry-Emery Ricci tensor*

$$\mathrm{Ric}_u := \mathrm{Ric} + H(u) \tag{2.14}$$

expressing the equation of gradient Ricci solitons and having the divergence

$$\mathrm{divRic}_u = \frac{1}{2}\left[(\Delta + S)(du) + d(R + \Delta u)\right]. \tag{2.15}$$

This tensor field is divergence-free if and only if

$$(2d\delta + S)(du) = -dR \tag{2.16}$$

and we will meet again in the following section. We finish this application with a generalization of Ricci solitons:

Definition 2.6 On the Riemannian manifold (M, g) endowed with $Z \in \mathcal{T}^0_{2,s}(M)$ the pair (u, λ) is a *Z-gradient soliton* if

$$H(u) + Z + \lambda g = 0. \tag{2.17}$$

(VI) Let $\eta \in \Omega^1(M)$ and $\xi \in \mathcal{X}(M)$ its g-dual. Consider $Z = \eta \otimes \eta$ and its $(1, 1)$-version $Z^\sharp(Z_\sharp) = \eta \otimes \xi$. Then $\nabla Z^\sharp = \nabla\eta \otimes \xi + \eta \otimes \nabla\xi$ which yields

$$\mathrm{div} Z = (div\xi)\eta + \nabla_\xi \eta. \tag{2.18}$$

Since $Z_\sharp : \omega \in \Omega^1(M) \rightarrow \omega(\xi)\eta \in \Omega^1(M)$, it results that condition (1.7) requires $(\mathrm{div}\xi)\eta + \nabla_\xi \eta$ be a multiple of η. The first term is already a multiple of η, hence we need the hypothesis

$$\nabla_\xi \eta = u\eta \tag{2.19}$$

for a given $u \in C^\infty(M)$, which can be called *the ξ-recurrence of η* since is a particular case of the recurrence $\nabla\eta = u\eta \otimes \eta$. Then

$$Z_\sharp^{-1} : \mathrm{div} Z \rightarrow \omega_Z \in \Omega^1(M), \quad \omega_Z(\xi) = \mathrm{div}\xi + u \tag{2.20}$$

and we derive:

Proposition 2.7 *Let $Z = \eta \otimes \eta$ be non-degenerate with η being ξ-recurrent with the factor $u \in C^\infty(M)$. Suppose there exists $\omega_Z \in \Omega^1(M)$ such that*

$$\omega_Z(\xi) = \mathrm{div}\xi + u. \tag{2.21}$$

Then Z admits last multipliers if and only if ω_Z is an exact 1-form and the corresponding Liouville equation is $d \ln f = -\omega_Z$.

Hence, the Jacobi form of this example is exactly ω_Z satisfying (2.21) and the recurrence (2.19) can be expressed as $\nabla_\xi \xi = u\xi$. An important particular case is that of a geodesic vector field, $\nabla_\xi \xi = 0$, for which its Jacobi form must satisfies $\omega_Z(\xi) = div\xi$.

(VII) Let $A, B \in \mathcal{X}(M)$ and $a, b \in \Omega^1(M)$ their g-dual. It is well-known that A and B define the skew-symmetric operator: $A \wedge_g B : X \in \mathcal{X}(M) \rightarrow g(A, X)B - g(B, X)A \in \mathcal{X}(M)$. For example, (M, g) has constant curvature k if and only if its curvature tensor $Riem$ satisfies ([27, p. 84]) $Riem(X, Y) = -kX \wedge_g Y$ for all vector fields X, Y.

The same vector fields define also a symmetric operator $Z^\sharp = \frac{1}{2}(A \otimes b + a \otimes B) : X \in \mathcal{X}(M) \rightarrow \frac{1}{2}[g(A, X)B + g(B, X)A] \in \mathcal{X}(M)$, and hence we can consider its $(0, 2)$-variant: $Z = \frac{1}{2}(a \otimes b + b \otimes a)$. If locally we have $A = A^i \frac{\partial}{\partial x^i}$, $B = B^j \frac{\partial}{\partial x^j}$ then $Z_{ij} = \frac{1}{2}(A_i B_j + A_j B_i)$. Its variant on 1-forms is $Z_\sharp : \omega \in \Omega^1(M) \rightarrow \frac{1}{2}[\omega(A)b +$

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$\omega(B)a] \in \Omega^1(M)$ and then $\eta \in \Omega^1(M)$ belongs to the domain of $Z_\sharp^{-1}$ if and only if is $C^\infty(M)$-combination of a and b. A straightforward computation gives the divergence:

$$\mathrm{div} Z = \frac{1}{2}[\nabla_A b + \nabla_B a + (\mathrm{div} A)b + (\mathrm{div} B)a]. \tag{2.22}$$

For $a = b$ we reobtain the application VI.

(VIII) Suppose that M is a hypersurface in N^{n+1} and let g be its first fundamental form and $Z = b$ its second fundamental form. Let A be the Weingarten (or shape) operator of M and suppose that A is invertible. The divergence of b with respect to g is

$$\mathrm{div} b = d(TrA) = ndH \tag{2.23}$$

with H the mean curvature. The condition (1.7) becomes

$$H \in Ker(d \circ A_\sharp^{-1} \circ d). \tag{2.24}$$

Hence we define *the Jacobi-shape form* of the hypersurface M as

$$\omega_M := A_\sharp^{-1}(dH) \tag{2.25}$$

while the Liouville equation is

$$d \ln f = -n\omega_M. \tag{2.26}$$

In conclusion, the CMC hypersurfaces admit as last multipliers the positive constant functions. For a general hypersurface let $\{e_1, \ldots, e_n\}$ its principal directions and $\{\lambda_1, \ldots, \lambda_n\}$ its principal curvatures. As in item vi) of Remarks 1.4 we obtain the Jacobi-shape form of M:

$$\omega_M = \sum_{i=1}^{n} \frac{e_i(H)}{\lambda_i} e^i \tag{2.27}$$

for $H = \frac{1}{n}\sum_{j=1}^{n} \lambda_j$.

(IX) A generalization of the previous application concerns with smooth maps. Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds and let $Z = \varphi^* h$ be the first fundamental form of φ. With the formula (1.28) we get that $f \in C_+^\infty(M)$ is a last multiplier for $\varphi^* h$ with respect to g if and only if

$$(\Delta_{g, f\varphi^* h} 1_M)_\sharp^g + \frac{1}{2} d(f \|d\varphi\|^2_{g \boxtimes_\varphi \varphi^* h}) = 0. \tag{2.28}$$

(X) Suppose that (M, g) supports a Riemannian (static) continuum body characterized by i) *the mass density* $\rho \in C^\infty(M)$; ii) *the mass force* $F \in \Omega^1(M)$. It is well known that the behavior of this continuous deformable medium is described by *the stress tensor* $\sigma \in \mathcal{T}^0_{2,s}(M)$ of Cauchy, see [23]. Hence, the equation of motion is described by the first Cauchy law of equilibrium

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$$\mathrm{div}\sigma + \rho F = 0 \tag{2.29}$$

and we suppose that the stress tensor is non-degenerate. It follows the Jacobi form of this body:

$$\omega_\sigma = -\sigma^{-1}(\rho F). \tag{2.30}$$

3 Examples of Jacobi-Ricci Forms

In this section, we discuss some explicit examples with computable Jacobi-Ricci form.

3.1 Rotationally Symmetric Metrics

Following [27, p. 118] we consider a general rotationally symmetric metric

$$g = dr^2 + \rho^2(r)ds^2_{n-1} \tag{3.1}$$

with ds^2_{n-1} the canonical metric of S^{n-1}. Its scalar curvature is [27, p.121]

$$R = -2(n-1)\frac{\ddot{\rho}}{\rho} + (n-1)(n-2)\frac{1-\dot{\rho}}{\rho^2}, \tag{3.2}$$

and then we are interested in the behavior of Q on $\frac{\partial}{\partial r}$ and from the same citation:

$$Q\left(\frac{\partial}{\partial r}\right) = -(n-1)\frac{\ddot{\rho}}{\rho}\frac{\partial}{\partial r}. \tag{3.3}$$

Hence for $n = 2$ we suppose that $\frac{\ddot{\rho}}{\rho} \neq 0$ and its Jacobi-Ricci form is

$$\omega_{\mathrm{Ric}} = d\ln\left(\frac{\ddot{\rho}}{\rho}\right)^2 \tag{3.4}$$

which yields the following:

Proposition 3.1 *A 2D rotationally symmetric metric (3.1) with $\ddot{\rho} \neq 0$ admits last multipliers having the expression*

$$f = f_C(r) = C\left(\frac{\rho}{\ddot{\rho}}\right)^2 \tag{3.5}$$

with $C > 0$.

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For $n \geq 3$ we obtain the Jacobi-Ricci form

$$\omega_{\text{Ric}} = d \ln \left(\frac{\ddot{\rho}}{\rho} \right)^2 + (2-n) \frac{2\dot{\rho}(\dot{\rho}-1) - \rho\ddot{\rho}}{\rho^2 \ddot{\rho}} dr \quad (3.6)$$

and then (M, g) is a Ricci-exact (Ricci-closed) modular manifold if and only if the 1-form $\left[\frac{2\dot{\rho}(\dot{\rho}-1)}{\rho^2\ddot{\rho}} - \frac{1}{\rho} \right] dr$ is an exact (closed) form.

3.2 Quasi-Constant Curvature Manifolds

As in application VI let a unit form $\eta \in \Omega^1(M)$ and $\xi = \eta^\sharp \in \mathcal{X}(M)$ its g-dual. The triple (M, g, ξ) with $n = dimM \geq 3$ is called *quasi-constant curvature manifold* if there exists $a, b \in C^\infty(M)$ such that the curvature tensor field is ([5, p. 237])

$$R(X, Y) = aX \wedge_g Y + b \left[\eta(X)Y^\flat - \eta(Y)X^\flat \right] \xi + b \left[\eta(Y)X - \eta(X)Y \right] \eta \quad (3.7)$$

with $X^\flat$ the g-dual form of X; we denote $M^n_{a,b}(\xi)$ this manifold. It follows the Ricci tensor field

$$S = [(n-1)a + b]I + (n-2)b\eta \otimes \xi \quad (3.8)$$

and the scalar curvature

$$R = (n-1)(na + 2b). \quad (3.9)$$

In order to obtain a computable Jacobi-Ricci form, we introduce the following type of $M^n_{a,b}(\xi)$:

Definition 3.2 The quasi-constant curvature manifold is called *special* if

(i) it is regular ([5, p. 238]): $a + b \neq 0$; and
(ii) da and db are parallel with η i.e., there exists non-zero $\alpha, \beta \in C^\infty(M)$ such that

$$\frac{da}{\alpha} = \frac{db}{\beta} = \eta. \quad (3.10)$$

We derive immediately the following:

Proposition 3.3 *The Jacobi-Ricci form of a special $M^n_{a,b}(\xi)$ is the closed 1-form*

$$\omega_{Ric} = \frac{n\alpha + 2\beta}{a+b}\eta = \frac{1}{a+b} d(na + 2b). \quad (3.11)$$

In conclusion, a special $M^n_{a,b}(\xi)$ is a closed modular manifold. In the particular case $a = b > 0$, we have that the special $M^n_{a,a}(\xi)$ is an exact modular manifold with $\omega_{\text{Ric}} = \frac{n+2}{2} d \ln a$ and its last multipliers have the form $f = f_C = \frac{C}{a}$ with $C > 0$.

3.3 Quasi-Einstein Manifolds

Inspired by (3.8) the triple (M^n, g, ξ) as above is called *quasi-Einstein manifold* exists $a, b \in C^\infty(M)$ such that the Ricci tensor field is

$$S = aI + b\eta \otimes \xi. \tag{3.12}$$

The given quasi-Einstein is called *special* if the conditions of definition 3.2 holds; the notion of *regular quasi-Einstein manifold* was introduced in [14, p. 363].

Since the scalar curvature of a quasi-Einstein manifold is $R = na + b$ it follows:

Proposition 3.4 *The Jacobi-Ricci form of a special quasi-Einstein $M^n_{a,b}(\xi)$ is the closed* 1*-form*

$$\omega_{Ric} = \frac{n\alpha + \beta}{a+b}\eta = \frac{1}{a+b}d(na+b). \tag{3.13}$$

In conclusion, a special quasi-Einstein $M^n_{a,b}(\xi)$ is a closed modular manifold. In the particular case $a = b > 0$, we have that the special quasi-Einstein $M^n_{a,a}(\xi)$ is an exact modular manifold with $\omega_{Ric} = \frac{n+1}{2} d \ln a$ and its last multipliers have the form $f = f_C = \frac{C}{a}$ with $C > 0$.

3.4 Ricci solitons

The vector field V of application IV) is a *generator* of a Ricci soliton on (M, g) if there exists a scalar λ such that ([14, p. 362])

$$\mathcal{L}_V g + 2\mathrm{Ric} + 2\lambda g = 0. \tag{3.14}$$

Then $(\mathcal{L}_V g)_\sharp = -2(S + \lambda I)$ and tracing (3.14) we obtain the divergence of V

$$\mathrm{div} V = -R - \lambda n. \tag{3.15}$$

The Lemma 1.10 of [8, p. 6] states that $V^\flat$ belongs to the kernel of $\Delta + S$ and then the Schrödinger–Ricci equation (2.9) admits $V^\flat$ as solution if and only if V is divergence-free or equivalently R is constant:

Proposition 3.5 *Let (M, g, V, λ) be a Ricci soliton with constant scalar curvature and non-degenerate Ricci tensor. Then $\omega_{Ric} = 0$ and the maps Q and $(\mathcal{L}_V g)^\sharp : (TM, g^C) \to (TM, g^C)$ are harmonic. More generally, if the given data are an almost Ricci soliton i.e., λ is a smooth function then the second harmonicity holds if and only if the function $R + n\lambda$ is a constant.*

For the gradient Ricci solitons, $V = \nabla u$, the formula (1.27) of [8, p. 8] (which is the same with (1.31) of page 9) states that

$$S(du) = \frac{1}{2} dR \tag{3.16}$$

and then the following:

Proposition 3.6 *The Jacobi-Ricci form of a gradient Ricci soliton* (M, g, u, λ) *is*

$$\omega_{\mathrm{Ric}} = d(2u) \tag{3.17}$$

and then (M, g) *is a Ricci-exact modular manifold with the last multipliers of* Ric *having the form*

$$f = f_C = C \exp(-u). \tag{3.18}$$

For the example of Gaussian soliton $(M, g) = (\mathbb{R}^n, can)$, we have $u(x) = -\frac{\lambda}{2}\|x\|^2$ which yields $f_C(x) = Ce^{\frac{\lambda}{2}\|x\|^2}$ with $\|\cdot\|$ the Euclidean n-norm and arbitrary scalar λ. Let us remark that the proper setting for the data (M, g, u) of this paper is the *smooth metric measure space* $(M, g, \exp(-u)d\mu_g)$ with $d\mu_g$ the canonical volume form (measure) induced by g; another usual name is that of *weighted manifold* conform [15,16]. It follows that for a gradient Ricci soliton its associated metric measure space has the volume form

$$\mu_g^u := \exp(-u)d\mu_g = f_1 d\mu_g \tag{3.19}$$

with f_1 the (unit) last multiplier of Ric from (3.18) and the Bakry-Emery Ricci tensor (2.14) is self-adjoint with respect to the L^2-inner product of functions using this measure. Also, the diffusion operator of this space, called *weighted Laplacian*

$$\Delta_u \cdot = e^u \mathrm{div}(e^{-u}\nabla\cdot) = \Delta - g(\nabla u, \nabla\cdot), \tag{3.20}$$

can be expressed in terms of last multipliers and weighted divergence (1.10) as

$$\Delta_u F = \frac{1}{f_1}\mathrm{div}(f_1 \nabla F) = \mathrm{div}_{f_1}(\nabla F) \tag{3.21}$$

for any smooth function F on M. For the example of a closed M, the Perelman's energy functional is [8, p. 191]

$$\mathcal{F}(g, u) := \int_M (R + \|\nabla u\|_g^2)e^{-u} d\mu_g \tag{3.22}$$

which

(a) in terms of last multipliers it becomes

$$\mathcal{F}(g, u) = \int_M (R + \|d \ln f_1\|_g^2) f_1 d\mu_g \tag{3.23}$$

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(b) can be generalized to arbitrary Z-solitons (2.17) through

$$\mathcal{F}(g, Z, u) := \int_M (Tr_g Z + \|\nabla u\|_g^2) e^{-u} d\mu_g. \tag{3.24}$$

Returning to the general case of proposition 3.5, let us remark that for a constant scalar curvature the tensor field $(\mathcal{L}_V g)^\sharp$ is zero in the steady situation: $\lambda = 0$. Indeed, with the Lemma 1.11 of [8, p. 6] we have

$$\Delta R + 2\|\mathrm{Ric}\|^2 = V(R) - 2\lambda R \tag{3.25}$$

and hence a constant R yields

$$S(du) = 0, \quad \|\mathrm{Ric}\|^2 = -\lambda R \tag{3.26}$$

which means that du is a zero of S and if $\lambda R \neq 0$ then R and λ have opposite sign; let us point out that this fact holds for a general (i.e., with R not a constant) shrinking ($\lambda < 0$) closed Ricci soliton from Proposition 1.13 of [8, p. 7]. For $\lambda = 0$, it results $Ric = 0$ i.e., (M, g) is Ricci-flat and the definition (3.14) gives that V is a Killing vector field.

In [4], it is proved that compact almost Ricci solitons with constant scalar curvature are gradient. Non-steady gradient Ricci solitons with constant scalar curvature are studied in [19] where a main consequence of the constancy of R is the fact that the potential function u is an isoparametric one meaning that its level sets are parallel hypersurfaces of constant principal curvatures and hence constant mean curvature. Then the last multipliers (3.18) of Ric are also isoparametric functions.

The general case (not necessary compact or gradient) of non-steady Ricci solitons with constant scalar curvature R on complete Riemannian geometries can be described with the classification provided by Theorem 8.2 of [2, p. 463]:

(I) expanding ($\lambda > 0$). We have $-n\lambda \leq R \leq 0$ and
(I1) if $R = -n\lambda$ then V is Killing vector field and (M, g) is Einstein,
(I2) if $R = 0$ then V is a homothetic vector field and (M, g) is Ricci-flat.
II) shrinking ($\lambda < 0$). We have $0 \leq R \leq -n\lambda$ and
(II1) if $R = 0$ then V is a homothetic vector field and (M, g) is flat,
(II2) if $R = -n\lambda$ then V is Killing vector field and (M, g) is a compact Einstein manifold.

It follows that a proper $(\mathcal{L}_V g)^\sharp$, i.e., not a constant multiple of Kronecker tensor, is attained for possible intermediary values $R \in (-n\lambda, 0)$, respectively, $R \in (0, -n\lambda)$. In the gradient case, from Theorem 1 of [19], it results that only the intermediary discrete values are possible: $R \in \{-(n-1)\lambda, \ldots, -\lambda\}$, respectively, $R \in \{-2\lambda, \ldots, -(n-1)\lambda\}$ which excludes the dimension $n = 2$ and fixes the value $R = -2\lambda$ for dimension $n = 3$ in the shrinking case; from (3.15) it results that the Laplacian of u is constant. Also, from Theorem 2 of the cited paper, the complete non-steady gradient Ricci solitons with non-degenerate Ricci tensor having the constant rank n are *rigid* which

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means that (M, g) is isometric to $N \times_\Gamma \mathbb{R}^k$ where N is Einstein, $\mathbb{R}^k$ has the Gaussian soliton structure and Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^k$.

Concerning the level sets $S^u_c : u = c \in \mathbb{R}$ of the potential function u of a gradient Ricci soliton (with non-constant R) let us remark from (3.18) that these coincide with the level sets $S^{f_C}_{Ce^{-c}}$ of the last multiplier f_C. Also, from (3.16) and the non-degeneration of S, it results that the level sets of u coincide with the level sets of R. Let II_c be the second fundamental form of S^u_c supposing that u is strictly convex. With the computations of [10] we have

$$II_c = \frac{-1}{\|\nabla u\|_g} \cdot H_u = \frac{-|f_1|}{\|d \ln f_1\|_g} H_u \tag{3.27}$$

which becomes for our setting:

$$II_c = \frac{2}{\|\omega_{\mathrm{Ric}}\|_g}(Ric + \lambda g). \tag{3.28}$$

3.5 Spheres

Let $S^n(r)$ be the n-dimensional sphere with its canonical metric g of constant curvature $c = \frac{1}{r^2}$. It is well known that its Laplacian spectrum has the first positive eigenvalue $\lambda_1 = n$ with the multiplicity n and eigenvectors $u \in C^\infty(S^n(r))$ called *first-order spherical harmonics*. These functions appear in the Obata characterization of the Euclidean sphere.

So, for a first spherical harmonic u we have

$$H(u) = -\frac{u}{r^2} g, \quad H(u)^{-1}_\sharp = -\frac{r^2}{u} I \tag{3.29}$$

and

$$\Delta u = -\frac{n}{r^2} u, \quad \mathrm{div} H(u) = -\frac{du}{r^2}. \tag{3.30}$$

It follows the Jacobi form associated to $H(u)$

$$\omega_{H(u)} = H(u)^{-1}_\sharp(\mathrm{div} H(u)) = \frac{du}{u} = d(\ln u) \tag{3.31}$$

and then $(S^n(r), g, H(u))$ is an exact-modular manifolds with the last multipliers for $H(u)$ having the form

$$f = f_C = \frac{C}{u}. \tag{3.32}$$

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4 Conformal and Curvature Deformations

Returning to the general setting let $u \in C^{\infty}_{+}(M)$ be fixed. The aim of this section is to compare the Jacobi form of the triples (M, g, Z) and $(M, \tilde{g} := ug, Z)$. Let us remark that the (1, 1)-version of Z with respect to $\tilde{g}$ is $\tilde{Z}_{\sharp} = \frac{1}{u}Z_{\sharp}$ and hence $\tilde{Z}_{\sharp}^{-1} = uZ_{\sharp}^{-1}$.

With the well-known formula for the difference between the Levi-Civita connection of $\tilde{g}$ and g ([27, p. 156]), we derive

$$2(\mathrm{div}_{\tilde{g}} Z - \mathrm{div}_g Z) = nZ_{\sharp}(d \ln u) - (TrZ^{\sharp})d \ln u \tag{4.1}$$

where $TrZ^{\sharp}$ is the trace of $Z^{\sharp}$; in a local chart we have $TrZ^{\sharp} = \sum_{i=1}^{n} Z^i_i$. Then the tilde Jacobi form $\tilde{\omega}_Z := \tilde{Z}_{\sharp}^{-1}(\mathrm{div}_{\tilde{g}} Z)$ is

$$\tilde{\omega}_Z = u\omega_Z + \frac{1}{2}[ndu - (TrZ^{\sharp})Z_{\sharp}^{-1}(du)] \tag{4.2}$$

which yields the following:

Proposition 4.1 *Suppose that $Z \in \mathcal{T}^0_{2,s}(M)$ is non-degenerate and traceless.*

i. *If (M, g, Z) is a closed modular manifold and du is parallel to ω_Z, i.e., $du \wedge \omega_Z = 0$, then $(M, \tilde{g}, Z)$ is also a closed modular manifold.*
ii. *In particular, suppose that (M, g, Z) is an exact modular manifold with the potential u. Then $(M, \tilde{g}, Z)$ is also an exact modular manifold with the potential $\frac{u^2+nu}{2}$.*

Let us remark that the subspace $\mathcal{T}^0_{2,s,t}(M) \subset \mathcal{T}^0_{2,s}(M)$ of traceless tensors appears naturally in our study. Indeed, it is well known that pointwise we have that $\mathcal{T}^0_{2,s}(M)$ splits into $O(T_x M)$-irreducible subspaces as $\mathcal{T}^0_{2,s}(M) = \mathcal{T}^0_{2,s,t}(M) \oplus \mathbb{R}g$; in the words of [27, p. 110]: *the homotheties and traceless matrices are perpendicular*.

In the second part of this section, we study the case of curvature deformation. Recall that g yields *the curvature operator*:

$$R^g : \mathcal{T}^0_{2,s}(M) \to \mathcal{T}^2_{0,s}(M), \quad Z = (Z_{ij}) \to R^g(Z) = (R^g(Z)_{ij} := R_{iabj}Z^{ab}). \tag{4.3}$$

The symmetries of the (0, 4)-curvature tensor field $Riem = (R_{ijkl})$ guarantee that this operator is proper defined; remark also that R^g is a g-self-adjoint operator on $\mathcal{T}^0_{2,s}(M)$.

For a fixed Z, the (1, 1)-variant of its curvature transformation $R^g(Z)$ is

$$R^g(Z)^k_j = g^{ki}R^g(Z)_{ij} = g^{ki}R_{ibcj}Z^{bc} = g^{ki}R_{jbci}Z^{bc} = R^k_{jbc}Z^{bc} \tag{4.4}$$

and then

$$R^g(Z)^k_{j|i} = R^k_{jbc|i}Z^{bc} + R^k_{jbc}Z^{bc}_{|i}. \tag{4.5}$$

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It follows the j-component of divergence of the curvature transformation:

$$[\mathrm{div} R^g(Z)]_j = \sum_{i=1}^{n}[R^i_{jbc|i} Z^{bc} + R^i_{jbc} Z^{bc}_{|i}]. \tag{4.6}$$

With the second Bianchi identity ([2, p. 17]) it follows that

$$\sum_{i=1}^{n} R^i_{jbc|i} = R_{jc|b} - R_{jb|c} \tag{4.7}$$

and due to the symmetry of Z we have

$$[\mathrm{div} R^g(Z)]_j = \sum_{i=1}^{n} R^i_{jbc} Z^{bc}_{|i} \tag{4.8}$$

which means globally

$$[\mathrm{div} R^g(Z)](X) = \mathrm{Trace}\left[(U, V) \to g(R(X, U)V, (\nabla \hat{Z})(U, V))\right] \tag{4.9}$$

with $\hat{Z} = (Z^{ab})$ the contravariant version of Z.

5 Curvature Last Multipliers and the General Case of Tensors

It is well known that the divergence of the Riemannian curvature tensor is ([27, p. 104])

$$(\mathrm{divRiem})(X, Y, Z) = (\nabla_X \mathrm{Ric})(Y, Z) - (\nabla_Y \mathrm{Ric})(X, Z) \tag{5.1}$$

for any vector fields X, Y, and Z. We introduce a new class of last multipliers:

Definition 5.1 The function $f \in C^{\infty}_{+}(M)$ is called *curvature last multiplier* for g if $f Riem$ is divergence-free or, in other words, the tensor field $f Riem$ is *conservative* or (M, g) has *harmonic curvature*.

A direct computation gives

$$\mathrm{div}(f\mathrm{Riem})(X, Y, Z) = f(\mathrm{divRiem})(X, Y, Z) + R(X, Y)Z(f) \tag{5.2}$$

and then f is a curvature last multiplier if and only if the following Liouville equation holds for any X, Y and Z:

$$\mathrm{Riem}(X, Y)Z(\ln f) = (\nabla_Y \mathrm{Ric})(X, Z) - (\nabla_X \mathrm{Ric})(Y, Z). \tag{5.3}$$

In particular, if Ric is a Codazzi tensor i.e., the right-hand side of (5.3) is zero then f is a first integral of the curvature: $\mathrm{Riem}(X, Y)Z(f) = 0$ for all X, Y, and Z; this

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equation appears for a problem concerning the vertical lift of a Killing potential in Proposition 11 of [11, p. 175]. For example, if (M, g) is a space-form M^n_c i.e., it has the constant curvature c then the only functions satisfying the last equations are the constants.

At this moment, we have discussed the last multipliers f of three types of tensor fields:

(1) vector field $X \in \mathcal{T}^1_0(M)$; the Liouville equation of f is ([12, p. 458])

$$X(\ln f) = -divX \tag{5.4}$$

which means

$$d(\ln f) \circ X = -\mathrm{div} X. \tag{5.5}$$

(2) endomorphism $F \in \mathcal{T}^1_1(M)$; the Liouville equation is the contravariant version of (1.5)

$$d(\ln f) \circ F = -\mathrm{div} F \tag{5.6}$$

and for a non-degenerate F one have the *Jacobi form*: $\omega_F := \mathrm{div} F \circ F^{-1}$.

(3) curvature $Riem \in \mathcal{T}^1_3(M)$; again the Liouville equation (5.3) reads as

$$d(\ln f) \circ \mathrm{Riem} = -\mathrm{divRiem}. \tag{5.7}$$

Concerning the gradient vector fields $X = \nabla u$ and complete metrics g with Theorem 2.18 from [2, p. 126], we have that any nonnegative and div_f-superharmonic $u \in C^2(M) \cap L^1(M, f dV_g)$ is constant if (M, g) is $\mathrm{div}_f|_{gradients}$-stochastically complete.

These cases yield the following general definition:

Definition 5.2 Let $T \in \mathcal{T}^1_k(M, g)$ be fixed and $f \in C^\infty_+(M)$. f is called *last multiplier of T with respect to g* if *the Liouville equation* holds

$$d(\ln f) \circ T = -\mathrm{div}_g T, \tag{5.8}$$

which means the vanishing of the *drift* (or *drifting*) *divergence*: $div_f T := \frac{1}{f}\mathrm{div}(fT)$.

The both members of Liouville equation belongs to $\mathcal{T}^0_k(M)$ and following the terms of item ii) of Remarks 1.4 we say that T is f-divergence-free with respect to g or f-conservative with respect to g. If f_1 and f_2 are two last multipliers it follows that the image of T is a subspace in the annihilator of the exact 1-form $d(\ln \frac{f_2}{f_1})$.

For example let $\alpha \in \Omega^k(M)$ and $T = \nabla\alpha \in \mathcal{T}^0_{k+1}(M, g) \simeq_g \mathcal{T}^1_k(M, g)$. The Weitzenböck formula is ([29, p. 303])

$$\Delta\alpha = -\mathrm{div} T + \rho(\alpha) \tag{5.9}$$

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and then the Liouville equation for T is

$$d(\ln f) \circ (\nabla \alpha) = \Delta \alpha - \rho(\alpha). \tag{5.10}$$

In particular, for a harmonic k-form we have $d(\ln f) \circ (\nabla \alpha) = -\rho(\alpha)$.

6 Last Multipliers with Respect to Dirichlet Forms

Returning to equation (5.5) if $X = \nabla u$ then we get the relation (3.2) of [12, p. 462]:

$$g(\nabla u, \nabla(\ln f)) = -\Delta u. \tag{6.1}$$

This relation permits to define last multipliers in the setting of Dirichlet forms.

More precisely, let M be a connected locally compact separable space and let μ be a positive Radon measure on M. Fix $\mathcal{E}$ a regular and strongly local Dirichlet form on M with domain $\mathcal{D} \subset L^2(M, d\mu)$ i.e., $\mathcal{E}$ is a positive, symmetric, closed bilinear form on $L^2(M, \mu)$ such that unit contractions operate on $\mathcal{E}$, [20]. This form $\mathcal{E}$ admits an *energy measure* Γ such that

$$\mathcal{E}(u, v) = \int_M d\Gamma(u, v) \tag{6.2}$$

for $u, v \in \mathcal{D}$. Let also A be the self-adjoint operator uniquely associated with the Dirichlet space $(M, \mathcal{E}, L^2(M, d\mu))$

$$\mathcal{E}(u, v) = (Au, v) := \int_M (-Au) v d\mu \tag{6.3}$$

for $u \in \mathcal{D}(A) = \mathcal{D}$ and $v \in \mathcal{D}$. The well-known example is that of Riemannian manifolds (M, g) where

$$d\Gamma(u, v) = g(\nabla u, \nabla v) d\mu_g, \quad A = \Delta \tag{6.4}$$

with $d\mu_g$ the Riemannian measure. Hence (6.1) can be written as

$$\mathcal{E}_g(u, 1) = \int_M (-\Delta u) d\mu_g = \int_M d\Gamma(u, \ln f) = \mathcal{E}_g(u, \ln f). \tag{6.5}$$

We arrive at the following general definition:

Definition 6.1 Let the Dirichlet space $(M, \mathcal{E}, L^2(M, d\mu))$ and $u \in \mathcal{D}$. The positive $m \in \mathcal{D}$ is called *a last multiplier* for u if

$$\mathcal{E}(u, 1) = \mathcal{E}(u, m). \tag{6.6}$$

For an arbitrary $v \in \mathcal{D}$ the previous relation becomes

$$\mathcal{E}(u, v) = \int_M v d\Gamma(u, m) \tag{6.7}$$

while the linearity of $\mathcal{E}$ gives the following form of (6.6) which we call *Liouville equation* for u

$$\mathcal{E}(u, m - 1) = 0. \tag{6.8}$$

If $\mathcal{E}$ possesses *the local property* then the hypothesis $supp[u] \cap supp[m - 1] = \emptyset$ implies (6.8) where, as usual, $supp[u]$ denotes the support of the measure $u \cdot \mu$.

Example 6.2 As in example 1 of [28, p. 57] on the manifold M let us consider a measure μ with positive smooth density with respect to the Lebesgue measure on each local chart. Fix also the smooth vector fields $\{X_1, ..., X_r\}$ and we define the operator

$$\Gamma(u, v) = \sum_{i=1}^{r} X_i(u) X_i(v) \tag{6.9}$$

and $\mathcal{E}$ through (6.2). Hence $m \in C_+^\infty(M)$ is a last multiplier for a fixed u if and only if

$$\sum_{i=1}^{r} X_i(u) X_i(m) = 0. \tag{6.10}$$

For example, if all X_i admit a common first integral m then m is an “universal last multiplier” i.e., last multiplier for all u. □

A main source of Dirichlet forms is provided by symmetric Markov diffusion semigroups as it is pointed out in [3]. Fix now a symmetric Markov semigroup $\mathbf{P} = (P_t)_{t \geq 0}$ with the *infinitesimal generator* given by

$$Lf := \lim_{t \searrow 0} \frac{1}{t} \left(P_t(f) - f\right). \tag{6.11}$$

The associated Bakry-Emery *carré du champ* is

$$\Gamma(f, g) := \frac{1}{2} \left(L(fg) - gLf - fLg\right) \tag{6.12}$$

and we recall the Definition 1.11.1 of [3, p. 43]:

Definition 6.3 L is a *diffusion operator* if

$$L\psi(f) = \psi'(f)Lf + \psi''(f)\Gamma(f, f) \tag{6.13}$$

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for every $\psi : \mathbb{R} \to \mathbb{R}$ of class at least C^2 and every suitable smooth function f.

The more used example of symmetric Markov diffusion semigroups is provided by a Riemannian manifold (M, g) with L being the Laplacian $\Delta = \Delta_g$ and $\Gamma_g(u, v) = g(\nabla u, \nabla v)$. Remark hence that the Eq. (6.1) reads

$$0 = f\Delta u + g(\nabla u, \nabla f) = fLu + \Gamma(u, f) \tag{6.14}$$

and then we arrive at the following notion of last multiplier:

Definition 6.4 Let $SMDS = (\mathbf{P}, L, \Gamma)$ be a symmetric Markov diffusion semigroup and a fixed u. Then f is a *last multiplier for* u *with respect to* $SMDS$ if

$$fLu + \Gamma(u, f) = 0. \tag{6.15}$$

which we call the *Liouville equation for* u.

Fix now a function $\psi : \mathbb{R} \to \mathbb{R}$ as in Definition 6.3 and search for f as being $\psi(u)$:

Proposition 6.5 *Let* $\Psi = \int \psi$ *be the antiderivative of* ψ. *Then* $f = \psi(u)$ *is a last multiplier of* u *with respect to the given SMDS if and only if* $\Psi(u)$ *is* L*-harmonic:* $L\Psi(u) = 0$.

Proof The diffusion property (6.12) yields for our f as follows:

$$L\Psi(u) = fLu + \psi'(u)\Gamma(u, u), \tag{6.16}$$

while the relation (1.11.5) of [3, p. 44] gives the chain rule

$$\psi'(u)\Gamma(u, u) = \Gamma(u, \psi(u) = f). \tag{6.17}$$

Hence the Liouville expression becomes

$$fLu + \Gamma(u, f) = L\Psi(u), \tag{6.18}$$

and we have the conclusion. □

Remark 6.6 In Proposition 3.1 of [12, p. 463], we obtain that in a Riemannian geometry (M, g) a given function u is last multiplier for its gradient $\nabla_g u$ if and only if u^2 is a harmonic function. It follows that this example is provided by the last proposition with ψ being the identity function.

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