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A COMPLEX APPROACH TO THE GRADIENT-TYPE DEFORMATION OF CONICS

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In the memory to Cristian Ida

Abstract

The gradient-type deformation of conics introduced recently by the author is studied with complex numbers.

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1 Introduction

The paper [1], devoted to Finsler geometry, starts with a deformation of a conic Γ based on the components of gradient vector field for the quadratic form defining Γ. This deformation is inspired by the scaling (linear) transformation of Computer Graphics: $(x, y) \in \mathbb{R}^2 \to (\lambda_x \cdot x, \lambda_y \cdot y) \in \mathbb{R}^2$, following [3, p. 136]. Using the well-known invariants from the Euclidean geometry of conics we give the classification of the new conics which depend on two scalars denoted α and β. Since the new conic, denoted $\tilde{\Gamma}$, is a degenerate one, we could interpret the map $\Gamma \to \tilde{\Gamma}$ as a "curve shortening" transformation.

In this short note we present an approach based on the complex geometry of the plane. More precisely, we complexify both the initial and the deformed curve as well as their invariants. The diagonal case $\alpha = \beta$ is particularly analyzed.

I dedicate this short note to the memory of Cristian Ida (02.19.1980-12.02.2016), a Romanian expert in complex geometry. We worked together on some papers concerning with the complex setting and our last study is [2].

2 A gradient-type deformation of conics

In the setting of two-dimensional Euclidean space $(\mathbb{R}^2, g_{can} = diag(1, 1))$ let us consider the conic Γ implicitly defined by $f \in C^\infty(\mathbb{R}^2)$ as: $\Gamma = \{(x, y) \in$

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$\mathbb{R}^2 \mid f(x,y) = 0\}$ where f is a quadratic function of the form $f(x,y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$ with $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$.

The gradient vector field of f, namely $\nabla f = \left(f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}\right)$, gives important properties of Γ; for example, the centers of Γ are exactly the critical points of ∇f. Inspired by this fact we introduce in [1] an associated conic:

Definition 1. *Fix the nonzero scalars α, β. The (α,β)-deformation of Γ is the conic:*

$$\tilde{\Gamma} = \Gamma_{\alpha,\beta} : \alpha \left[\frac{1}{2} f_x\right]^2 + \beta \left[\frac{1}{2} f_y\right]^2 = 0. \tag{1}$$

Examples 2. ([1]): i) Fix other nonzero scalars a, b. The ellipse $E(a,b)$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ and the hyperbola $H(a,b)$: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ have the same (α,β)-deformation:

$$E_{\alpha,\beta} = H_{\alpha,\beta} : \frac{\alpha x^2}{a^4} + \frac{\beta y^2}{b^4} = 0 \tag{2}$$

which is the origin $O(0,0)$ for $\alpha\beta > 0$ and two secant lines through O if $\alpha\beta < 0$. These lines are orthogonal if and only if: $\frac{\alpha}{\beta} = -\frac{a^4}{b^4}$ and consists in the pair of canonical bisectrices: $B_{\pm} : y = \pm x$. In particular, the $(1,-1)$-deformation of the unit circle S^1 or more generally, the $(a^4, -b^4)$-deformation of $E(a,b)$, respectively $H(a,b)$, deserves this case.
ii) For $p > 0$ let the parabola $P(p) : y^2 - 2px = 0$. Its (α,β)-deformation is:

$$P_{\alpha,\beta} : \alpha p^2 + \beta y^2 = 0 \tag{3}$$

which is the empty set for $\alpha\beta > 0$ and consists in two parallel lines for $\alpha\beta < 0$.
iii) Consider again the ellipse $E(a,b)$ with $a > b > 0$. The family of all *confocal* conics with $E(a,b)$ is given by: Γ_λ : $\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} - 1 = 0$ for $\lambda \in \mathbb{R} \setminus \{a,b\}$. The $(a-\lambda, b-\lambda)$-deformation of Γ_λ:

$$(\Gamma_\lambda)_{(a-\lambda,b-\lambda)} : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} = 0 \tag{4}$$

is exactly *the homogeneous part* of Γ_λ. Also, E_{a^2,b^2} is the homogeneous part of $E(a,b)$. $\square$

In order to classify the (α,β)-deformations we recall the algebraic invariants associated to Γ:

$$\Delta = \begin{vmatrix} r_{11} & r_{12} & r_{10} \\ r_{12} & r_{22} & r_{20} \\ r_{10} & r_{20} & r_{00} \end{vmatrix}, D = \delta + r_{11}r_{00} - r_{10}^2 + r_{22}r_{00} - r_{20}^2, I = r_{11} + r_{22}, \delta = r_{11}r_{22} - r_{12}^2. \tag{5}$$

The main result of the first Section of [1] is:

Theorem 1. *The conic $\Gamma_{\alpha,\beta}$ is a degenerate one. It is given by:*
1) $\delta \neq 0$, $\alpha\beta > 0$: $\Gamma_{\alpha,\beta}$ is a point, 2) $\delta \neq 0$, $\alpha\beta < 0$: $\Gamma_{\alpha,\beta}$ consists in two secant lines,
3) $\delta = 0$, $\alpha\beta > 0$: $\Gamma_{\alpha,\beta}$ is the empty set, 4) $\delta = 0$, $\alpha\beta < 0$: $\Gamma_{\alpha,\beta}$ consists in two parallel lines.

The proof uses the following relations:

$$\begin{cases} \tilde{r}_{11} = \alpha r_{11}^2 + \beta r_{12}^2, \tilde{r}_{12} = r_{12}(\alpha r_{11} + \beta r_{22}), \tilde{r}_{22} = \alpha r_{12}^2 + \beta r_{22}^2, \\ \tilde{r}_{10} = \alpha r_{11} r_{10} + \beta r_{12} r_{20}, \tilde{r}_{20} = \alpha r_{12} r_{10} + \beta r_{22} r_{20}, \tilde{r}_{00} = \alpha r_{10}^2 + \beta r_{20}^2. \end{cases} \tag{6}$$

Remark 1. *i) Returning to Examples it follows that* (2) *and* (3) *are the "canonical forms" of* (α, β)*-deformations.*
ii) If instead of the usual Euclidean metric of plane we consider the (semi-)Riemannian metric $g_{\alpha,\beta} = diag(\alpha, \beta)$ *then the function defining* $\tilde{\Gamma}$*, namely* $\alpha f_x^2 + \beta f_y^2$*, is the square norm of the gradient field* ∇f *with respect to* $g_{\alpha,\beta}$*.*
iii) In fact the conic $\tilde{\Gamma}$ *belongs to the pencil of conics* $(\Gamma_1 = \left[\frac{1}{2} f_x\right]^2, \Gamma_2 = \left[\frac{1}{2} f_y\right]^2)$ *which are both degenerate (consisting in double lines) and this serves as another explanation of the first sentence of the theorem above.* □

The case $\alpha = \beta$ for which $\tilde{f} = \frac{\alpha}{4} \|\nabla f\|_{can}^2$ deserves a special attention since it provides a relationship between some of the above invariants:

Proposition 1. *([1, p. 89]) For the conic* Γ *with unique center (i.e.* $\delta \neq 0$*) we define* $T = \frac{I^2}{\delta}$*. Then:*

$$\tilde{T} = (T - 2)^2. \tag{7}$$

We remark that $I^2 = (r_{11} + r_{22})^2 \geq 2|r_{11} r_{22}| \geq r_{11} r_{22} \geq r_{11} r_{22} - r_{12}^2 = \delta$ and the case of all equalities implies the impossible case $r_{11} = r_{12} = r_{22} = 0$. So, the fixed value 1 is impossible but we point out the birth of a parabola, namely the graph of the real function $x \to (x - 2)^2$.

3 The complex approach

The aim of this section is to study the application $\Gamma \to \tilde{\Gamma}$ by using the complex structure of the plane. More precisely, with the usual notation $z = x + iy \in \mathbb{C}$ we derive the complex expression of Γ:

$$\Gamma : F(z, \bar{z}) := Az^2 + Bz\bar{z} + \bar{A}\bar{z}^2 + Cz + \bar{C}\bar{z} + r_{00} = 0 \tag{8}$$

with:

$$A = \frac{r_{11} - r_{22}}{4} - \frac{r_{12}}{2} i \in \mathbb{C}, \quad 2B = r_{11} + r_{22} = I \in \mathbb{R}, \quad C = r_{10} - r_{20} i \in \mathbb{C}. \tag{9}$$

It follows that the usual rotation performed to eliminate the mixed term xy has the purpose to reduce/rotate A in the real line while the translation which eliminates the term y has a similar purpose with respect to C. The inverse relationship between f and F is:

$$r_{11} = B + 2\Re A, \quad r_{22} = B - 2\Re A, \quad r_{12} = -2\Im A, \quad r_{10} = \Re C, \quad r_{20} = -\Im C \tag{10}$$

with $\Re$ and $\Im$ respectively the real and imaginary part.

The expression of the invariants of Γ in terms of A, B, C is:

$$I = 2B, \quad \delta = B^2 - 4|A|^2, \quad D = \delta + 2r_{00}I - |C|^2 \tag{11}$$

$$\Delta = r_{00}(B^2 - 4|A|^2) - B|C|^2 + 2\Re C(\Re A \Re C + \Im A \Im C) + 2\Im C(\Re C \Im A - \Re A \Im C). \tag{12}$$

By using (6) we derive the transformation of the complex coefficients:

$$\tilde{A} = \frac{\alpha}{4}(B+2\Re A)^2 - \frac{\beta}{4}(B-2\Re A)^2 + (\beta-\alpha)(\Im A)^2 + \Im A[\alpha(B+2\Re A)+\beta(B-2\Re A)]i \tag{13}$$

$$\tilde{B} = \frac{\alpha+\beta}{2}B^2 + 2(\alpha-\beta)B\Re A + 2(\alpha+\beta)|A|^2 = \frac{\alpha}{2}|B+2A|^2 + \frac{\beta}{2}|B-2A|^2 \tag{14}$$

$$\tilde{C} = \alpha(B+2\Re A)\Re C + 2\beta \Im A \Im C + [\alpha \Re C \Im A + \beta \Im C(B - 2\Re A)]i. \tag{15}$$

For the considered particular case $\alpha = \beta$ we obtain:

$$\tilde{A} = 2\alpha BA, \quad \tilde{B} = \alpha(B^2 + 4|A|^2) \tag{16}$$

$$\frac{1}{\alpha}\tilde{C} = B\Re C + 2(\Re A \Re C + \Im A \Im C) + [\Re C \Im A + \Im C(B - 2\Re A)]i \tag{17}$$

and there is an amazing difference between the simple expressions of $\tilde{A}, \tilde{B}$ and the complexity of that of $\tilde{C}$. Also, the expression of T from proposition 1 is:

$$T = \frac{4B^2}{B^2 - 4|A|^2}. \tag{18}$$

The quadratic invariant δ is the determinant of the hermitian matrix:

$$\Gamma^c = \begin{pmatrix} B & 2\bar{A} \\ 2A & B \end{pmatrix} \tag{19}$$

which is a special one, the entries of the main diagonal being equal; hence their set is a three-dimensional subspace in the four-dimensional real linear space of 2×2 hermitian matrices. The considered (α, α)-deformation implies the square map:

$$\Gamma^c = \begin{pmatrix} B & 2\bar{A} \\ 2A & B \end{pmatrix} \to \tilde{\Gamma}^c = \begin{pmatrix} B^2 + 4|A|^2 & 4B\bar{A} \\ 4BA & B^2 + 4|A|^2 \end{pmatrix} = (\Gamma^c)^2. \tag{20}$$

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