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RESEARCH PAPER

Space-Like Slant Curves in Three-Dimensional Normal Almost Paracontact Geometry

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Abstract The aim of this paper is to study (spacelike) slant curves of three-dimensional normal almost paracontact manifolds as natural generalization of Legendre curves. Such a curve is characterized by means of the scalar product between its normal vector field and the characteristic vector field of the ambient space. In the particular case of a helix, we show that it has a proper (non-harmonic) mean curvature vector field. The general expressions of the curvature and torsion of these curves and the associated Lancret invariant are computed as well as the corresponding variants for the quasi-para-Sasakian case. Two examples (one of us and one of Joanna Wełyczko) are discussed for a normal not quasi-para-Sasakian 3-manifold.

Keywords Normal almost paracontact manifold · (spacelike) slant · Legendre curve · Lancret invariant · (generalized) helix

Mathematics Subject Classification 53C15 · 53C25 · 53C40 · 53C42 · 53C50

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1 Introduction

The paracontact geometry appears as a natural counter-part of the contact geometry in Kaneyuki and Williams (1985); see also Blaga and Crasmareanu (2015) and Crasmareanu and Pişcoran (2015). Comparing with the huge literature in almost contact geometry, it seems that there are necessary new studies in almost paracontact geometry; a very interesting paper connecting these fields is Cappelletti Montano (2010). The present work is another step in this direction, more precisely from the point of view of slant curves.

A nice notion of classical differential geometry of curves is that of curve of constant slope, also called cylindrical helix. This is a curve in the Euclidean space $\mathbb{E}^3$ for which the tangent vector field has a constant angle with a fixed direction called the axis. The second name corresponds to the fact that there exists a cylinder on which the curve moves in such a way that it cuts each ruling at a constant angle. The classical characterization of these curves is the Bertrand–Lancret-de Saint Venant Theorem (Barros (1997)): the curve γ in $\mathbb{E}^3$ is of constant slope if and only if the ratio of the torsion τ and the curvature k is constant. More precisely, for a cylindrical helix, we have the constant ratio $\frac{\cos\theta}{|\sin\theta|} = \frac{\tau}{k}$ and then, inspired by the title of Barros (1997), we define the Lancret invariant as $\mathrm{Lancret}(\gamma) = \frac{\cos\theta}{|\sin\theta|}$. By computing k and τ in terms of θ, we get the result above, and therefore, the expression of Lancret invariant in the three-dimensional Euclidean geometry is

$$\mathrm{Lancret}(\gamma) = \frac{\tau}{k}. \tag{1.1}$$

An interesting generalization of this class of curves is that of slant curve in almost contact metric geometry. This

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concept was introduced in Cho et al. (2006) with the constant angle θ between the tangent vector field and the Reeb vector field. The particular case of $\theta = \frac{\pi}{2}$ (or $\theta = \frac{3\pi}{2}$) is very important, since we recover the Legendre curves of Baikoussis and Blair (1994). For the same general expression $\frac{\cos\theta}{|\sin\theta|}$, Theorem 3.1. of Cho et al. (2006, p. 362) gives the form of Lancret invariant in three-dimensional Sasakian geometry:

$$\mathrm{Lancret}(\gamma) = \frac{\tau \pm 1}{k}. \tag{1.2}$$

Although the literature on Legendre curves is rich (Blair et al. 1995; Belkhelfa et al. 2002; Camci et al. 2008; Fetcu 2008; Lee 2010; Smoczyk 2003; Welyczko 2007, 2009), slant curves have been studied until now only for the Sasakian geometry in Cho et al. (2006), for the contact pseudo-Hermitian geometry in Cho and Lee (Cho and Lee) and Özgür and Güvenç (2012), for warped products in Călin and Crasmareanu (2013a), for normal almost contact metric geometry in Călin and Crasmareanu (2013b), for Bianchi–Cartan–Vranceanu metrics in Călin and Crasmareanu (2014), and for the f-Kenmotsu geometry in Călin et al. (2012). Therefore, the purpose of this paper is to begin a study of slant curves in normal almost paracontact manifolds, which are provided by the integrability of the almost paracomplex structure naturally associated with the given almost paracontact structure. Since in dimension 3, the paracontact metric is a Minkowski–Lorentzian one, we restrict our search into the class of space-like curves but also provides a series of formulae for the general case of Frenet curves. A particular case of slant curves, namely, magnetic curves, is studied recently by the first author in a special type of three-dimensional paracontact geometry in Călin and Crasmareanu (2016).

Our work is structured as follows: the first section is a very brief review of (normal) almost paracontact geometry. The next section is focused on the study of slant (particularly Legendre) curves in this setting; although Legendre curves are carefully analyzed in Welyczko (2009), we derived new properties of them. Therefore, we obtain a characterization of slant curves similar to Proposition 4.1. of [Cho et al. (2006), p. 362] and the form of Lancret invariant is derived in our Proposition 3.5. The present expression for Lancret invariant is more complex, involving a square root in addition to curvature and torsion, and the derivative of two functions along the curve: first is provided by the paracontact structure, while the second is defined by the curve and the structural endomorphism. Another main result of this section is that a helix slant curve has a proper (non-harmonic) curvature vector field. The last section is devoted to examples following the lines of Welyczko (2009). Recently, some remarkable classes of three-dimension paracontact structures are studied: homogeneous in Calvaruso (2012), respectively, Walker in Calvaruso (2012).

Let us note that another paper on the same subject is Welyczko (2014), and hence, there exist some overlaps between the cited paper and our work mainly in the general formulae. Some highlights of our paper are as follows: we restricts the study to the space-like curves with possible interpretations in the underlying Minkowski geometry, we develop a Lancret-type invariant, and we study the case of slant curves with proper mean harmonic vector field.

2 Normal Almost Paracontact Geometry

Let M be a $(2n+1)$-dimensional smooth manifold, φ a tensor field of (1, 1)-type called the structural endomorphism, ξ a vector field called sthe characteristic vector field, η a 1-form called the paracontact form, and g a pseudo-Riemannian metric on M of signature $(n+1, n)$. We say that (φ, ξ, η, g) defines an almost paracontact metric structure on M if Zamkovoy (2008, p. 38):

1. $\eta(\xi) = 1$, $\varphi^2 = I - \eta \otimes \xi$, $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$;
2. φ induces on the $2n$-dimensional distribution $\mathcal{D} := \ker \eta$ an almost paracomplex structure P, i.e., $P^2 = 1$ and the eigensubbundles T^+, T^-, corresponding to the eigenvalues 1, -1 of P, respectively, have equal dimension n; hence, $\mathcal{D} = T^+ \oplus T^-$;
3. $g(\varphi\cdot, \varphi\cdot) = -g + \eta \otimes \eta$.

For a list of examples of almost paracontact metric structures, see Dacko and Olszak (2007) and Ivanov et al. (2010, p. 84). From the definition, it follows that η is the g-dual of ξ, i.e., $\eta(X) = g(X, \xi)$ and ξ is an unitary vector field: $g(\xi, \xi) = 1$. Let ∇ be the Levi–Civita connection of g.

The Nijenhuis tensor field with respect to the tensor field φ, denoted by N_φ, is given by

$$N_\varphi(X, Y) = [\varphi(X), \varphi(Y)] + \varphi^2([X, Y]) - \varphi([\varphi(X), Y]) - \varphi([X, \varphi(Y)]), \ \forall X, Y \in \Gamma(TM). \tag{2.1}$$

Definition 2.1 The almost paracontact metric manifold $M(\varphi, \xi, \eta, g)$ is said to be *normal* if the almost paracomplex structure J on the manifold $M \times \mathbb{R}$, given by

$$J\left(X, \lambda\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\varphi(X) + \lambda\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right), \ X \in \Gamma(TM), \quad t \in \mathbb{R}, \tag{2.2}$$

is integrable (i.e., its Nijenhuis tensor field is zero), where λ is a real valued function on $M \times \mathbb{R}$.

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The condition Eq. (2.2) is equivalent to

$$N_\varphi - 2\mathrm{d}\eta \otimes \xi = 0. \tag{2.3}$$

In the following, we restrict to the dimension 3 for which the normality is equivalent to Welyczko (2009, p. 379) or Bejan and Crasmareanu (2014):

$$\begin{cases} \nabla_X \xi = \alpha(X - \eta(X)\xi) + \beta\varphi(X), \\ (\nabla_X \varphi)Y = \alpha(g(\varphi X, Y)\xi - \eta(Y)\varphi X) + \beta(g(X,Y)\xi - \eta(Y)X) \end{cases} \tag{2.4}$$

where $\alpha = \frac{1}{2}\mathrm{div}\xi$ and $\beta = \frac{1}{2}\mathrm{trace}(\varphi\nabla\xi)$. An important consequence of (1.4*a*) is that ξ is *a geodesic vector field*:

$$\nabla_\xi \xi = 0. \tag{2.5}$$

3 Slant Curves in Three-Dimensional Normal Almost Paracontact Geometry

In dimension 3, the metric g becomes a Minkowski–Lorentzian one, having the signature (2, 1). For a Frenet curve $\gamma : I \subset \mathbb{R} \to M_3$, we denote the Frenet frame as usual ($T = \gamma', N, B$) and the Frenet equations are Welyczko (2009, p. 381)

$$\nabla_T T = k\varepsilon_2 N, \ \nabla_T N = -k\varepsilon_1 T + \tau\varepsilon_3 B, \ \nabla_T B = -\tau\varepsilon_2 N, \tag{3.1}$$

where $k \geq 0$ denotes the curvature and $\tau \geq 0$ the torsion. Here, the g-norms of the Frenet vectors are as follows: $\|T\|^2 = \varepsilon_1, \|N\|^2 = \varepsilon_2, \|B\|^2 = \varepsilon_3$ with $\varepsilon_i = \pm 1$ for $1 \leq i \leq 3$. With the discussion of Kühnel (2002, p. 35), we have $\varepsilon_3 = -\varepsilon_1\varepsilon_2$.

Definition 3.1 The structural function of γ is the map $c_\gamma : I \to \mathbb{R}$ given by

$$c_\gamma(s) = g(T(s), \xi) = \eta(T(s)). \tag{3.2}$$

Inspired by Cho et al. (2006, p. 361), the curve γ is called a slant curve, or more precisely c-slant curve, if c_γ is a constant function, $c_\gamma = c \in \mathbb{R}$. In the particular case of $c = 0$, the curve γ is called Legendre curve (Welyczko 2009).

In the following, we suppose that γ is non-geodesic, i.e., $k > 0$, and from Eq. (2.5), we get that γ cannot be an integral curve of ξ which means $c \neq \pm 1$. Then, we define the *Lancret invariant of* γ as

$$\mathrm{Lancret}(\gamma) = \frac{c}{\sqrt{|1 - c^2|}}. \tag{2.2L}$$

A characterization of slant curves and an "a-priori" estimate is given by:

Proposition 3.2 *The Frenet curve γ is a c-slant curve if and only if, along γ the following relation holds*:

$$\eta(N) = -\frac{\alpha(\varepsilon_1 - c^2)}{\varepsilon_2 k}. \tag{3.3}$$

A necessary condition for γ to be c-slant is:

$$(\varepsilon_1 - c^2)[\alpha^2(1 - \varepsilon_1 c^2) - \varepsilon_2 k^2] \geq 0. \tag{3.4}$$

Proof Let us take the covariant derivative in the relation Eq. (3.2) along γ:

$$\begin{aligned} 0 = c'_\gamma(s) &= g(k\varepsilon_2 N, \xi) + g(T, \alpha(T - \eta(T)\xi) + \beta\varphi(T)) \\ &= k\varepsilon_2\eta(N) + \alpha(\varepsilon_1 - c^2) \end{aligned}$$

which yields (3.3), since φ is a g-skew-symmetric operator. For the second part, let us recall after Kühnel (2002, p. 34) that the decomposition of a vector field in the g-orthonormal frame $\{e_1, e_2, e_3 = e_1 \times_M e_2\}$ is

$$X = \varepsilon_1 g(X, e_1)e_1 + \varepsilon_2 g(X, e_2)e_2 - \varepsilon_1\varepsilon_2 g(X, e_3)e_3, \tag{3.5}$$

where $\times_M$ is the Minkowski vector product (see the cited book) and $\varepsilon_i = g(e_i, e_i) = \pm 1$ for $1 \leq i \leq 2$. Then, the expression of ξ in the Frenet frame is

$$\xi = \varepsilon_1 cT + \varepsilon_2\left(-\frac{\alpha(\varepsilon_1 - c^2)}{\varepsilon_2 k}\right)N - \varepsilon_1\varepsilon_2\eta(B)B$$

and since ξ is an unitary vector field, we get that

$$1 = \varepsilon_1 c^2 + \varepsilon_2\frac{\alpha^2(\varepsilon_1 - c^2)^2}{k^2} - \varepsilon_1\varepsilon_2\eta(B)^2$$

and then

$$\eta^2(B) = \frac{\varepsilon_1 - c^2}{k^2}[\alpha^2(1 - \varepsilon_1 c^2) - \varepsilon_2 k^2].$$

From $\eta^2(B) \geq 0$, we get Eq. (3.4). □

Remark 3.3

1. It is important to point out that the characterization Eq. (3.3), as well as the condition Eq. (3.4), does not depend on β.
2. If $\frac{\alpha}{k}$ is a constant, then γ is a slant helix in the sense of Ali and López (2011); see also Choi et al. (2012).

To simplify the computations in what follows we make a choice for the causal character of our vector fields. More precisely, we set

$$\varepsilon_1 = +1, \quad \varepsilon_2 = -1 \tag{3.6}$$

which means that: (1) γ is parametrized by arc-length; (2) ξ, $T = \gamma'$, and B are space like, while N is timelike. Therefore, γ is a space-like curve. Our choice is motivated by

1. Lemma 2.19 of Kühnel (2002, p. 34) which states that a second variant $\varepsilon_1 = -1$ is a more difficult case;
2. possible physical interpretations for these curves as world-lines for what we can call "slant particles" in the underlying Minkowski geometry.

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The condition (3.4) reads now:

$$(1-c^2)[\alpha^2(1-c^2)+k^2]\geq 0 \tag{3.7}$$

and then we restrict also to the case $c\in(-1,1)$.

From (1.4*a*), we have: $\nabla_{\gamma'}\xi=\alpha(\gamma'-c\xi)+\beta\varphi(\gamma')$, and then, we consider another orthonormal frame field in *TM* along γ:

$$F_1=T=\gamma', F_2=\frac{\varphi(\gamma')}{\sqrt{1-c^2}}, F_3=\frac{\xi-c\gamma'}{\sqrt{1-c^2}}. \tag{3.8}$$

We have: $g(F_1,F_1)=g(F_3,F_3)=-g(F_2,F_2)=+1$ which means that $F_1=\gamma'$ and F_3 are spacelike, while F_2 is timelike. The decomposition of ξ with respect to this frame is

$$\xi=cF_1+\sqrt{1-c^2}F_3. \tag{3.9}$$

The equations of motion for this orthonormal field of frames are consequence of Proposition 3.2 from Welyczko (2014, p. 969):

Proposition 3.4 *If γ is a space-like c-slant curve in the normal almost paracontact metric manifold M_3 with $c\in(-1,1)$ then*:

$$\begin{cases}\nabla_{\gamma'}F_1=-\delta\sqrt{1-c^2}F_2-\alpha\sqrt{1-c^2}F_3\\ \nabla_{\gamma'}F_2=-\delta\sqrt{1-c^2}F_1+(c\delta+\beta)F_3\\ \nabla_{\gamma'}F_3=\alpha\sqrt{1-c^2}F_1+(c\delta+\beta)F_2\end{cases} \tag{3.10}$$

where

$$\delta=\frac{1}{1-c^2}g(\nabla_{\gamma'}\gamma',\varphi(\gamma')). \tag{3.11}$$

Also

$$\nabla_{\gamma'}\xi=\alpha(1-c^2)F_1+\beta\sqrt{1-c^2}F_2-\alpha c\sqrt{1-c^2}F_3. \tag{3.12}$$

We are ready for the first main result of this paper:

Proposition 3.5 *The curvature and torsion of a space-like c-slant curve in M_3 with $c\in(-1,1)$ are*

$$\begin{cases}k=\sqrt{(1-c^2)(\delta^2-\alpha^2)}\\ \tau=\left|\beta+c\delta+\dfrac{\alpha'\delta-\alpha\delta'}{\delta^2-\alpha^2}\right|.\end{cases} \tag{3.13}$$

The associated Lancret invariant is:

$$\text{Lancret}_{\pm}(\gamma)=\frac{(\alpha'\delta-\alpha\delta')(\delta^2-\alpha^2)^{-\frac{1}{2}}+(\pm\tau-\beta)(\delta^2-\alpha^2)^{\frac{1}{2}}}{\delta k}. \tag{3.14}$$

Proof The relation (2.13*a*) follows from (2.10*a*) taking into account the definition of the curvature. More precisely, from

$$k\varepsilon_2N=-kN=\nabla_{\gamma'}\gamma'=-\sqrt{1-c^2}(\delta F_2+\alpha F_3) \tag{3.15}$$

we derive that $\delta\neq 0$, since N and F_3 have different causal characters. From $g(kN,kN)=-k^2$, we get (2.13*a*), and then

$$k'=\frac{(1-c^2)(\delta\delta'-\alpha\alpha')}{k}. \tag{3.16}$$

Differentiating the above relation (3.15) along γ, we deduce that

$$\tau B=\frac{1}{\sqrt{\delta^2-\alpha^2}}\left(\beta+c\delta+\frac{\alpha'\delta-\alpha\delta'}{\delta^2-\alpha^2}\right)(\alpha F_2+\delta F_3). \tag{3.17}$$

which yields (2.13*b*).

Let us point out that since $k>0$ the fraction from the expression (2.13*b*) is well-defined. From formulae (3.13), we have

$$\sqrt{1-c^2}=\frac{k}{\sqrt{\delta^2-\alpha^2}},\quad c=\frac{1}{\delta}\left[\pm\tau-\beta+\frac{\alpha'\delta-\alpha\delta'}{\delta^2-\alpha^2}\right] \tag{3.18}$$

which yields (3.14) with the sign corresponding to the sign of $\mu=\beta+c\delta+\frac{\alpha'\delta-\alpha\delta'}{\delta^2-\alpha^2}$. □

With respect to the Frenet frame, we have the decompositions:

$$F_2=-\frac{\delta}{\sqrt{\delta^2-\alpha^2}}N+\frac{\alpha\mu}{|\mu|\sqrt{\delta^2-\alpha^2}}B,\quad F_3=-\frac{\alpha}{\sqrt{\delta^2-\alpha^2}}N+\frac{\delta\mu}{|\mu|\sqrt{\delta^2-\alpha^2}}B \tag{3.19}$$

$$\xi=cT-\frac{\alpha\sqrt{1-c^2}}{\sqrt{\delta^2-\alpha^2}}N+\frac{\delta\mu\sqrt{1-c^2}}{|\mu|\sqrt{\delta^2-\alpha^2}}B \tag{3.20}$$

$$\nabla_{\gamma'}\xi=\alpha(1-c^2)T+\frac{\sqrt{1-c^2}}{\sqrt{\delta^2-\alpha^2}}(\beta\delta+c\alpha^2)N-\frac{\alpha\mu\sqrt{1-c^2}}{|\mu|\sqrt{\delta^2-\alpha^2}}(\beta+c\delta)B. \tag{3.21}$$

An important consequence is that the norm of $\nabla_{\gamma'}\xi$ depends only of the restrictions of α and β to γ:

$$\|\nabla_{\gamma'}\xi\|=\sqrt{(1-c^2)|\alpha^2-\beta^2|}. \tag{3.22}$$

Remark 3.6

(i) Some particular cases are as follows:

1. (quasi-para-Sasakian) For $\alpha=0$ and $\beta\neq 0$, we get the quasi-para-Sasakian case Welyczko (2009, p. 380) and we obtain

$$k=|\delta|\sqrt{1-c^2},\quad \tau=|\beta+c\delta|,\quad \text{Lancret}_{\pm}(\gamma)=\frac{\pm\tau-\beta}{k}\cdot\frac{|\delta|}{\delta}. \tag{3.23}$$

The slant curve is a helix, i.e., k and τ are constants, if and only if β and δ are constants. In particular, for $\beta = -1$, we have the para-Sasakian case:

$$k = |\delta|\sqrt{1-c^2}, \quad \tau = |c\delta - 1|, \qquad \mathrm{Lancret}_{\pm}(\gamma) = \frac{\pm\tau + 1}{k} \cdot \frac{|\delta|}{\delta}. \tag{3.24}$$

The second equation of (3.24) is the para-contact version of formula (0.2) from the Introduction. If δ is a constant, then γ is Bertrand curve, i.e., we find two real numbers x, y, such that $xk + y\tau = 1$. The characterization (3.3) says that N is orthogonal on ξ. In addition, $\|\nabla_{\gamma'}\xi\| = |\beta|\sqrt{1-c^2}$.

2. For $c = 0$, we get the case of Legendre curve studied by Welyczko (2009) and our formulae (3.13) reduce to (4.1) and (4.2) of her paper. A new result for this case is that $\|\nabla_{\gamma'}\xi\| = \sqrt{|\alpha^2 - \beta^2|}$, and then, for $\alpha = \pm\beta$, it results that ξ is parallel along its Legendre curves.

(ii) For a general $\varepsilon_{1,2}$, we have

$$\delta = \frac{1}{|\varepsilon_1 - c^2|} g\left(\nabla_{\gamma'}\gamma', \varphi(\gamma')\right) \tag{3.25}$$

$$k = \sqrt{|\varepsilon_1(\alpha^2 - \varepsilon_1\delta^2)(1 - \varepsilon_1 c^2)|}, \qquad \tau = \left|\varepsilon_1\beta + \delta c + \frac{\alpha\delta' - \alpha'\delta}{\alpha^2 - \varepsilon_1\delta^2}\right|. \tag{3.26}$$

(iii) Returning with k of (2.13a) in condition (3.7), we obtain in the left-hand side the expression $(1 - c^2)^2\delta^2$ which is indeed strictly positive.

(iv) A helix with $\tau = 0$ is a circle. From (3.24), it results that in the para-Sasakian case with $\delta = \frac{1}{c}$ (for $c \neq 0$, i.e., non-Legendre curve), we get a circle with $k = \frac{\sqrt{1-c^2}}{|c|}$.

Denote by h the second fundamental form of γ and by H its mean curvature field. We know that

$$H = \mathrm{trace}(h) = h(T,T) = \nabla_T T. \tag{3.27}$$

Then, γ is called a curve with proper mean curvature vector field if there exists $\lambda \in C^\infty(\gamma)$, so that

$$\Delta H = \lambda H. \tag{3.28}$$

In particular, if $\lambda = 0$, then γ is known as a curve with harmonic mean curvature vector field. Here, the Laplace operator Δ acts on the vector valued function H and it is given by

$$\Delta H = -\nabla_T\nabla_T\nabla_T T. \tag{3.29}$$

Making use of the Frenet equations, we can rewrite (3.28) as

$$-3k'kT + (k'' + k^3 + k\tau^2)N + (2k'\tau + k\tau')B = \lambda(-kN). \tag{3.30}$$

It follows that both k and τ are constants, and the function λ becomes a constant too, namely

$$\lambda = -k^2 - \tau^2. \tag{3.31}$$

For our framework, we state the following:

Proposition 3.7 *A non-geodesic space-like c-slant curve γ in a quasi-para-Sasakian M_3 has a proper mean curvature vector field if and only if γ is a helix and then*

$$\lambda = -2c\beta\delta - \beta^2 - \delta^2 < 0. \tag{3.32}$$

In particular, a helix Legendre curve has

$$\lambda_L = -\delta^2 - \beta^2 < 0. \tag{3.33}$$

For the para-Sasakian case, we have

$$\lambda = 2c\delta - 1 - \delta^2, \quad \lambda_L = -1 - \delta^2. \tag{3.34}$$

Proof Recall that γ is a *helix* if both k and τ are constants. We compute λ of (3.31) using (3.23) and obtain the required formulae. Let us remark that the constancy of k gives the constancy of δ in the quasi-para-Sasakian setting and that γ cannot have harmonic mean curvature vector field. □

4 Examples

Let N be an open connected subset of $\mathbb{R}^2$, (a, b) an open interval in $\mathbb{R}$ and let us consider the manifold $M = N \times (a, b)$. Let (x, y) be the coordinates on N induced from the cartesian coordinates on $\mathbb{R}^2$ and let z be the coordinate on (a, b) induced from the cartesian coordinate on $\mathbb{R}$. Thus, (x, y, z) are the coordinates on M. Now, we choose the functions:

$$\omega_1, \omega_2 : N \to \mathbb{R}, \quad \sigma : M \to \mathbb{R}_+^*, \quad f : M \to \mathbb{R} \tag{4.1}$$

and following the idea from Welyczko (2007), we define a normal almost paracontact metric structure (φ, ξ, η, g) on M as follows:

$$\varphi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y} - \omega_2\frac{\partial}{\partial z}, \quad \varphi\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x} - \omega_1\frac{\partial}{\partial z}, \qquad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \quad \eta = dz + \omega_1 dx + \omega_2 dy, \tag{4.2}$$

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$$g = [g_{ij}] = \begin{bmatrix} \omega_1^2 - \sigma e^{2f} & \omega_1\omega_2 & \omega_1 \\ \omega_1\omega_2 & \omega_2^2 + \sigma e^{2f} & \omega_2 \\ \omega_1 & \omega_2 & 1 \end{bmatrix}. \tag{4.3}$$

It follows that

$$\alpha = \frac{1}{2\sigma}\frac{\partial \sigma}{\partial z} + \frac{\partial f}{\partial z}, \quad \beta = \frac{e^{-2f}}{2\sigma}\left(-\frac{\partial \omega_1}{\partial y} + \frac{\partial \omega_2}{\partial x}\right). \tag{4.4}$$

Let us point out that in Welyczko (2007), the function f is considered only as $f = f(x, y)$, and then, in the expression of α, the term in $\frac{\partial f}{\partial z}$ disappears. Thus, (M_3, g) is quasi-para-Sasakian if and only if $\sigma = \exp(\frac{1}{f^2})$.

If we denote $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$, then γ is a space-like c-slant curve if and only if

$$\begin{cases} \omega_1\gamma_1' + \omega_2\gamma_2' + \gamma_3' = c \\ (\omega_1^2 - \sigma e^{2f})(\gamma_1')^2 + (\omega_2^2 + \sigma e^{2f})(\gamma_2')^2 + (\gamma_3')^2 + 2\omega_1\omega_2\gamma_1'\gamma_2' + 2\omega_1\gamma_1'\gamma_3' + 2\omega_2\gamma_2'\gamma_3' = 1. \end{cases} \tag{4.5}$$

However, (3.5b) becomes

$$(\omega_1\gamma_1' + \omega_2\gamma_2' + \gamma_3')^2 + \sigma e^{2f}\left[-(\gamma_1')^2 + (\gamma_2')^2\right] = 1 \tag{4.6}$$

and then, γ is a space-like c-slant curve if and only if

$$\omega_1\gamma_1' + \omega_2\gamma_2' + \gamma_3' = c, \quad -(\gamma_1')^2 + (\gamma_2')^2 = \frac{1 - c^2}{\sigma} e^{-2f}. \tag{4.7}$$

Now, we can derive the general expression of space-like slant curves for this general class of manifolds:

Proposition 4.1 *Let γ be a non-geodesic space-like c-slant curve in $(M, \varphi, \xi, \eta, g)$. Then, γ is given by*

$$\gamma(s) = \sqrt{1 - c^2}\Bigg(\int_{s_0}^{s} \frac{e^{-f}}{\sqrt{\sigma}}\zeta(t)\mathrm{d}t, \frac{cs}{\sqrt{1 - c^2}} - \int_{s_0}^{s} \frac{e^{-f}}{\sqrt{\sigma}}(\omega_1 \sinh u(t) + \omega_2 \cosh u(t))\mathrm{d}t\Bigg) \tag{4.8}$$

where $\zeta(s) = (\sinh u(s), \cosh u(s))$ is an arbitrary parametrization of the hyperbolic unit circle $y^2 - x^2 = 1$. In the quasi-para-Sasakian case we have

$$\gamma(s) = \sqrt{1 - c^2}\Bigg(\int_{s_0}^{s} \zeta(t)\mathrm{d}t, \frac{cs}{\sqrt{1 - c^2}} - \int_{s_0}^{s} (\omega_1 \sinh u(t) + \omega_2 \cosh u(t))\mathrm{d}t\Bigg) \tag{4.8qpS}$$

Proof From (3.7b), it follows the existence of a function $u = u(s)$, such that

$$\gamma_1' = \frac{\sqrt{1 - c^2}}{\sqrt{\sigma}} e^{-f} \sinh u(s), \quad \gamma_2' = \frac{\sqrt{1 - c^2}}{\sqrt{\sigma}} e^{-f} \cosh u(s) \tag{4.9}$$

which, replaced in (3.7a) yields

$$\gamma_3' = c - \frac{\sqrt{1 - c^2}}{\sqrt{\sigma}} e^{-f}(\omega_1 \sinh u(s) + \omega_2 \cosh u(s)). \tag{4.10}$$

Therefore, the conclusion follows directly. □

The first Example from Welyczko (2009, p. 384) is recovered with $N = \mathbb{R}^2$, $(a, b) = (0, +\infty)$ and

$$\omega_1 = f = 0, \quad \omega_2 = 2x, \quad \sigma = 2z \tag{4.11}$$

which yields

$$\alpha = \beta = \frac{1}{2z} \tag{4.12}$$

and then, M is not quasi-para-Sasakian. From (3.22), it results that for a Legendre curve γ in this manifold, we have that ξ is parallel along γ.

Now, we make the choice of Example 1 of Welyczko (2009, p. 384): $\gamma_1 = 0$. It follows that on γ, we have $\omega_2 = 0$, and thus, $\gamma_3 = cs$. From (3.7b), we get a non-Legendre curve on $I = (0, +\infty)$

$$\gamma_2' = \sqrt{\frac{1 - c^2}{2c}} \cdot \frac{1}{\sqrt{s}} \tag{4.13}$$

which implies $c \in (0, 1)$ and the solution

$$\gamma_2 = \sqrt{\frac{2(1 - c^2)s}{c}}. \tag{4.14}$$

Hence, the space-like c-slant curve is

$$\gamma(s) = \left(0, \sqrt{\frac{2(1 - c^2)s}{c}}, cs\right). \tag{4.15}$$

The corresponding ε are as follows: $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = +1$ which means that T and N are spacelike, while B is timelike. A long but straightforward computation using the formulae for ∇ of Welyczko (2009) gives

$$\delta(s) = \frac{1}{2s} \tag{4.16}$$

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and then

$$k = \frac{1 - c^2}{2cs}, \quad \tau = \frac{1 + c^2}{2cs}. \tag{4.17}$$

Therefore, this curve is a generalized helix, since

$$\frac{\tau}{k} = \text{constant} = \frac{1 + c^2}{1 - c^2}. \tag{4.18}$$

Because both k and τ depend on s, this curve is not a helix and has not a proper mean curvature vector field.

In the same manifold $(M_3, \varphi, \xi, \eta, g)$, the curve $\gamma_b : (0, +\infty) \to M_3$ from Welyczko (2009, p. 385):

$$\gamma_b = \left(\rho\sqrt{t}, -2\rho\sqrt{t}, 2\rho^2 t\right) \tag{4.19}$$

with $\rho = 3^{-\frac{1}{4}}$, it is proved to be a (spacelike) Legendre one with non-constant k and τ; also $k \neq \tau$. This curve is again a generalized helix which is a helix since $\alpha(= \tau) = \frac{\sqrt{3}}{4t} \neq k = \frac{\sqrt{2}}{4t}$. Moreover, since $\alpha = \beta$, it results according to our Remarks 2.6.(i)2 that ξ is parallel along γ_b.

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