Conservation laws generated by pseudosymmetries with applications to Hamiltonian systems

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Abstract

In this paper we extend a result of Gerald L. Jones which give conservation laws for ordinary differential equations. Applications to Hamiltonian systems are given.

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Introduction

The very well-known way to obtain conservation laws for a system of differential equations is Noether theorem which associates to every symmetry a conservation law. G. L. Jones gives in [3] another method based on a weaker generalization of notion of symmetry, namely *pseudosymmetries.* The advantages of this method are that it does not require any integration(if there are associated some natural invariants, see the Hamiltonian case below) and does not require, as Noether theorem, that the differential equations follow from a variational principle.

In this paper we present a generalization of Jones result and applications to a special type of divergence-free(i.e. solenoidal) vector fields and to Hamiltonian systems.

1 From pseudosymmetries to conservation laws

Let M be a smooth, n-dimensional manifold, $C^{\infty}(M)$ the ring of real-valued smooth functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Omega^{p}(M)$ the $C^{\infty}(M)$-module of p-differential forms, $1 \leq p \leq n$.

For $X \in \mathcal{X}(M)$ with local expression $X = X^{i}(x)\frac{\partial}{\partial x^{i}}$ one consider the system of differential equations which give the flow of X:

$$(1.1) \quad \dot{x}^{i}(t) = \frac{dx^{i}}{dt}(t) = X^{i}\left(x^{1}(t), \ldots, x^{n}(t)\right), \qquad i = 1, \ldots, n.$$

A solution of (1.1) is called *integral curve* of X.

Definition 1.1 A function $f \in C^{\infty}(M)$ is called *conservation law*(or *first integral,* or *constant of motion,* or *invariant function*) for X or (1.1) if f is constant along the solutions of (1.1) that is $\frac{d(f \circ c)}{dt}(t) = 0$ for every integral curve $c(t)$ of X.

Because for $f \in C^{\infty}(M)$ its rate of change along (1.1) is: $\frac{df}{dt} = \frac{\partial f}{\partial x^{i}}\dot{x}^{i} = \frac{\partial f}{\partial x^{i}}X^{i} = \mathcal{L}_{X}f$ where the right-hand side means *the Lie derivative of f with respect to X* we get:

Proposition 1.2 *$f \in C^{\infty}(M)$ is conservation law for (1.1) if and only if:*

$$(1.2) \quad \mathcal{L}_{X}f = 0.$$

For our approach is necessary the following:

Definition 1.3 (i) $Y \in \mathcal{X}(M)$ is called *symmetry for X* if:

$$(1.3) \quad \mathcal{L}_{X}Y = 0.$$

(ii) If $Y \in \mathcal{X}(M)$ is fixed then $Z \in \mathcal{X}(M)$ is called *Y-pseudosymmetry for X* if there exists $f \in C^{\infty}(M)$ such that:

$$(1.4) \quad \mathcal{L}_{X}Z = fY.$$

(iii) $\omega \in \Omega^{p}(M)$ is called *invariant form for X* if:

$$(1.5) \quad \mathcal{L}_{X}\omega = 0.$$

Remark 1.4 (i) A 0-pseudosymmetry is obviously a symmetry.
(ii) A X-pseudosymmetry for X is called *pseudosymmetry for X* in ([3, p.

1055]).
(iii) If in (1.4) f is constant then $\mathcal{L}_X Z$ is symmetry for X.
(iv) If in (1.4) f is not constant then $\mathcal{L}_X Z$ is symmetry for X if and only if f is conservation law for X.

The result which give the association between pseudosymmetries and conservation laws is:

Theorem 1.5 *Let $X \in \mathcal{X}(M)$ be a fixed vector field and $\omega \in \Omega^p(M)$ be a p-form invariant for X. If $Y \in \mathcal{X}(M)$ is symmetry for X and $S_1, \ldots, S_{p-1} \in \mathcal{X}(M)$ are $(p-1)$ Y-pseudosymmetries for X then:*

$$(1.6) \qquad \phi = \omega(S_1, \ldots, S_{p-1}, Y)$$

or locally:

$$(1.7) \qquad \phi = S_1^{i_1} \ldots S_{p-1}^{i_{p-1}} Y^{i_p} \omega_{i_1 i_2 \ldots i_p}$$

is a conservation law for X. Particularly, if $Y, S_1, \ldots, S_{p-1}$ are symmetries for X then ϕ given by (1.6) is conservation law.

Proof Applying the properties of Lie derivatives one have:

$$\mathcal{L}_X \phi = (\mathcal{L}_X S_1)^{i_1} \ldots + S_1^{i_1} (\mathcal{L}_X S_2)^{i_2} \ldots + \ldots + \ldots (\mathcal{L}_X Y)^{i_p} \omega_{\ldots} + \ldots Y^{i_p} (\mathcal{L}_X \omega)_{i_1 \ldots i_p}.$$

In this relation each of the first $p-1$ terms has a factor of the form:

$$\mathcal{L}_X S_j = \lambda_j Y$$

so that ω is contracted with two factors of Y and then each term vanishes by the antisymmetry of ω. The p-th term and $p+1$-th term vanishes since (1.3) and (1.5). $\square$

Remark 1.6 (i) If the pseudosymmetries are linearly dependent then $\phi = 0$ by the antisymmetry of ω.
(ii) For $Y = X$ one obtain the main result of G. L. Jones([3, p. 1056]).
(iii) If $p = 1$ one obtain theorem 2.5.10 of ten Eikelder([2, p. 48]).
(iv) The fact that the pseudosymmetries (1.4) with f =constant can be used to integrate planar($n = 2$) vector fields can be found in [5, p. 37-38, relation 8.13].

2 Vector fields with a special invariant 2-form

Let $M = \mathbf{R}^{2m} = \{(x,y) = (x^i, y^i)_{i=1,\ldots,m}\}$ that is $n = 2m$, and let X with the form:

$$(2.1) \qquad X = X^i(x,y)\frac{\partial}{\partial x^i} + \tilde{X}^i(x,y)\frac{\partial}{\partial y^i}.$$

Let us consider the 2-form $\omega = (\omega_{ij})$ given by:

$$(2.2) \qquad \omega = \begin{pmatrix} 0_m & 1_m \\ -1_m & 0_m \end{pmatrix}$$

where 0_m is the null matrix and 1_m is the identity matrix of order m.

A straightforward computation give:

$$(2.3a) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial \tilde{X}^i}{\partial x^j} - \frac{\partial \tilde{X}^j}{\partial x^i}$$

$$(2.3b) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{\partial X^j}{\partial y^i} - \frac{\partial X^i}{\partial y^j}$$

$$(2.3c) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = \frac{\partial X^j}{\partial x^i} + \frac{\partial \tilde{X}^i}{\partial y^j}$$

$$(2.3d) \qquad (\mathcal{L}_X\omega)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^j}\right) = -\frac{\partial X^i}{\partial x^j} - \frac{\partial \tilde{X}^j}{\partial y^i}.$$

As consequence we obtain:

Proposition 2.1 *If X with expression* (2.1) *satisfy:*

$$(2.4a) \qquad \frac{\partial X^i}{\partial y^j} = \frac{\partial X^j}{\partial y^i}$$

$$(2.4b) \qquad \frac{\partial \tilde{X}^i}{\partial x^j} = \frac{\partial \tilde{X}^j}{\partial x^i}$$

$$(2.4c) \qquad \frac{\partial X^j}{\partial x^i} + \frac{\partial \tilde{X}^i}{\partial y^j} = 0$$

for all $i, j = 1, \ldots, m$ then ω is 2-form invariant for X.

Applying the theorem 1.5 we have:

Proposition 2.2 *If X satisfy* $(2.4a)-(2.4c)$, $Y \in \mathcal{X}(M)$ *is symmetry for X and $S \in \mathcal{X}(M)$ is Y-pseudosymmetry for X then:*

$$(2.5) \quad \phi = \omega(S, Y)$$

is a conservation law for X where ω is given by (2.2). *Particularly if Y, S are symmetries for X then ϕ given by* (2.5) *is conservation law.*

If $Y = Y^i \frac{\partial}{\partial x^i} + \tilde{Y}^i \frac{\partial}{\partial y^i}$ and $S = S^i \frac{\partial}{\partial x^i} + \tilde{S}^i \frac{\partial}{\partial y^i}$ then:

$$\phi = \left(S, \tilde{S}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} Y \\ \tilde{Y} \end{pmatrix} = \left(-\tilde{S}, S\right) \begin{pmatrix} Y \\ \tilde{Y} \end{pmatrix} = S\tilde{Y} - \tilde{S}Y$$

that is:

$$(2.6) \quad \phi = S^i \tilde{Y}^i - \tilde{S}^i Y^i$$

with summation after $i = 1, \ldots, m$.

Corollary 2.3 *If X satisfy* $(2.4a)-(2.4c)$ *and $S \in \mathcal{X}(M)$ is a pseudosymmetry for X then:*

$$(2.7) \quad \phi = \omega(S, X) = S^i \tilde{X}^i - \tilde{S}^i X^i$$

is a conservation law for X.

Remark 2.4 (i) If in relation $(2.4c)$ one makes $i = j$ and take the sum after $i = 1, \ldots, m$ then *the divergence of X* vanishes:

$$(2.8) \quad divX := \sum_{i=1}^{m} \left(\frac{\partial X^i}{\partial x^i} + \frac{\partial \tilde{X}^i}{\partial y^i} \right) = 0$$

that is X is *divergence-free*(or *solenoidal* or *source-free*) vector field.
(ii) Every solenoidal vector field in dimension $n = 2$, that is $m = 1$, satisfy $(2.4a)-(2.4c)$. Applications for divergenceless vector fields in a odd dimension, namely $n = 3$, are given in [3, p. 1056].

3 Applications to Hamiltonian systems in $\mathbf{R}^{2m}$

Let X be given by (2.1).

Definition 3.1 X is called *Hamiltonian vector field* if there exists $H \in C^{\infty}(M)$, usually called *Hamiltonian,* such that the flow system (1.1) of X is exactly the system of Hamilton equations for H, i.e.:

$$(3.1a) \quad \dot{x}^i = X^i = \frac{\partial H}{\partial y^i}$$

$$(3.1b) \quad \dot{y}^i = \tilde{X}^i = -\frac{\partial H}{\partial x^i}.$$

In other words X is *the gradient of* H *with respect to* ω that is:

$$(3.2) \quad X^a = \omega^{ab} \frac{\partial H}{\partial x^a}$$

where $a = 1, \ldots, 2m$, $X^{m+i} = \tilde{X}^i$ and $x^{m+i} = y^i$.

The key result is:

Proposition 3.2 *If* X *is Hamiltonian vector field then* X *satisfy* $(2.4a) - (2.4c)$*. The Hamilton equations* $(3.1a) - (3.1b)$ *can be write:*

$$(3.3) \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \omega \cdot \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix}$$

with ω *given by* (2.2).

Corollary 3.3 *A Hamiltonian vector field is solenoidal.*

Applying remark 2.4(ii) we get:

Proposition 3.4([5, p. 45]) *If* $n = 2$ *then* $X \in \mathcal{X}(M)$ *is Hamiltonian if and only if is solenoidal.*

One can apply the results of previously section and one obtain:

Proposition 3.5 *Let* X *be a Hamiltonian vector field and* $Y \in \mathcal{X}(M)$ *a symmetry for* X*. If* $S \in \mathcal{X}(M)$ *is* Y*-pseudosymmetry for* X *then:*

$$(3.4) \quad \phi = S^i \tilde{Y}^i - \tilde{S}^i Y^i$$

is a conservation law for the Hamiltonian system (3.1) *or* (3.2)*. Particularly, if* Y, S *are symmetries for* X *then* ϕ *is conservation law.*

Corollary 3.6(Jones) *If* X *is Hamiltonian vector field and* $S \in \mathcal{X}(M)$ *is pseudosymmetry for then:*

$$(3.5) \quad \phi = -S^i \tilde{X}^i + \tilde{S}^i X^i = S^i \frac{\partial H}{\partial x^i} + \tilde{S}^i \frac{\partial H}{\partial y^i} = \mathcal{L}_S H$$

is a conservation law for the associate Hamiltonian system.

Remark 3.7

(i) G. L. Jones obtains the last result in [3, p. 108] using the properties of *canonical transformations* for Hamiltonian systems. Our approach does not require these properties.

(ii) If S is a symmetry for the Hamiltonian H, that is:

$$(3.6) \quad \mathcal{L}_S H = 0$$

then ϕ given by (3.5) vanishes and the method is ineffectual. Then to get a non-zero conservation law one must find a pseudosymmetry of X which is not a symmetry for H or to apply proposition 3.5.

4 Applications to general Hamiltonian systems

Let (M, ω) be a symplectic manifold of dimension $n = 2m$.

Definition 4.1 $X \in \mathcal{X}(M)$ is said to be a *Hamiltonian vector field* if there exists $H \in C^\infty(M)$ such that:

$$(4.1) \quad i_X \omega = -dH$$

i.e. X is the ω-dual of H, which is called a *Hamiltonian.*

The following version of proposition 3.2 is classical.

Proposition 4.2 *If X is Hamiltonian vector field on the symplectic manifold (M, ω) then ω is invariant 2-form for X.*

Proof

$$\mathcal{L}_X \omega = d i_X \omega + i_X d\omega = d i_X \omega = d(-dH) = 0. \qquad \Box$$

Because ω is nondegenerate:

$$(4.2) \quad \Omega := \omega^m$$

is a volume form on M. Then for every $X \in \mathcal{X}(M)$ its *divergence* is the function $div_\omega X \in C^\infty(M)$ uniquely determined by:

$$(4.3) \quad \mathcal{L}_X \Omega = (div_\omega X)\, \Omega.$$

Applying proposition 4.2 one obtain:

Proposition 4.3 *The divergence of a Hamiltonian vector field vanishes.*

One can apply the main result of section 1.

Therem 4.4 *Let X be a Hamiltonian vector field and $Y \in \mathcal{X}(M)$ a symmetry of X. If $S \in \mathcal{X}(M)$ is a Y -pseudosymmetry of X then:*

$$(4.4) \qquad \phi = \omega(S, Y)$$

is a conservation law for the Hamiltonian flow defined by X.

5 An example

Let the 2-dimensional istropic harmonic oscillator:

$$\begin{aligned} (5.1a) \qquad \ddot{q}^1 + \omega^2 q^1 &= 0 \\ (5.1b) \qquad \ddot{q}^2 + \omega^2 q^2 &= 0 \end{aligned}$$

a toy model for many methods to finding conservation laws.

The Lagrangian is:

$$(5.2) \qquad L = \frac{1}{2}\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2\right] - \frac{\omega^2}{2}\left[\left(q^1\right)^2 + \left(q^2\right)^2\right]$$

and then applying the conservation of energy H(L is time-independent) we have two conservation laws:

$$\begin{aligned} (5.3a) \qquad \phi_1 &= \left(\dot{q}^1\right)^2 + \omega^2\left(q^1\right)^2 \\ (5.3b) \qquad \phi_2 &= \left(\dot{q}^2\right)^2 + \omega^2\left(q^2\right)^2. \end{aligned}$$

A straightforward computation give the Noetherian conservation law([1, p. 192]):

$$(5.4) \qquad \phi_3 = q^2\dot{q}^1 - q^1\dot{q}^2.$$

But we can obtain a nonnoetherian conservation law with symmetries. The Hamiltonian vector field of (5.1) is:

$$(5.5) \qquad X = \dot{q}^1\frac{\partial}{\partial q^1} + \dot{q}^2\frac{\partial}{\partial q^2} - \omega^2 q^1\frac{\partial}{\partial \dot{q}^1} - \omega^2 q^2\frac{\partial}{\partial \dot{q}^2}$$

which is solenoidal and another calculus give that:

$$(5.6) \quad Y = \dot{q}^2 \frac{\partial}{\partial q^1} + \dot{q}^1 \frac{\partial}{\partial q^2} - \omega^2 q^2 \frac{\partial}{\partial \dot{q}^1} - \omega^2 q^1 \frac{\partial}{\partial \dot{q}^2}$$

is a symmetry for X. Also, because X is total 1-homogeneous, i.e. with respect to all variables $(q, \dot{q})$ it result that:

$$(5.7) \quad Z = q^1 \frac{\partial}{\partial q^1} + q^2 \frac{\partial}{\partial q^2} + \dot{q}^1 \frac{\partial}{\partial \dot{q}^1} + \dot{q}^2 \frac{\partial}{\partial \dot{q}^2}$$

is symmetry for X. We have: $\mathcal{L}_Y H = 0, \quad \mathcal{L}_Z H = 2H$ and then: $\phi = \omega(X, Y) = 0, \quad \phi = \omega(X, Z) = 2H$ i.e. we not have new conservation law. But:

$$(5.8) \quad \phi_4 = \omega(Y, Z) = \dot{q}^1 \dot{q}^2 + \omega^2 q^1 q^2$$

is a new conservation law given by theorem 4.4. We remark that ϕ_4 represent the energy of a new Lagrangian of (5.1), that is:

$$(5.9) \quad L^* = \dot{q}^1 \dot{q}^2 - \omega^2 q^1 q^2$$

a result very important from the point of view of Inverse Problem of Analytical Mechanics([4]). Our Lagrangian L^* appear in [4, p. 122].

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