Libertas Mathematica (new series)
Volume 37(2017), No. 1, 1-15
doi:99.9999/xxxxx

Conjugate covariant derivatives on vector bundles and duality

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Abstract: The notion of *conjugate connections*, discussed in [2] for a given manifold M and its tangent bundle, is extended here to covariant derivatives on an arbitrary vector bundle E endowed with quadratic endomorphisms. A main property of pairs of such covariant derivatives, namely the duality, is pointed out. As generalization, the case of anchored (particularly Lie algebroid) covariant derivatives on E is considered. As applications we study the Finsler bundle of M as well as the Finsler connections on the slit tangent bundle of a Finsler geometry.

Keywords: (Finsler) vector bundle; quadratic endomorphism; (conjugate) covariant derivatives; duality; mean covariant derivative; anchored bundle.

MSC2010: 53C05, 53C07, 53C60

Introduction

In a series of papers, namely [7]-[9], we have studied the geometry of pairs of linear connections on a manifold M which are conjugated with respect to a tensor field of $(1,1)$-type $T \in \mathcal{T}^1_1(M)$ satisfying the reduced quadratic equation: $T^2 = \varepsilon I$. Here I is the usual Kronecker tensor field $I = (\delta^i_j)$ and $\varepsilon \in \{0, -1, +1\}$. Recently, an unified approach for the non-degenerate case $\varepsilon = \pm 1$ was presented in [2]. Related to the almost product case is the study [6] concerned with golden structures and [10] dealing with metallic structures.

Let us remark that all the previous studies involve only the tangent bundle TM of M. The aim of the present work is to extend these objects to an arbitrary vector bundle E endowed with a non-degenerate quadratic endomorphism λ and a given covariant derivative ∇. An important remark is that the main property mentioned above, namely the duality of conjugate linear connections $(\nabla, \nabla^{\lambda})$, continues to hold in this general setting. Two new features of the present paper are: i) the use of local expressions for all the involved objects, which leads to a better picture; for example in Section 2, for the structural and virtual tensor fields of a pair (linear connection, endomorphism), ii) a special attention is given to the mean covariant derivative ∇^0 which parallelizes the given λ. More generally, we use ∇^0 and the first associated Obata operator in order to determine the whole class $C(\lambda)$ of covariant derivatives with respect to which λ is parallel. The second Obata operator is used to express the variation of the curvature tensor field. We finish the first section with the study of anchored, particularly Lie algebroid, covariant derivatives on E.

The above computations are applied in the cases of two particular geometries involving again the tangent bundle. The first concerns the so-called Finsler bundle $TM \times_M TM$ while the second is a proper Finsler geometry provided by a 1-homogeneous function $F : TM \to \mathbb{R}_+$ with non-degenerate square. Let us remark that the class of almost complex and almost product connections in vector bundles endowed with such endomorphisms are discussed also in [19] but our study follows a different path: we unify the treatment of these geometries and in this way we firstly determine the mean covariant derivative from an arbitrary pair (∇, ∇^λ) and secondly we derive the set $C(\lambda)$. A strong motivation for an unified treatment of almost complex and almost product geometries comes from the relationship (1.20) of curvatures of the conjugate covariant derivatives; the same relation holds for both geometries i.e. it does not depends on ε. For the usual case of the tangent bundle we associate to ∇ two tensor fields of $(1,2)$-type called *the structural* and *the virtual tensor field* of the pair (λ, ∇) respectively. In the last section we compute these tensor fields in the case of a Finsler connection.

1 Conjugate covariant derivatives for ε-endomorphisms

Let $\pi : E \to N$ be a vector bundle of (paracompact) base N^n and fibre $\mathbb{R}^k$. As usual, $\mathcal{X}(N) = \Gamma(TN)$ and $\Gamma(\pi)$ denote the $C^\infty(N)$-module of sections for the vector bundle $\tau_N : TN \to N$ and π, respectively; in fact we mainly follow the notations of [21]. A classical notion for this setting is:

Definition 1.1. ([21, p. 277]) A *covariant derivative operator* on π is an $\mathbb{R}$-bilinear map $\nabla : \mathcal{X}(N) \times \Gamma(\pi) \to \Gamma(\pi)$, $(X, s) \to \nabla_X s$, such that:
i) ∇ is tensorial in its first variable: $\nabla_{fX} s = f \nabla_X s$,
ii) ∇ is a derivation in its second variable: $\nabla_X(fs) = X(f) \cdot s + f \nabla_X s$,
for all $f \in C^\infty(N)$, $X \in \mathcal{X}(N)$ and $s \in \Gamma(\pi)$.

Let now $\varepsilon \in \{\pm 1\}$ and $\lambda \in \Gamma(End(\pi)) \cong End(\Gamma(\pi))$. Recall from [21, p. 284] that ∇ induces a covariant derivative operator $\hat{\nabla}$ on the bundle $End(E)$ through the relation:

$$(\hat{\nabla}_X \lambda) s := \nabla_X(\lambda(s)) - \lambda(\nabla_X s), \tag{1.1}$$

for all $X \in \mathcal{X}(N)$ and $s \in \Gamma(\pi)$.

For the given λ the natural problem is to obtain the class of all ∇ such that $\hat{\nabla}\lambda = 0$; let us denote by $\mathcal{C}(\lambda)$ the set of these covariant derivatives. In the following we restrict to a particular remarkable type of such endomorphisms:

Definition 1.2. λ is an ε-*endomorphism* if: $\lambda^2 = \varepsilon 1_{\Gamma(E)}$.

From now on we suppose that the fixed λ is an ε-endomorphism. In order to study the above problem we follow the method of [2] and [7]-[9] by introducing:

Definition 1.3. The λ-*conjugate* of ∇ is $\nabla^\lambda : \mathcal{X}(N) \times \Gamma(\pi) \to \Gamma(\pi)$ given by:

$$\nabla^\lambda_X s := \varepsilon\lambda(\nabla_X(\lambda s)). \tag{1.2}$$

More generally, the λ-*conjugate* of ∇ is $\nabla^\lambda : \mathcal{X}(N) \times \Gamma(\pi) \to \Gamma(\pi)$ given by:

$$\nabla^\lambda := \lambda^{-1} \circ \nabla \circ (1_{\mathcal{X}(N)}, \lambda), \tag{1.2gen}$$

for any invertible endomorphism λ.

It follows immediately that ∇^λ is also a covariant derivative operator on π. Let us remark that defining the *mean covariant derivative* of ∇ and ∇^λ:

$$\nabla^0 = \frac{1}{2}\left(\nabla + \nabla^\lambda\right) \tag{1.3}$$

we obtain the desired solution of the problem, namely: $\hat{\nabla}^0\lambda = 0$; see also the Proposition 1.5 below. It follows that a study of ∇^λ is necessary and in the following we provide some properties of this conjugate covariant derivative. A first important property is the duality announced in the title:

Proposition 1.4. $\left(\nabla^\lambda\right)^\lambda = \nabla$, *hence* $\left(\nabla^\lambda\right)^0 = \nabla^0$.

Proof. A direct consequence of the definition is:

$$\nabla^\lambda_X(\lambda s) = \lambda(\nabla_X s) \tag{1.4}$$

and hence:

$$\left(\nabla^\lambda\right)^\lambda_X s = \varepsilon\lambda(\nabla^\lambda_X(\lambda s)) = \varepsilon\lambda\left[\lambda(\nabla_X s)\right] = \varepsilon^2 \nabla_X s$$

which implies the conclusion. □

The next property concerns with the hat-versions of the involved ∇'s:

Proposition 1.5. $\hat{\nabla}^\lambda\lambda = -\hat{\nabla}\lambda$ *and then* $\nabla \in \mathcal{C}(\lambda)$ *if and only if* $\nabla^\lambda \in \mathcal{C}(\lambda)$, *that is* $\nabla^\lambda = \nabla = \nabla^0$.

Proof. We have directly from the definitions:

$$\left(\hat{\nabla}^{\lambda}_{X}\lambda\right)(s)=\nabla^{\lambda}_{X}(\lambda s)-\lambda(\nabla^{\lambda}_{X}s)=\lambda(\nabla_X s)-\varepsilon^2(\nabla_X\lambda s)=-(\hat{\nabla}_X\lambda)(s)$$

which is the claimed equation. □

In the following we express locally the above objects. Let $h=(U,u^{\alpha};\alpha=1,...,n)$ be a local chart on N and suppose that $E|_U:=\pi^{-1}(U)$ has a trivialization. By the same arguments as in [21, p. 279] it follows that there exists a local frame field $S_U=\{s_i;i=1,...,k\}$ of π over U. So, any local section of $\Gamma_U(\pi)$ can be uniquely written as a $C^{\infty}(U)$-linear combination of elements of S_U. In particular:

$$\nabla^{U}_{\frac{\partial}{\partial u^{\alpha}}}s_i=\Gamma^{j}_{\alpha i}s_j \tag{1.5}$$

for some smooth functions $\Gamma^{j}_{\alpha i}\in C^{\infty}(U)$. Following the cited book we call these functions *the Christoffel symbols* of ∇ with respect to the chart h and the local frame field S_U. Also we have the local expression of λ:

$$\lambda(s_i)=\lambda^{j}_{i}s_j \tag{1.6}$$

with $\lambda^{j}_{i}\in C^{\infty}(U)$. Let us denote by $\overset{\lambda}{\Gamma}{}^{j}{}_{\alpha i}$ ($\overset{0}{\Gamma}{}^{j}{}_{\alpha i}$) the Christoffel symbols of ∇^{λ} (∇^{0}) with respect to the same pair (h,S_U). A straightforward computation yields the change $\Gamma\rightarrow\overset{\lambda}{\Gamma}$:

$$\overset{\lambda}{\Gamma}{}^{j}{}_{\alpha i}=\varepsilon\lambda^{j}_{k}\left(\frac{\partial\lambda^{k}_{i}}{\partial u^{\alpha}}+\Gamma^{k}_{\alpha l}\lambda^{l}_{i}\right). \tag{1.7}$$

Recall the usual local expression of the ∇-covariant derivative of λ:

$$\lambda^{k}_{i|\alpha}:=\frac{\partial\lambda^{k}_{i}}{\partial u^{\alpha}}+\Gamma^{k}_{\alpha l}\lambda^{l}_{i}-\Gamma^{l}_{\alpha i}\lambda^{k}_{l}. \tag{1.8}$$

Then the conjugate and the mean covariant derivatives can be expressed in the following way:

$$\begin{cases}\overset{\lambda}{\Gamma}{}^{j}{}_{\alpha i}=\varepsilon\lambda^{j}_{k}\left(\lambda^{k}_{i|\alpha}+\Gamma^{l}_{\alpha i}\lambda^{k}_{l}\right)=\varepsilon\lambda^{j}_{k}\lambda^{k}_{i|\alpha}+\Gamma^{j}_{\alpha i},\\ \overset{0}{\Gamma}{}^{j}{}_{\alpha i}=\frac{\varepsilon}{2}\lambda^{j}_{k}\lambda^{k}_{i|\alpha}+\Gamma^{j}_{\alpha i}=-\frac{\varepsilon}{2}\lambda^{k}_{i}\lambda^{j}_{k|\alpha}+\Gamma^{j}_{\alpha i}.\end{cases} \tag{1.9}$$

The relationship in formulae (1.3) and (1.9) can be represented as follows:

∇	$\overset{+\frac{\varepsilon}{2}\lambda^{j}_{k}\lambda^{k}_{i\|\alpha}}{\longrightarrow}$	∇^{0}	$\overset{-\frac{\varepsilon}{2}\lambda^{j}_{k}\lambda^{k}_{i\|\alpha}}{\longleftarrow}$	∇^{λ}

. (1.9fig)

If $\omega = (\omega^j_i)$ is the connection 1-form of ∇ and $\overset{\lambda}{\omega}$ is the connection 1-form of $\overset{\lambda}{\nabla}$ then, from $\omega^j_i = \Gamma^j_{\alpha i} du^\alpha$ it follows that:

$$\overset{\lambda}{\omega}{}^j_i - \omega^j_i = \varepsilon \lambda^j_k \lambda^k_{i|\alpha} du^\alpha = -\varepsilon \lambda^k_i \lambda^j_{k|\alpha} du^\alpha. \tag{1.10}$$

Recall also the curvature of ∇:

$$R\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right) s_i = R^j_{\alpha\beta i} s_j, \tag{1.11}$$

where:

$$R^j_{\alpha\beta i} = \frac{\partial \Gamma^j_{\beta i}}{\partial u^\alpha} - \frac{\partial \Gamma^j_{\alpha i}}{\partial u^\beta} + \Gamma^k_{\beta i}\Gamma^j_{\alpha k} - \Gamma^k_{\alpha i}\Gamma^j_{\beta k}. \tag{1.12}$$

Example 1.6. A) Suppose that E is the tangent bundle TN, hence λ is a tensor field of $(1,1)$-type on N and $n = k$ while $\alpha \in \{1, ..., n\}$.
i) The case $\varepsilon = -1$, corresponding to almost complex geometry, was studied in [7]; n must be an even positive integer.
ii) The case $\varepsilon = +1$, corresponding to almost product geometry, was studied in [8].
iii) From the expression (1.9) we derive also a relation between the torsions of ∇ and ∇^λ, ∇^0:

$$\overset{\lambda}{T}{}^j{}_{\alpha i} = T^j_{\alpha i} + \varepsilon \lambda^j_k (\lambda^k_{i|\alpha} - \lambda^k_{\alpha|i}), \quad \overset{0}{T}{}^j{}_{\alpha i} = T^j_{\alpha i} + \frac{\varepsilon}{2}\lambda^j_k (\lambda^k_{i|\alpha} - \lambda^k_{\alpha|i}). \tag{1.13}$$

iv) In the setting of G-structures suppose that λ is integrable. Then there exists an atlas on N such that the components of λ are constant and hence:

$$\overset{\lambda}{\Gamma}{}^j{}_{\alpha i} = \varepsilon \lambda^j_k \Gamma^k_{\alpha l} \lambda^l_i, \quad \overset{0}{\Gamma}{}^j{}_{\alpha i} = \frac{\varepsilon}{2}\lambda^j_k \Gamma^k_{\alpha l}\lambda^l_i + \frac{1}{2}\Gamma^j_{\alpha i} \tag{1.14}$$

which yields:

$$\overset{\lambda}{R}{}^j{}_{\alpha\beta i} = \varepsilon \lambda^j_k R^k_{\alpha\beta l} \lambda^l_i, \quad \overset{0}{R}{}^j{}_{\alpha\beta i} = \frac{\varepsilon}{2}\lambda^j_k R^k_{\alpha\beta l}\lambda^l_i + \frac{1}{2}R^j_{\alpha\beta i}. \tag{1.15}$$

We recover the result of Proposition 2.1 of [2], namely ∇ is flat if and only if ∇^λ is also flat; this result holds even if λ is not-integrable as well as for ∇^0. The relationships between the torsions of ∇ and ∇^λ, ∇^0 are:

$$\overset{\lambda}{T}{}^j{}_{\alpha i} = \varepsilon \lambda^j_k T^k_{\alpha l} \lambda^l_i, \quad \overset{0}{T}{}^j{}_{\alpha i} = \frac{1}{2}T^j_{\alpha i} + \frac{\varepsilon}{2}\lambda^j_k T^k_{\alpha l}\lambda^l_i. \tag{1.16}$$

B) Almost complex structures on the vertical bundle associated to the tangent bundle are studied in [4], while almost product structures on the same bundle are studied in [17]. □

Returning to the general case we describe now the $C^\infty(N)$-affine module $C(\lambda)$. The Kronecker endomorphism $I \in \Gamma(End(\pi)) \cong End(\Gamma(\pi))$ acts locally as:

$$I(s_i) = \delta_i^j s_j \tag{1.17}$$

and λ has two associated $(2,2)$-tensor fields, called *Obata operators*:

$$\Omega_{ij}^{hk} = \frac{1}{2}\left(\delta_i^h \delta_j^k + \varepsilon \lambda_i^h \lambda_j^k\right), \quad \Psi_{ij}^{hk} = \frac{1}{2}\left(\delta_i^h \delta_j^k - \varepsilon \lambda_i^h \lambda_j^k\right). \tag{1.18}$$

A straightforward computation yields:

Proposition 1.7. *1. The generic element* ∇^g *of* $C(\lambda)$ *has the expression:*

$$\overset{g}{\Gamma^j}_{\alpha i} = \overset{0}{\Gamma^j}_{\alpha i} + \Omega_{ia}^{lj} X_{\alpha l}^a, \tag{1.19}$$

with arbitrary $X = (X_{\alpha l}^a)$.
2. If $\nabla \in C(\lambda)$ *then* Ω *and* Ψ *are also covariant constant with respect to* ∇.

The second Obata operator is useful to express globally the first equation in (1.15) through:

$$\overset{\lambda}{R} = R - 2\Psi(R), \tag{1.20}$$

a relation obtained in [11] for $E = TN$ and conjugate connections with respect to non-degenerate $(0,2)$-tensor fields; see also [12]. In conclusion $\overset{0}{R} = R - \Psi(R)$. Let us remark that from $\nabla^g \in \mathcal{C}(\lambda)$ it results that $\overset{g}{R}$ commutes with λ:

$$\overset{g}{R}(\cdot,\cdot) \circ \lambda = \lambda \circ \overset{g}{R}(\cdot,\cdot). \tag{1.21}$$

Another approach for the pair $(\nabla^\lambda, \nabla^0)$ is expressed in terms of *quasi-covariant derivatives*, more precisely λ-*covariant derivatives*, which are maps D as in Definition 1.1 with the second condition replaced by:
iiλ) $D_X(fs) = fD_X s + X(f) \cdot \lambda(s)$.

It is easy to see that any covariant derivative ∇ yields a λ-covariant derivative D^∇ through:

$$D_X^\nabla := \nabla_X \circ \lambda \tag{1.22}$$

and then:

$$\nabla^\lambda = \varepsilon \lambda \circ D^\nabla, \quad \nabla_X^0 = \frac{\varepsilon}{2}(\lambda \circ D_X^\lambda + D^\lambda \circ \lambda) \tag{1.23}$$

for every $X \in \mathcal{X}(N)$. Also, it follows that: $\nabla_X = \varepsilon D_X^\nabla \circ \lambda$.

We finish this section with a slight generalization concerning algebroid covariant derivatives. Namely, let $(\mathcal{A}, N, \tau : \mathcal{A} \to N, \rho)$ be an *anchored vector bundle* over N of rank r, i.e. $\rho : \mathcal{A} \to TN$ is a morphism of vector bundles over the identity of N, called *anchor*, [13, p. 7]. Following the cited book we consider:

Definition 1.8. An $\mathcal{A}$-*covariant derivative* on π is an $\mathbb{R}$-bilinear map $D : \Gamma(\tau) \times \Gamma(\pi) \to \Gamma(\pi)$, $(\xi, s) \to D_\xi s$ satisfying:
i) D is tensorial in the first variable: $D_(f\xi)s = fD_\xi s$,
ii) D is a derivation in the second variable: $D_\xi(fs) = fD_\xi s + (\rho \circ \xi)(f)s$, for all $f \in C^\infty(N)$, $\xi \in \Gamma(\tau)$ and $s \in \Gamma(\pi)$.

We note that this notion also occurs in the infinite dimensional setting of Banach vector bundles, but under another name, in [3]. We can consider the same problem for λ as above in the setting of anchored covariant derivatives and all the definitions and results 1.3-1.5 hold with X replaced by ξ. Let us express locally these objects. Let $h = (U, u^\alpha), \alpha = 1, ..., n$ be a local chart on N and the trivialization of τ and π respectively:
i) $\mathcal{A}|_U := \tau^{-1}(U)$ has a trivialization $s_U^{\mathcal{A}} = \{e_A, A = 1, ..., r\}$,
ii) $E|_U$ has a trivialization as above.
The trivialization $s_U^{\mathcal{A}}$ yields:
1) the linear fibre coordinates z^A i.e. $\mathcal{A}|_U$ has the local coordinates (u^α, z^A),
2) the smooth functions ([13, p. 7]) $\rho_A^\alpha := \dot{u}^\alpha \circ \rho \circ e_A \in C^\infty(U)$. Hence: $\rho(e_A) = \rho_A^\alpha \frac{\partial}{\partial u^\alpha}$.

A fixed $\mathcal{A}$-covariant derivative D has the local expression: $D_{e_A}^U s_i := \Gamma_{Ai}^j s_j$ with $\Gamma_{Ai}^j \in C^\infty(U)$. Then its conjugate $\mathcal{A}$-covariant derivative D^λ is the generalization of (1.7):

$$\overset{\lambda}{\Gamma}{}^j{}_{Ai} = \varepsilon\lambda_k^j \left(\rho_A^\alpha \frac{\partial \lambda_i^k}{\partial u^\alpha} + \Gamma_{Al}^k \lambda_i^l\right). \tag{1.24}$$

Also, we consider the generalization of (1.8):

$$\lambda_{i|A}^k := \rho_A^\alpha \frac{\partial \lambda_i^k}{\partial u^\alpha} + \Gamma_{Al}^k \lambda_i^l - \Gamma_{Ai}^l \lambda_l^k \tag{1.25}$$

and hence (1.9) generalizes to:

$$\begin{cases} \overset{\lambda}{\Gamma}{}^j{}_{Ai} = \varepsilon\lambda_k^j \left(\lambda_{i|A}^k + \Gamma_{Ai}^l \lambda_l^k\right) = \varepsilon\lambda_k^j \lambda_{i|A}^k + \Gamma_{Ai}^j, \\ \overset{0}{\Gamma}{}^j{}_{Ai} = \frac{\varepsilon}{2}\lambda_k^j \lambda_{i|A}^k + \Gamma_{Ai}^j = -\frac{\varepsilon}{2}\lambda_i^k \lambda_{k|A}^j + \Gamma_{Ai}^j. \end{cases} \tag{1.26}$$

We remark that if $\mathcal{A}$ is the usual tangent bundle of N then (1.24) – (1.26) reduce to the formulae (1.7) – (1.9) since ρ is the Kronecker endomorphism (1.17).

In order to introduce the curvature we suppose that, in addition, $\mathcal{A}$ is a *Lie algebroid* i.e. $\Gamma(\tau)$ is endowed with a Lie bracket $[\cdot, \cdot]_{\mathcal{A}}$ following [13, p. 7]. Then the curvature tensor field of D is ([13, p. 48]):

$$R(\xi, \eta)s := D_\xi D_\eta s - D_\eta D_\xi s - D_{[\xi,\eta]_{\mathcal{A}}} s \tag{1.27}$$

with local coefficients:

$$R(e_A, e_B)s_i := R^j_{ABi}s_j, \quad R^j_{ABi} \in C^\infty(U). \tag{1.28}$$

A straightforward computation yields:

$$R^j_{ABi} := \rho^\alpha_A \frac{\partial \Gamma^j_{Bi}}{\partial u^\alpha} - \rho^\alpha_B \frac{\partial \Gamma^j_{Ai}}{\partial u^\alpha} + \Gamma^k_{Bi}\Gamma^j_{Ak} - \Gamma^k_{Ai}\Gamma^j_{Bk} - \Theta^C_{AB}\Gamma^j_{Ci} \tag{1.29}$$

where the coefficients Θ are provided by the $\mathcal{A}$-Lie bracket:

$$\Theta^C_{AB} := z^C \circ [e_A, e_B]_{\mathcal{A}}. \tag{1.30}$$

In the particular case $\tau = \pi$ we can also define *the torsion* of D, ([13, p. 48]):

$$T(\xi, \eta) := D_\xi \eta - D_\eta \xi - [\xi, \eta]_{\mathcal{A}} \tag{1.31}$$

and hence:

$$T(e_A, e_B) := T^C_{AB}e_C, \quad T^C_{AB} := \Gamma^C_{AB} - \Gamma^C_{BA} - \Theta^C_{AB}. \tag{1.32}$$

2 The case of a Finsler bundle endowed with an ε-endomorphism

Let now M be a smooth manifold of dimension m and $\tau = \tau_M : TM \to M$ be its tangent bundle. A local chart $h_M = (U, x^i; i = 1, ..., m)$ on M induces a local chart $(\tau^{-1}(U), x^i, y^i)$ on $TM|_U$, where $u \in TM|_U$ is expressed as $u = y^i \frac{\partial}{\partial x^i}$.

The vector bundle in the preceding section is $\pi : E = TM \times_M TM \to N = TM$ with [21, p. 179]:

$$TM \times_M TM = \{(u, v) \in TM \times TM; \tau(u) = \tau(v)\} \tag{2.1}$$

and $\pi(u, v) = u$. The fiber of π over $u \in TM$ is $\pi^{-1}(u) = \{u\} \times T_{\tau(u)}M$ and hence $k = m$ and $n = 2m$. Usually, π is called *the Finsler bundle of* M and its sections are called *Finsler vector fields*. The $C^\infty(TM)$-module $\Gamma(\pi)$ of Finsler vector fields is canonically isomorphic with the $C^\infty(TM)$-module of sections of τ_M along τ_M:

$$\Gamma_{\tau_M}(TM) := \{\underline{X} \in C^\infty(TM, TM); \tau_M \circ \underline{X} = \tau_M\}. \tag{2.2}$$

Hence the Finsler vector fields are of the form:

$$\tilde{X} : u \in TM \to \tilde{X}(u) = (u, \underline{X}(u)) \in TM \times_M TM \tag{2.3}$$

or briefly $\tilde{X} = (1_{TM}, \underline{X})$. A remarkable Finsler vector field $\mathbb{C}$ corresponds to $\underline{X} = 1_{TM} \in \Gamma_{\tau_M}(TM)$ and then:

$$\mathbb{C} = (1_{TM}, 1_{TM}) : u \in TM \to (u, u) \in TM \times_M TM. \tag{2.4}$$

In a compact form we have the *Liouville vector field* $\mathbb{C} = y^i \frac{\partial}{\partial y^i} \in \mathcal{X}(TM)$.

The local chart h_M of M determines for $u \in \tau^{-1}(U) \subset TM|_U$ the basis $\{\bar{\partial}_i(u)\}$ of the fibre $\pi^{-1}(u)$ given by:

$$\bar{\partial}_i(u) := \left(u, \frac{\partial}{\partial x^i}|_{\tau(u)}\right). \tag{2.5}$$

Hence $S|_U = \{s_i = \bar{\partial}_i; i = 1, ..., n\}$ is a local frame field of π over U.

Fix now an ε-endomorphism λ of this π; hence λ is a particular case of a *Finsler tensor field* of $(1,1)$-type. The equations:

$$\lambda(\bar{\partial}_i) = \lambda^j_i \bar{\partial}_j \tag{2.6}$$

give its components $\lambda^j_i \in C^\infty(\tau^{-1}(U))$:

$$\lambda^j_i = \lambda^j_i(u^\alpha) = \lambda^j_i(x^1, ..., x^m, y^1, ..., y^m). \tag{2.7}$$

Fix now the covariant differential operator ∇ for π. The local expression of ∇:

$$\nabla^U_{\frac{\partial}{\partial x^i}} \bar{\partial}_j = \Gamma^k_{ij} \bar{\partial}_k, \quad \nabla^U_{\frac{\partial}{\partial y^i}} \bar{\partial}_j = C^k_{ij} \bar{\partial}_k \tag{2.8}$$

generates the pair of Christoffel symbols: i) horizontal: $\Gamma^k_{ij} = \Gamma^k_{ij}(x, y)$, ii) vertical: $C^k_{ij} = C^k_{ij}(x, y)$. The conjugate ∇^λ has the pair $(\overset{\lambda}{\Gamma^k}{}_{ij}, \overset{\lambda}{C^k}{}_{ij})$:

$$\overset{\lambda}{\Gamma^k}{}_{ij} = \varepsilon \lambda^k_a \left(\frac{\partial \lambda^a_j}{\partial x^i} + \Gamma^a_{il} \lambda^l_j\right), \quad \overset{\lambda}{C^k}{}_{ij} = \varepsilon \lambda^k_a \left(\frac{\partial \lambda^a_j}{\partial y^i} + C^a_{il} \lambda^l_j\right). \tag{2.9}$$

Then ∇ yields two covariant derivatives: one horizontal $|$ and one vertical $_|$. For λ we have:

$$\lambda^k_{j|i} := \frac{\partial \lambda^k_j}{\partial x^i} + \Gamma^k_{il} \lambda^l_j - \Gamma^l_{ij} \lambda^k_l, \quad \lambda^k_{j_| i} := \frac{\partial \lambda^k_j}{\partial y^i} + C^k_{il} \lambda^l_j - C^l_{ij} \lambda^k_l \tag{2.10}$$

and then:

$$\overset{\lambda}{\Gamma^k}{}_{ij} = \varepsilon \lambda^k_a \left(\lambda^a_{j|i} + \Gamma^l_{ij} \lambda^a_l\right), \quad \overset{\lambda}{C^k}{}_{ij} = \varepsilon \lambda^k_a \left(\lambda^a_{j_| i} + C^l_{ij} \lambda^a_l\right). \tag{2.11}$$

Also, the mean covariant derivative ∇^0 has the pair $(\overset{0}{\Gamma^k}{}_{ij}, \overset{0}{C^k}{}_{ij})$:

$$\overset{0}{\Gamma^k}{}_{ij} = \frac{\varepsilon}{2} \lambda^k_a \lambda^a_{j|i} + \Gamma^k_{ij}, \quad \overset{0}{C^k}{}_{ij} = \frac{\varepsilon}{2} \lambda^k_a \lambda^a_{j_| i} + C^k_{ij}. \tag{2.12}$$

Example 2.1. Suppose that λ is a tensor field on the base M; then $\lambda = \lambda(x)$. It follows that:

$$\overset{\lambda}{C}{}^k{}_{ij} = \varepsilon\lambda^k_a C^a_{il}\lambda^l_j, \quad \overset{0}{C}{}^k{}_{ij} = \frac{\varepsilon}{2}\lambda^k_a C^a_{il}\lambda^l_j + \frac{1}{2}C^k_{ij} \tag{2.13}$$

and then, the vertical part of both ∇^0 and ∇^λ follows the path of formula (1.14). In particular, if the vertical part of ∇ vanishes then also the vertical parts of ∇^λ and ∇^0 vanish; in particular this is the case discussed in [2] when Γ lives also on the base M i.e. $\Gamma = \Gamma(x)$.

Now, with $m = 2$ and following the example 1.6 part iv) we consider:

$$\lambda = \overset{\varepsilon}{\lambda} := \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}. \tag{2.14}$$

The first equations (2.13) become:

$$\overset{\lambda}{C}{}^1{}_{ij} = \varepsilon C^2_{il}\lambda^l_j, \quad \overset{\lambda}{C}{}^2{}_{ij} = C^1_{il}\lambda^l_j \tag{2.15}$$

or, in more details:

$$\overset{\lambda}{C}{}^1{}_{i1} = C^2_{i2}, \quad \overset{\lambda}{C}{}^1{}_{i2} = \varepsilon C^2_{i1}, \quad \overset{\lambda}{C}{}^2{}_{i1} = \varepsilon C^1_{i2}, \quad \overset{\lambda}{C}{}^2{}_{i2} = C^1_{i1}. \tag{2.16}$$

□

The corresponding result of Proposition 1.7 for this setting is:

Proposition 2.2. *The generic element of* $C(\lambda)$ *has the expression:*

$$\overset{g}{\Gamma}{}^k{}_{ij} = \overset{0}{\Gamma}{}^k{}_{ij} + \Omega^{lk}_{ja}X^a_{il}, \quad \overset{g}{C}{}^k{}_{ij} = \overset{0}{C}{}^k{}_{ij} + \Omega^{lk}_{ja}Y^a_{il}, \tag{2.17}$$

with arbitrary $X = (X^a_{il})$ *and* $Y = (Y^a_{il})$.

Let us finish this section with the simple case of the tangent bundle τ_M; then $\lambda \in \mathcal{T}^1_1(M)$. Two tensor fields of $(1,2)$-type are associated in [2] to a given linear connection ∇: *the structural* and *the virtual* tensor field, denoted respectively by C^λ_∇ and B^λ_∇. After a short computation of their initial expression we write them in more useful form as:
1) the structural tensor field:

$$C^\lambda_\nabla(X,Y) := \frac{1}{2}\{(\nabla_{\lambda X}\lambda)Y - \lambda[(\nabla_X\lambda)(Y)]\}, \tag{2.18}$$

2) the virtual tensor field:

$$B^{\lambda}_{\nabla}(X,Y) := \frac{1}{2}\{(\nabla_{\lambda X}\lambda)Y + \lambda[(\nabla_X\lambda)(Y)]\}. \tag{2.19}$$

Their utility is provided by the relation (30) in [2]:

$$\nabla^{\lambda} = \nabla + \varepsilon(B^{\lambda}_{\nabla} - C^{\lambda}_{\nabla}) \Rightarrow \nabla^0 = \nabla + \frac{\varepsilon}{2}(B^{\lambda}_{\nabla} - C^{\lambda}_{\nabla}). \tag{2.20}$$

Suppose that locally we write:

$$C^{\lambda}_{\nabla}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = (C^{\lambda}_{\nabla})^k_{ij}\frac{\partial}{\partial x^k}, \quad B^{\lambda}_{\nabla}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = (B^{\lambda}_{\nabla})^k_{ij}\frac{\partial}{\partial x^k}. \tag{2.21}$$

Since the equation (1.8) is expressed by:

$$(\nabla_{\frac{\partial}{\partial x^i}}\lambda)(\frac{\partial}{\partial x^j}) = \lambda^k_{j|i}\frac{\partial}{\partial x^k}, \tag{2.22}$$

we derive:

$$(C^{\lambda}_{\nabla})^k_{ij} = \frac{1}{2}(\lambda^l_i\lambda^k_{j|l} - \lambda^k_l\lambda^l_{j|i}), \quad (B^{\lambda}_{\nabla})^k_{ij} = \frac{1}{2}(\lambda^l_i\lambda^k_{j|l} + \lambda^k_l\lambda^l_{j|i}). \tag{2.23}$$

3 Finsler geometry endowed with an ε-endomorphism

Recall from [5] that a *Finsler fundamental function* on M is a map $F : TM \to \mathbb{R}_+$ with the following properties:
F1) F is smooth on the slit tangent bundle $T_0M := TM \setminus O$ and continuous on the null section O of τ_M,
F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$,
F3) the matrix $(g_{ij}) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is invertible and its associated quadratic form is positive definite. The tensor field $g = \{g_{ij}(x, y); 1 \leq i, j \leq m\}$ is called *the Finsler metric* and the homogeneity of F implies:

$$F^2(x, y) = g_{ij}y^i y^j = y_i y^i, \tag{3.1}$$

where $y_i = g_{ij}y^j$. The pair (M, F) is called *Finsler manifold.* In particular, if g does not depend on y, we recover the Riemannian geometry.

On $N := T_0M$ we have two distributions:
i) $V(T_0M) := \ker \pi_*$, called *the vertical distribution*; does not depends of F. It is integrable and has the basis $\{\frac{\partial}{\partial y^i}; 1 \leq i \leq m\}$. A remarkable section of it is *the*

Liouville vector field $\mathbb{C} = y^i \frac{\partial}{\partial y^i}$.
ii) $H(T_0M)$ with the basis $\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial y^i} - N_i^j \frac{\partial}{\partial y^j}\}$, where:

$$N_j^i = \frac{1}{2}\frac{\partial \gamma_{00}^i}{\partial y^j} \tag{3.2}$$

and $\gamma_{00}^i = \gamma_{jk}^i y^j y^k$ is built from the usual Christoffel symbols:

$$\gamma_{jk}^i = \frac{1}{2} g^{ia}\left(\frac{\partial g_{ak}}{\partial x^j} + \frac{\partial g_{ja}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^a}\right). \tag{3.3}$$

$H(T_0M)$ is often called the *Cartan* (or canonical) *nonlinear connection* of the geometry (M, F) and a remarkable section of it is *the geodesic spray*:

$$S_F = y^i \frac{\delta}{\delta x^i}. \tag{3.4}$$

The Finslerian connections are triples $\Gamma = (N_i^k, F_{ij}^k(x,y), C_{ij}^k(x,y))$ where $F_{\cdot\cdot}^{\cdot}$ behave like the coefficients of a linear connection on M and C is a d-tensor field on T_0M. Such a Finslerian connection yields the covariant derivative $\Gamma\Delta$ on T_0M given by:

$$\begin{cases} \Gamma\Delta_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} := F_{ij}^k \frac{\delta}{\delta x^k}, & \Gamma\Delta_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} := F_{ij}^k \frac{\partial}{\partial y^k} \\ \Gamma\Delta_{\frac{\partial}{\partial y^j}} \frac{\delta}{\delta x^i} := C_{ij}^k \frac{\delta}{\delta x^k}, & \Gamma\Delta_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} := C_{ij}^k \frac{\partial}{\partial y^k}. \end{cases} \tag{3.5}$$

There are four remarkable Finslerian connections, [5, p. 227]:
-Cartan $Ca = (N_i^k, F_{ij}^k, C_{ij}^k)$,
-Chern-Rund $CR = (N_i^k, F_{ij}^k, 0)$,
-Berwald $B = (N_i^k, G_{ij}^k, 0)$,
-Hashiguchi $H = (N_i^k, G_{ij}^k, C_{ij}^k)$,
where $G^k = N_j^i y^j$ and:

$$F_{ij}^k = \frac{1}{2} g^{ka}\left(\frac{\delta g_{aj}}{\delta x^i} + \frac{\delta g_{ia}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^a}\right), C_{ij}^k = \frac{1}{2} g^{ka}\frac{\partial g_{ij}}{\partial y^a}, G_{ij}^k = \frac{\partial^2 G^k}{\partial y^i \partial y^j} = \gamma_{00}^i. \tag{3.6}$$

Let now λ be an ε-endomorphism given locally by $\lambda = (\lambda_i^j(x,y))$ as in (2.7). Γ, or equivalently $\Gamma\Delta$, yields two covariant derivatives: one horizontal $|$ and one vertical $_|$ which for λ are:

$$\lambda_{j|i}^k := \frac{\delta \lambda_j^k}{\delta x^i} + F_{il}^k \lambda_j^l - F_{ij}^l \lambda_l^k, \quad \lambda_{j_| i}^k := \frac{\partial \lambda_j^k}{\partial y^i} + C_{il}^k \lambda_j^l - C_{ij}^l \lambda_l^k. \tag{3.7}$$

Then Γ has:

1) a conjugate Finsler connection $\Gamma^{\lambda} = (N^k_i, \overset{\lambda}{F}{}^k_{ij}(x,y), \overset{\lambda}{C}{}^k_{ij}(x,y))$ with coefficients as in (2.11),

2) a mean Finsler connection $\Gamma^{0} = (N^k_i, \overset{0}{F}{}^k_{ij}(x,y), \overset{0}{C}{}^k_{ij}(x,y))$ with coefficients as in (2.12). λ is covariant constant with respect to this Finsler connection:

$$\lambda^k_{\underset{0}{j|i}} = \lambda^k_{\underset{0}{j|i}} = 0. \tag{3.8}$$

Example 3.1. The almost complex case ($\varepsilon = -1$) of Γ^0 appears in [20, p. 15] while the almost product case ($\varepsilon = +1$) in [18, p. 61], see also [1]. An important remark of these works is that the set $C(\lambda)$ derived from (2.12) is a commutative group. □

Returning to the general case let us remark that the Finsler connection Γ defines the pairs: $(\overset{h}{C^{\lambda}}\Gamma, \overset{h}{B^{\lambda}}\Gamma)$, $(\overset{v}{C^{\lambda}}\Gamma, \overset{v}{B^{\lambda}}\Gamma)$ with:

$$2(\overset{h}{C^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\delta \lambda^k_j}{\delta x^l} - \lambda^k_l \frac{\delta \lambda^l_j}{\delta x^i} + \varepsilon F^k_{ij} + \lambda^l_i F^k_{ls}\lambda^s_j - \lambda^k_l[\lambda^s_i F^l_{sj} + \lambda^s_j F^l_{si}] \tag{3.9}$$

$$2(\overset{h}{B^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\delta \lambda^k_j}{\delta x^l} + \lambda^k_l \frac{\delta \lambda^l_j}{\delta x^i} - \varepsilon F^k_{ij} + \lambda^l_i F^k_{ls}\lambda^s_j - \lambda^k_l[\lambda^s_i F^l_{sj} - \lambda^s_j F^l_{si}] \tag{3.10}$$

$$2(\overset{v}{C^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\partial \lambda^k_j}{\partial y^l} - \lambda^k_l \frac{\partial \lambda^l_j}{\partial y^i} + \varepsilon C^k_{ij} + \lambda^l_i C^k_{ls}\lambda^s_j - \lambda^k_l[\lambda^s_i C^l_{sj} + \lambda^s_j C^l_{si}] \tag{3.11}$$

$$2(\overset{v}{B^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\partial \lambda^k_j}{\partial y^l} + \lambda^k_l \frac{\partial \lambda^l_j}{\partial y^i} - \varepsilon C^k_{ij} + \lambda^l_i C^k_{ls}\lambda^s_j - \lambda^k_l[\lambda^s_i C^l_{sj} - \lambda^s_j C^l_{si}]. \tag{3.12}$$

Hence, for the case $\Gamma \in \{Chern - Rund,\ Berwald\}$ we have:

$$2(\overset{v}{C^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\partial \lambda^k_j}{\partial y^l} - \lambda^k_l \frac{\partial \lambda^l_j}{\partial y^i}, \quad 2(\overset{v}{B^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\partial \lambda^k_j}{\partial y^l} + \lambda^k_l \frac{\partial \lambda^l_j}{\partial y^i} \tag{3.13}$$

and these tensor fields are zero if λ is a basic endomorphism, i.e. $\lambda \in \mathcal{T}^1_1(M)$, or an integrable one, i.e. with constant components in a preferential atlas on M.

Let $(dx^i, \delta y^i := dy^i + N^i_j dx^j)$ be the dual of the Berwald basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$. A final remark is that the given λ yields four endomorphisms of N:

$$A_{\pm}(\lambda) := \lambda^j_i \frac{\delta}{\delta x^j} \otimes dx^i \pm \lambda^j_i \frac{\partial}{\partial y^j} \otimes \delta y^i, \quad B_{\pm}(\lambda) := \lambda^j_i \frac{\delta}{\delta x^j} \otimes \delta y^i \pm \lambda^j_i \frac{\partial}{\partial y^j} \otimes dx^i. \tag{3.14}$$

We note that $A_{\pm}(\lambda)$ and $B_{+}(\lambda)$ are exactly ε-endomorphisms on N while $B_{-}(\lambda)$ is an $(-\varepsilon)$-endomorphism on N.

Example 3.2. Suppose that $\varepsilon = +1$ and $\lambda = \delta = (\delta_i^j)$. Then: $A_+(\delta) = 1_{TN}$ is the Kronecker tensor field of tangent bundle TN; $A_-(\delta)$ is (together with the Sasaki-type metric G_F on TN induced by (g_{ij})) the almost para-Kähler structure P_F from [15, p. 1880] while $B_-(\delta)$ is (again together with G_F) the almost Kähler structure Ψ_F from [16, p. 243]. For the general setting of tangent manifolds, particularly tangent bundles, endowed with nonlinear connections, the almost product structure $A_-(\delta)$ appears in [14, p. 14] and it is well-known that $A_-(\delta)$ is integrable if and only if the corresponding nonlinear connection is without curvature i.e. flat. Also, for any Finsler connection Γ we have $\overset{v}{C^\delta}_\Gamma = \overset{v}{B^\delta}_\Gamma = \overset{h}{C^\delta}_\Gamma = \overset{h}{B^\delta}_\Gamma = 0$. □

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