International Journal of Geometric Methods in Modern Physics
Vol. 15 (2018) 1850082 (9 pages)

DOI: 10.1142/S0219887818500822

Wick–Tzitzeica solitons and their Monge–Ampér equation

Mircea Crasmareanu
Faculty of Mathematics, University "Al. I. Cuza"
Iaşi 700506, Romania
mcrasm@uaic.ro

Laurian-Ioan Pişcoran
North University Center of Baia Mare
Technical University of Cluj Napoca
Department of Mathematics and Computer Science
Victoriei 76, Baia Mare 430122, Romania
plaurian@yahoo.com

Received 28 November 2017
Accepted 28 December 2017
Published

We apply the Wick rotation to the Monge–Ampère equation of Tzitzeica graphs and we introduce the Wick–Tzitzeica solitons as complex functions solving the new equation. Some known Tzitzeica surfaces yields examples of these new solitons and we analyze them. To a second Wick–Tzitzeica soliton, we associate a homogeneous ODE system of gradient type which is Nambu–Poisson.

Keywords: Tzitzeica equation; Tzitzeica surface; Wick rotation; Wick–Tzitzeica soliton; Monge–Ampère equation.

Mathematics Subject Classification 2010: 53A05, 53A15, 53Z05, 35Q51

1. Introduction: A Review on Tzitzeica Geometries

Let $M \subset \mathbb{R}^3$ be a orientable and regular surface in the Euclidean 3-dimensional space with the Cartesian coordinates (x, y, z); we denote by $K(p)$ the Gaussian curvature in the point $p \in M$. From a historical point of view, the first centro-affine invariant of M was introduced by Georges Tzitzeica in [13] as the function $\mathrm{Tzitzeica}(M) : M \to \mathbb{R}$:

$$\mathrm{Tzitzeica}(M)(p) := \frac{K(p)}{d^4(p)}, \tag{1.1}$$

where $d(p) := d(O, T_pM)$ is the Euclidean distance from the origin $O \in \mathbb{R}^3$ to tangent space T_pM; several historical details are presented in [1]. Hence, he introduced a remarkable class of surfaces (and later hypersurfaces in the same manner) by asking the constancy of this function (the constant being called below as *Tzitzeica value*) and these are called *Tzitzeica surfaces* from a long time.

This class of surfaces is intimately related to two classes of remarkable partial differential equations (PDEs):

(1) *Monge–Ampère equations* since for an explicit expression of M, namely $z = z(x, y)$, the right-hand side of (1.1) is a Monge–Ampère expression:

$$\text{Tzitzeica}(M)(x, y, z) = \frac{z_{xx}z_{yy} - z_{xy}^2}{(xz_x + yz_y - z)^4}(= \text{constant}), \tag{1.2}$$

It follows that if $z = z(x, y)$ is a Tzitzeica graph then a linear deformation (equivalently centro-affine transformation) $\tilde{z}(x, y) := z(x, y) + \alpha x + \beta y$ with $\alpha, \beta \in \mathbb{R}$ is also a Tzitzeica graph with the same Tzitzeica value. We remark here that not all Tzitzeica surfaces are expressed globally as a graph.

(2) The so-called *Tzitzeica equation* for M given in asymptotic coordinates (u, v) (for the hyperbolic case $K < 0$, i.e. $\text{Tzitzeica}(M) < 0$) since then the compatibility relation of the Gauss–Weingarten equations is an equation in a function $h = h(u, v)$:

$$(\ln h)_{uv} = h - h^{-2}. \tag{1.3}$$

Although the Tzitzeica equation (1.3) was derived by Tzitzeica himself and extensively studied, especially form a solitonic point of view [4], there are few examples of Tzitzeica surfaces, see [12, Chap. 13]; remark that we fixed (1.3) as Tzitzeica equation since throughout the mathematics literature there are a few equations that are referred to as the Tzitzeica equation depending on how the surface is defined. Our study below restricts to two surfaces also found by Tzitzeica:

$$M_1 : xyz = 1, \quad M_2 : z(x^2 + y^2) = 1, \tag{1.4}$$

which are generalized in arbitrary dimension in [5]. Their Tzitzeica value is

$$\begin{aligned} \text{Tzitzeica}(M_1) &= \frac{1}{27} > 0, \\ \text{Tzitzeica}(M_2) &= -\frac{4}{27} < 0. \end{aligned} \tag{1.5}$$

Let us remark also that Tzitzeica himself gives at [13, p. 1258] the generalization of M_1 with arbitrary coefficients:

$$M_1^{\text{general}} : (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)(a_3x + b_3y + c_3z) = 1, \tag{1.6}$$

which is an algebraic surface of order 3.

Another very interesting Tzitzeica surface was introduced in [2]:

$$\begin{aligned} M_3 : z &= -\frac{3 + xy}{x + y}, \\ \text{Tzitzeica}(M_3) &= -\frac{1}{108} < 0, \end{aligned} \tag{1.7}$$

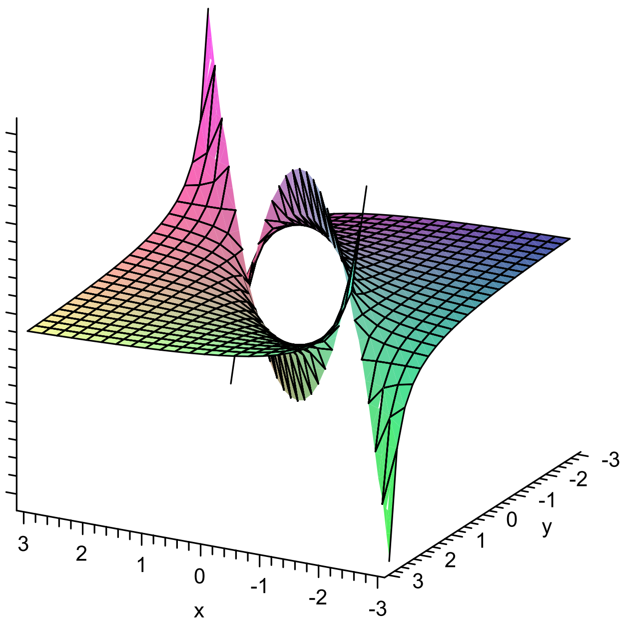

Fig. 1. M_3.

AQ: Fig. 1 is not cited in text. Kindly cite.

which can be called as *Euler–Tzitzeica surface* due to its relationship with the Euler line in triangle geometry. In [4], we present the pictures of M_1 and M_2; now we include the plot of M_3:

The Maple software yields its Gaussian curvature:

$$K(x,y) = \frac{-3(x+y)^4}{(x^4+2x^3y+3x^2y^2-3x^2+2xy^3-3y^2+y^4+9)^2}. \tag{1.8}$$

Also, a transformation of coordinates:

$$T : x = -\bar{x}+\bar{y}+\bar{z}, \quad y = \bar{x}-\bar{y}+\bar{z}, \quad z = \bar{x}+\bar{y}-\bar{z} \tag{1.9}$$

yields a new equation for M_3:

$$M_3 : \bar{x}^2+\bar{y}^2+\bar{z}^2-(\bar{x}-\bar{y})^2-(\bar{y}-\bar{z})^2-(\bar{z}-\bar{x})^2 = -3. \tag{1.10}$$

The inverse of the transformation T is

$$T^{-1} : \bar{x} = \frac{1}{2}(y+z), \quad \bar{y} = \frac{1}{2}(z+x), \quad \bar{z} = \frac{1}{2}(x+y). \tag{1.11}$$

For example, two points of M_3 with integer coordinates are $P_3(1,1,-2)$ and $P'_3(-1,-1,2)$ which are the intersection of M_3 with the plane $\pi : x+y+z=0$; the curvature in these points is $K(P_3) = K(P'_3) = -\frac{1}{3}$. Their image through T^{-1} are the points $\bar{P}_3(-\frac{1}{2},-\frac{1}{2},1)$, $\bar{P}'_3(\frac{1}{2},\frac{1}{2},-1)$ which are the intersections of $\bar{M}_3$ with π. A point on M_1 with integer coordinates is $P_1(1,1,1)$ while similar points on M_2 are $P_2(1,0,1)$, $P'_2(0,1,1)$.

In order to enlarge the study of Tzitzeica two-dimensional geometries, we complexify the Monge–Ampère equation (1.2) through a Wick rotation. More precisely, we have as model the duality between Born–Infeld solitons and minimal surfaces stated in [9] and realized through a Wick rotation. On this way, we introduce a new class of solitons, called *Wick–Tzitzeica*, which are generally specking special functions $\varphi : \Omega \subseteq \mathbb{C} \to \mathbb{C}$. Other solitonic tools in the study of Tzitzeica surfaces are presented in our previous work [4] in relationship with [14–16].

In the following section, we study the Wick–Tzitzeica solitons associated to the $M_{1,2,3}$ and several remarks concerning these complex functions are provided. A last section is devoted to some homogeneous ODE systems of gradient type and that corresponding to the Wick–Tzitzeica soliton φ_2 is described in the Nambu–Poisson formalism of multiple Hamiltonians.

2. Wick Rotation for Tzitzeica–Monge–Ampère Equation

We return now to a Tzitzeica graph $M : z = g(x, y)$ and its Tzitzeica–Monge–Ampère equation:

$$g_{xx}g_{yy} - g_{xy}^2 = \text{Tzitzeica}(g)(xg_x + yg_y - g)^4 \tag{2.1}$$

for which a Wick rotation $g(x, y) = \varphi(x, iy)$ is applied. After a straightforward computation, we arrive at the following definition.

Definition 2.1. Let λ be a given real number. The λ-*Wick–Tzitzeica equation* is

$$\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy} = \lambda(x\varphi_x + y\varphi_y - \varphi)^4 \tag{2.2}$$

and a solution $(x, y) \in D \subseteq \mathbb{R}^2 \to \varphi(x, y) \in \mathbb{C}$ is called *Wick–Tzitzeica soliton.*

Example 2.2. (i) The examples of introduction yield the following Wick–Tzitzeica solitons:

$$\begin{aligned} &\varphi_1(x, y) = \frac{i}{xy} = ig_1(x, y), \quad \varphi_2(x, y) = \frac{1}{x^2 - y^2}, \\ &\varphi_3(x, y) = -\frac{xy + 3i}{ix + y}. \end{aligned} \tag{2.3}$$

The corresponding λ_i is Tzitzeica(M_i). We note that the $O(2)$-invariance of the Tzitzeica graph M_2 yields the Lorentz $O(1, 1)$-invariance of φ_2; the centro-affine invariants in Minkowski geometry are discussed in [3]. Remark that φ_1 and φ_3 are complex-valued while φ_2 is real-valued; also we point out that the change of variables $x = \frac{1}{2}(\tilde{x} + \tilde{y})$, $y = \frac{1}{2}(\tilde{x} - \tilde{y})$ yields that $\varphi_2(x, y) = g_1(\tilde{x}, \tilde{y})$ which means that the graph of φ_2 is exactly M_1. We express also these functions in the complex variable $z = x + iy$:

$$\varphi_1(z, \bar{z}) = \frac{4}{\bar{z}^2 - z^2}, \quad \varphi_2(z, \bar{z}) = \frac{2}{z^2 + \bar{z}^2}, \quad \varphi_3(z, \bar{z}) = \frac{z^2 - \bar{z}^2 - 12}{4\bar{z}}. \tag{2.4}$$

Let us point out that also from the complex analysis point of view the pair $(\frac{1}{\varphi_2(x,y)}, \frac{2i}{\varphi_1(x,y)})$ is respectively the real and the complex part of the holomorphic function $F(z) = z^2$, hence there are (Euclidean) harmonic maps, i.e. belongs to the kernel of the Euclidean Laplacian $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\partial\overline{\partial}$. Remark that $(\frac{1}{g_1} = xy, \frac{1}{g_2} = x^2 + y^2)$ are hyperbolic harmonic maps, i.e. belongs to the kernel of the hyperbolic Laplacian $\Delta_h := \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. The third Tzitzeica graph satisfies: $\Delta_h(\frac{1}{g_3}) = \frac{(y-x)[3-(x^2+xy+y^2)]}{(xy+3)^3}$.

(ii) If we allow in (1.6) the coefficients a, b and c to be complex, we get that M_2 belongs also to M_1^{general}(complex). The graph of φ_2 belongs to M_1^{general}(real).

Example 2.3. (i) The common solutions of (2.1) and (2.2) are the linear functions: $\varphi_{\alpha,\beta}(x, y) = \alpha x + \beta y$ with coefficients $\alpha, \beta \in \mathbb{R}$ and $\lambda = \text{Tzitzeica} = 0$.

(ii) In [12, p. 320], is given the hyperbolic paraboloid is given $P_h : z = \sqrt{1 + axy}$ as Tzitzeica surface with $\text{Tzitzeica}(P_h) = -\frac{a^2}{4} < 0$; in fact, all quadrics with center are Tzitzeica surfaces. We derive the associated Wick–Tzitzeica soliton:

$$\varphi_a(x, y) = (1 - aixy)^{\frac{1}{2}} = \left(1 + \frac{a}{\varphi_1(x, y)}\right)^{\frac{1}{2}}, \quad \varphi_a(z, \bar{z}) = \frac{1}{2}[4 - a(z^2 - \bar{z}^2)]^{\frac{1}{2}}. \tag{2.5}$$

It follows a polynomial relationship between two Wick–Tzitzeica solitons: $\varphi_1(\varphi_a^2 - 1) = a$. For the usual Tzitzeica graphs $g_{1,2,3}$ from the introduction, we have the polynomial relation: $g_1 g_3^2(g_1 + 2g_2) = g_2(3g_1 + 1)^2$.

Remark 2.4. (i) The Wick rotation is usually performed by rotating the time coordinate in the Lorentz–Minkowski space-time to imaginary values. Here, we apply it in a $(1 + 1)$-geometry based on coordinates (x, y) and choosing the second coordinate, namely y, to rotate. We note that the Tzitzeica–Monge–Ampère equation is invariant to the involution $x \leftrightarrow y$, hence the same results are obtained if the first coordinate x is rotated.

(ii) The case of $\varphi_1 = ig_1$ yields the question if this situation is an exception. A simple computation yields that a Wick–Tzitzeica soliton is purely complex, i.e. $\varphi^c = ig(x, y)$, if and only if g is a Tzitzeica graph since g satisfies (2.1); the associated λ^c is $-\text{Tzitzeica}(g)$. This fact yields new examples of Wick–Tzitzeica solitons:

$$\varphi_2^c = \frac{i}{x^2 + y^2} = \frac{i}{z\bar{z}}, \quad \varphi_3^c = -i\frac{3 + xy}{x + y} = \frac{2(\bar{z}^2 - z^2 - 12i)}{(1 - i)z + (1 + i)\bar{z}}, \tag{2.6}$$

$$\varphi_a^c = i\sqrt{1 + axy} = \frac{1}{2}\sqrt{-4 + ai(z^2 - \bar{z}^2)}. \tag{2.7}$$

Hence, we have another polynomial relation between two Wick–Tzitzeica solitons: $\varphi_1[(\varphi_a^c)^2 + 1] = -ai$.

(iii) More generally, let us start with a Monge–Ampère equation for real valued $g : D \subseteq \mathbb{R}^n \to \mathbb{R}$:

$$\det D^2 g = f(g, \nabla g) \tag{2.8}$$

and we associate *the Wick–Monge–Ampère equation* for $\varphi = ig : D \to \mathbb{C}$:

$$\det D^2 \varphi = i^n f\left(\frac{\varphi}{i}, \frac{\nabla\varphi}{i}\right). \tag{2.9}$$

Supposing that f is r-homogeneous:

$$f(\lambda u, \lambda v) = \lambda^r f(u, v) \tag{2.10}$$

we get that (2.9) becomes the equation:

$$\det D^2 \varphi = i^{n-r} f(\varphi, \nabla\varphi). \tag{2.11}$$

Let us remark that the f of Tzitzeica–Monge–Ampère equation (2.1) is homogeneous with $r = 4$. The case of $n - r$ being multiple of 4 can be considered as *self-Wick*; these solitons can be considered as pairs $(g, \varphi = ig) : D \to (\mathbb{R}, i\mathbb{R})$ satisfying the same Monge–Ampère equation (2.8).

If, instead of complex deformation $g \in \mathbb{R} \to \varphi := ig \in i\mathbb{R}$, we consider the classical Wick rotation $g(x^1, \ldots, x^{n-1}, x^n) = \varphi(x^1, \ldots, x^{n-1}, ix^n)$ then (2.8) becomes

$$-\det D^2 \varphi = f(\varphi, \nabla\varphi) \tag{2.8Wick}$$

AQ: Kindly check.

which is an equation in $(x^1, \ldots, x^{n-1}, ix^n)$.

(iv) For the complex expression of a function $\varphi(z, \bar{z})$ above, we compute also the determinant:

$$\Delta(\varphi) := \begin{vmatrix} \dfrac{\partial^2\varphi}{\partial z^2} & \dfrac{\partial^2\varphi}{\partial z\partial\bar{z}} \\ \dfrac{\partial^2\varphi}{\partial z\partial\bar{z}} & \dfrac{\partial^2\varphi}{\partial\bar{z}^2} \end{vmatrix}.$$

We obtain

$$\begin{aligned} &\Delta(\varphi_1) = \frac{3}{4}\varphi_1^4, \quad \Delta(\varphi_2) = -3\varphi_2^4, \\ &\Delta(\varphi_a) = -16a^2\varphi_a^{-4}, \quad \Delta(\varphi_3) = -48\left(\frac{\partial^2\varphi_3}{\partial z^2}\right)^4. \end{aligned} \tag{2.12}$$

We remark the mild character of $\varphi_{1,2,a}$. Also, $\Delta(\varphi_2^c) = -5(\varphi_2^c)^4$, $\Delta(\varphi_a^c) = \frac{a^2}{16}(\varphi_a^c)^{-4}$.

(v) In [11, p. 195], the geometrical theory of Monge–Amperè general equation is presented:

$$Az_{xx} + 2Bz_{xy} + Cz_{yy} + D + E(z_{xx}z_{yy} - z_{xy}^2) = 0 \tag{2.13}$$

as well as their classification in *elliptic/hyperbolic* class according to the positivity/negativity of

$$\Delta := AC - B^2 - DE.$$

Our (2.2) has $A = B = C = 0$, $D = \lambda(xz_x + yz_y - z)^4$ and $E = -1$. Then

$$\Delta(\lambda) = \lambda(xz_x + yz_y - z)^4 \tag{2.14}$$

and hence the Wick–Tzitzeica equation associated to M_1 is elliptic while the Wick–Tzitzeica equations associated to M_2 and M_3 are hyperbolic. The ellipticity of M_1 means the convexity of $g_1 : \mathbb{R}^2_{+,+} := \{(x,y) \in \mathbb{R}^2; x > 0, y > 0\} \to \mathbb{R}$, $g_1(x,y) = \frac{1}{xy}$. Recall, after [8, pp. 531 and 532], that a convex C^2-function $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ yields *a Monge–Ampère measure* μ_u:

$$\mu_u(E) := \int_E \det D^2 u(x) dx \tag{2.15}$$

for any Borel set $E \subset \Omega$. Since

$$\det D^2 g_1(x,y) = \frac{3}{x^4 y^4}, \tag{2.16}$$

we get for its Monge–Ampère measure on products $E = (a,b) \times (c,d) \subset \mathbb{R}^2_{+,+}$:

$$\mu_{g_1}((a,b) \times (c,d)) = \frac{1}{3}\left(\frac{1}{a^3} - \frac{1}{b^3}\right)\left(\frac{1}{c^3} - \frac{1}{d^3}\right). \tag{2.17}$$

3. The Homogeneous ODE Systems Associated to g_2 and φ_2

Let $u_1(x,y,z) = xyz$ be the natural 3D function associated to M_1, namely M_1 is the level set of u_1 and value 1. A very interesting appearance of u_1 is as inverse Jacobi multiplier for the 3D quadratic Lotka–Volterra systems, conform [10, pp. 5 and 6]. Also, in [15, p. 113], the gradient flow of u_1 is discussed:

$$\begin{cases} \dot{x} = yz \\ \dot{y} = zx \\ \dot{z} = xy. \end{cases} \tag{3.1}$$

The same system appears in [6, p. 128, 7, p. 30] under the name of *Nahm system of static $SU(2)$-monopoles*. In a similar manner, we have for $u_2(x,y,z) = z(x^2 + y^2)$ of M_2 the ODE system:

$$\begin{cases} \dot{x} = 2xz \\ \dot{y} = 2yz \\ \dot{z} = x^2 + y^2 \end{cases} \tag{3.2}$$

while for Wick potential $u_2^{\varphi}(x,y,z) = z(x^2 - y^2)$, we get

$$\begin{cases} \dot{x} = 2xz \\ \dot{y} = -2yz \\ \dot{z} = x^2 - y^2. \end{cases} \tag{3.3}$$

Is easy to derive a pair of first integrals for these last two systems:

$$\begin{aligned} F_1^{g_2} &= \frac{y}{x}, \quad F_2^{g_2} = x^2 + y^2 - 2z^2 = \frac{1}{\varphi_2(x,z)} + \frac{1}{\varphi_2(y,z)}; \\ F_1^{\varphi_2} &= xy = \frac{i}{\varphi_1}, \quad F_2^{\varphi_2} = F_2^{g_2}. \end{aligned} \tag{3.4}$$

We remark that $F_1^{\varphi_2}$ and $F_2^{\varphi_2}$ are quadratic functions and using the formalism of [6], it follows that the system (3.3) admits a Nambu–Poisson description with $F_1^{\varphi_2}$ and $F_2^{\varphi_2}$ as Hamiltonians. The function $\frac{1}{g_2}(x,y) = x^2 + y^2$ is pointed out in [10, p. 9] as being the inverse Jacobi multiplier for the linear differential system:

$$\begin{cases} \dot{x} = \mu x - y \\ \dot{y} = x + \mu y \end{cases}, \quad \mu \neq 0 \leftrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

which is not of Poisson type.

For completeness, we consider also $u_3(x,y,z) = xy + yz + zx$ and M_3 appears as the level set of u_3 corresponding to the value -3. The gradient system of u_3 is again homogeneous:

$$\begin{cases} \dot{x} = y + z = \varphi_{1,1}(y,z) = 2\bar{x} \\ \dot{y} = z + x = \varphi_{1,1}(x,z) = 2\bar{y} \\ \dot{z} = x + y = \varphi_{1,1}(x,y) = 2\bar{z} \end{cases} \tag{3.5}$$

and admits a time-dependent first integral:

$$\mathcal{F} = e^{-2t}(x + y + z). \tag{3.6}$$

A compact form of the system (3.5) is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2T^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{3.7}$$

Acknowledgments

The first author has been supported by COST Action CA15117 (CANTATA). The second author has been supported by the Technical University of Cluj-Napoca through the research Contract no. 2011/12.07.2017, Internal Competition CICDI-2017.

References

[1] A. F. Agnew, A. Bobe, W. G. Boskoff and B. D. Suceava, Gheorghe Ţiţeica and the origins of affine differential geometry, *Historia Math.* **36**(2) (2009) 161–170.

[2] A. F. Agnew, A. Bobe, W. G. Boskoff, L. Homentcovschi and B. D. Suceava, The equation of Euler's line yields a Tzitzeica surface, *Elem. Math.* **64**(2) (2009) 71–80.

[3] A. Bobe, W. G. Boskoff and M. G. Ciuca, Tzitzeica-type centro-affine invariants in Minkowski spaces, *An. Ştiint. Univ. Ovidius Constanta Ser. Mat.* **20**(2) (2012) 27–34.

[4] W. G. Boskoff, M. Crasmareanu and L.-I. Piscoran, Tzitzeica equations and Tzitzeica surfaces in separable coordinate systems and the Ricci flow tensor field, *Carpathian J. Math.* **33**(2) (2017) 141–151.

[5] O. Constantinescu and M. Crasmareanu, A new Tzitzeica hypersurface and cubic Finslerian metrics of Berwald type, *Balkan J. Geom. Appl.* **16**(2) (2011) 27–34.

[6] M. Crasmareanu, First integrals generated by pseudosymmetries in Nambu–Poisson mechanics, *J. Nonlinear Math. Phys.* **7**(2) (2000) 126–135.

[7] M. Crasmareanu, Quadratic homogeneous ODE systems of Jordan-rigid body type, *Balkan J. Geom. Appl.* **7**(2) (2002) 27–42.

[8] G. De Philippis and A. Figalli, The Monge–Ampère equation and its link to optimal transportation, *Bull. Amer. Math. Soc.* **51**(4) (2014) 527–580.

[9] R. Dey and R. K. Singh, Born-Infeld solitons, maximal surfaces, and Ramanujan's identities, *Arch. Math.* **108**(5) (2017) 527–538.

[10] I. A. García and B. Hernández-Bermejo, Inverse Jacobi multiplier as a link between conservative systems and Poisson structures, *J. Phys. A, Math. Theor.* **50**(32) (2017) Article ID: 325204, 17 pp.

[11] T. A. Ivey and J. M. Landsberg, *Cartan for Beginners*: *Differential Geometry via Moving Frames and Exterior Differential Systems*, 2nd edn. Graduate Studies in Mathematics, Vol. 175 (American Mathematical Society, Providence, RI, 2016).

[12] S. N. Krivoshapko and V. N. Ivanov, *Encyclopedia of Analytical Surfaces* (Springer, Cham, 2015).

[13] G. Tzitzeica, Sur une Nouvelle Classe de Surfaces, *C. R. Acad. Sci. Paris* **144** (1907) 1257–1259.

[14] T. Ţurcanu and C. Udrişte, Tzitzeica geometry of soliton solutions for quartic interaction PDE, *Balkan J. Geom. Appl.* **21**(1) (2016) 103–112.

[15] C. Udrişte, Tzitzeica theory-opportunity for reflection in mathematics, *Balkan J. Geom. Appl.* **10**(1) (2005) 110–120.

[16] C. Udrişte, V. Arsinte and C. Cipu, Tzitzeica and sine-Gordon solitons, *Balkan J. Geom. Appl.* **16**(1) (2011) 150–154.