We apply the Wick rotation to the Monge–Ampère equation of Tzitzeica graphs and we introduce the Wick–Tzitzeica solitons as complex functions solving the new equation. Some known Tzitzeica surfaces yields examples of these new solitons and we analyze them. To a second Wick–Tzitzeica soliton, we associate a homogeneous ODE system of gradient type which is Nambu–Poisson.

Keywords: Tzitzeica equation; Tzitzeica surface; Wick rotation; Wick–Tzitzeica soliton; Monge–Ampère equation.

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1. Introduction: A Review on Tzitzeica Geometries

Let $M \subset \mathbb{R}^3$ be a orientable and regular surface in the Euclidean 3-dimensional space with the Cartesian coordinates $(x,y,z)$; we denote by $K(p)$ the Gaussian curvature in the point $p \in M$. From a historical point of view, the first centro-affine invariant of $M$ was introduced by Georges Tzitzeica in \cite{Tzitz1927} as the function $Tzitzeica(M) : M \to \mathbb{R}$:

$$Tzitzeica(M)(p) := \frac{K(p)}{d^4(p)}, \quad (1.1)$$

where $d(p) := d(O, T_p M)$ is the Euclidean distance from the origin $O \in \mathbb{R}^3$ to tangent space $T_p M$; several historical details are presented in \cite{Crasmareanu2008}. Hence, he introduced a remarkable class of surfaces (and later hypersurfaces in the same manner) by...
asking the constancy of this function (the constant being called below as \textit{Tzitzeica value}) and these are called \textit{Tzitzeica surfaces} from a long time.

This class of surfaces is intimately related to two classes of remarkable partial differential equations (PDEs):

1. \textit{Monge–Ampère equations} since for an explicit expression of $M$, namely $z = z(x, y)$, the right-hand side of (1.1) is a Monge–Ampère expression:

\[
\text{Tzitzeica}(M)(x, y, z) = \frac{z_{xx}z_{yy} - z_{xy}^2}{(x^2 + y^2 - z)^4} (= \text{constant}), \tag{1.2}
\]

It follows that if $z = z(x, y)$ is a Tzitzeica graph then a linear deformation (equivalently centro-affine transformation) $\tilde{z}(x, y) := z(x, y) + \alpha x + \beta y$ with $\alpha, \beta \in \mathbb{R}$ is also a Tzitzeica graph with the same Tzitzeica value. We remark here that not all Tzitzeica surfaces are expressed globally as a graph.

2. The so-called \textit{Tzitzeica equation} for $M$ given in asymptotic coordinates $(u, v)$ (for the hyperbolic case $K < 0$, i.e. $\text{Tzitzeica}(M) < 0$) since then the compatibility relation of the Gauss–Weingarten equations is an equation in a function $h = h(u, v)$:

\[
(\ln h)_{uv} = h - h^{-2}. \tag{1.3}
\]

Although the Tzitzeica equation (1.3) was derived by Tzitzeica himself and extensively studied, especially form a solitonic point of view \cite{4}, there are few examples of Tzitzeica surfaces, see \cite[Chap. 13]{12}; remark that we fixed (1.3) as Tzitzeica equation since throughout the mathematics literature there are a few equations that are referred to as the Tzitzeica equation depending on how the surface is defined. Our study below restricts to two surfaces also found by Tzitzeica:

\[
M_1 : xyz = 1, \quad M_2 : z(x^2 + y^2) = 1, \tag{1.4}
\]

which are generalized in arbitrary dimension in \cite{5}. Their Tzitzeica value is

\[
\text{Tzitzeica}(M_1) = \frac{1}{27} > 0, \tag{1.5}
\]

\[
\text{Tzitzeica}(M_2) = -\frac{4}{27} < 0.
\]

Let us remark also that Tzitzeica himself gives at \cite[p. 1258]{13} the generalization of $M_1$ with arbitrary coefficients:

\[
M_1^{\text{general}} : (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)(a_3x + b_3y + c_3z) = 1, \tag{1.6}
\]

which is an algebraic surface of order 3.

Another very interesting Tzitzeica surface was introduced in \cite{2}:

\[
M_3 : z = \frac{3 + xy}{x + y}, \tag{1.7}
\]

\[
\text{Tzitzeica}(M_3) = -\frac{1}{108} < 0.
\]
which can be called as Euler–Tzitzeica surface due to its relationship with the Euler line in triangle geometry. In [4], we present the pictures of $M_1$ and $M_2$; now we include the plot of $M_3$:

The Maple software yields its Gaussian curvature:

$$K(x, y) = \frac{-3(x + y)^4}{(x^4 + 2x^3y + 3x^2y^2 - 3x^2 + 2xy^2 - 3y^2 + y^4 + 9)^2}. \quad (1.8)$$

Also, a transformation of coordinates:

$$T : x = -\bar{x} + \bar{y} + \bar{z}, \quad y = \bar{x} - \bar{y} + \bar{z}, \quad z = \bar{x} + \bar{y} - \bar{z} \quad (1.9)$$

yields a new equation for $M_3$:

$$M_3 : \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - (\bar{x} - \bar{y})^2 - (\bar{y} - \bar{z})^2 - (\bar{z} - \bar{x})^2 = -3. \quad (1.10)$$

The inverse of the transformation $T$ is

$$T^{-1} : \bar{x} = \frac{1}{2}(y + z), \quad \bar{y} = \frac{1}{2}(z + x), \quad \bar{z} = \frac{1}{2}(x + y). \quad (1.11)$$

For example, two points of $M_3$ with integer coordinates are $P_3(1, 1, -2)$ and $P_3'(1, -1, 2)$ which are the intersection of $M_3$ with the plane $\pi : x + y + z = 0$; the curvature in these points is $K(P_3) = K(P_3') = -\frac{1}{4}$. Their image through $T^{-1}$ are the points $\bar{P}_3(-\frac{1}{2}, -\frac{1}{2}, 1), \bar{P}_3'(\frac{1}{2}, \frac{1}{2}, -1)$ which are the intersections of $\bar{M}_3$ with $\pi$. A point on $M_1$ with integer coordinates is $P_1(1, 1, 1)$ while similar points on $M_2$ are $P_2(1, 0, 1), P_2'(0, 1, 1)$.
In order to enlarge the study of Tzitzeica two-dimensional geometries, we complexify the Monge–Ampère equation (1.2) through a Wick rotation. More precisely, we have as model the duality between Born–Infeld solitons and minimal surfaces stated in [10] and realized through a Wick rotation. On this way, we introduce a new class of solitons, called Wick–Tzitzeica, which are generally specking special functions \( \varphi : \Omega \subseteq \mathbb{C} \to \mathbb{C} \). Other solitonic tools in the study of Tzitzeica surfaces are presented in our previous work [11] in relationship with \([14–16]\).

In the following section, we study the Wick–Tzitzeica solitons associated to the \( M_{1,2,3} \) and several remarks concerning these complex functions are provided. A last section is devoted to some homogeneous ODE systems of gradient type and that corresponding to the Wick–Tzitzeica soliton \( \varphi_2 \) is described in the Nambu–Poisson formalism of multiple Hamiltonians.

2. Wick Rotation for Tzitzeica–Monge–Ampère Equation

We return now to a Tzitzeica graph \( M : z = g(x, y) \) and its Tzitzeica–Monge–Ampère equation:

\[
g_{xx}g_{yy} - g_{xy}^2 = \text{Tzitzeica}(g)(xg_x + yg_y - g)^4
\]

for which a Wick rotation \( g(x, y) = \varphi(x, iy) \) is applied. After a straightforward computation, we arrive at the following definition.

Definition 2.1. Let \( \lambda \) be a given real number. The \( \lambda \)-Wick–Tzitzeica equation is

\[
\varphi_{xx} - \varphi_{xy}\varphi_{yy} = \lambda(x\varphi_x + y\varphi_y - \varphi)^4
\]

and a solution \((x, y) \in D \subseteq \mathbb{R}^2 \to \varphi(x, y) \in \mathbb{C}\) is called Wick–Tzitzeica soliton.

Example 2.2. (i) The examples of introduction yield the following Wick–Tzitzeica solitons:

\[
\begin{align*}
\varphi_1(x, y) &= \frac{i}{xy} = ig_1(x, y), & \varphi_2(x, y) &= \frac{1}{x^2 - y^2}, \\
\varphi_3(x, y) &= -\frac{xy + 3i}{ix + y}
\end{align*}
\]

The corresponding \( \lambda_i \) is Tzitzeica(\( M_i \)). We note that the \( O(2) \)-invariance of the Tzitzeica graph \( M_2 \) yields the Lorentz \( O(1, 1) \)-invariance of \( \varphi_2 \); the centro-affine invariants in Minkowski geometry are discussed in [2]. Remark that \( \varphi_1 \) and \( \varphi_3 \) are complex-valued while \( \varphi_2 \) is real-valued; also we point out that the change of variables \( x = \frac{1}{2}(\bar{z} + y), \ y = \frac{1}{2}(\bar{z} - \bar{y}) \) yields that \( \varphi_2(x, y) = g_1(\bar{x}, \bar{y}) \) which means that the graph of \( \varphi_2 \) is exactly \( M_1 \). We express also these functions in the complex variable \( z = x + iy \):

\[
\begin{align*}
\varphi_1(z, \bar{z}) &= \frac{4}{z^2 - \bar{z}^2}, & \varphi_2(z, \bar{z}) &= \frac{2}{z^2 + \bar{z}^2}, & \varphi_3(z, \bar{z}) &= \frac{z^2 - \bar{z}^2 - 12}{4\bar{z}}.
\end{align*}
\]
Wick–Tzitzeica solitons and their Monge–Ampère equation

(i) The Wick rotation is usually performed by rotating the time coordinate $t$ into $\tau = it$. Let us point out that also from the complex analysis point of view the pair $\{\varphi_1(x,y)\}$ is respectively the real and the complex part of the holomorphic function $F(z) = z^2$, hence there are (Euclidean) harmonic maps, i.e. belongs to the kernel of the Euclidean Laplacian $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\delta F$. Remark that $(\varphi_{g_1} = xy, \varphi_{g_2} = x^2 + y^2)$ are hyperbolic harmonic maps, i.e. belongs to the kernel of the hyperbolic Laplacian $\Delta_h := \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. The third Tzitzeica graph satisfies:

\[
\Delta_h(\varphi_{g_2}) = \frac{(\varphi_{g_2} - (x^2 + y^2))}{(x^2 + y^2)}.
\]

(ii) If we allow in (1.6) the coefficients $a$, $b$ and $c$ to be complex, we get that $M_2$ belongs also to $M_1^{\text{general}}(\text{complex})$. The graph of $\varphi_2$ belongs to $M_1^{\text{general}}(\text{real})$.

Example 2.3. (i) The common solutions of (2.1) and (2.2) are the linear functions:

\[
\varphi_{a,\beta}(x,y) = ax + \beta y \quad \text{with coefficients } a, \beta \in \mathbb{R} \quad \text{and } \lambda = \text{Tzitzeica} = 0.
\]

(ii) In [12] p. 320, is given the hyperbolic paraboloid is given $P_h : z = \sqrt{1 + axy}$ as Tzitzeica surface with Tzitzeica($P_h$) $= -\frac{a^2}{1} < 0$; in fact, all quadrics with center are Tzitzeica surfaces. We derive the associated Wick–Tzitzeica soliton:

\[
\varphi_a(x,y) = (1 - axy)^\frac{1}{2} = \left(1 + \frac{a}{\varphi_1(x,y)}\right)^{\frac{1}{2}}, \quad \varphi_a(z,\bar{z}) = \frac{1}{2}[4 - a(z^2 - \bar{z}^2)]^{\frac{1}{2}}.
\]

(2.5)

It follows a polynomial relationship between two Wick–Tzitzeica solitons:

\[
\varphi_1(\varphi_2^2 - 1) = a.
\]

For the usual Tzitzeica graphs $g_{1,2,3}$ from the introduction, we have the polynomial relation: $g_1 g_2^3(g_1 + 2g_2) = g_2(3g_1 + 1)^2$.

Remark 2.4. (i) The Wick rotation is usually performed by rotating the time coordinate in the Lorentz–Minkowski space-time to imaginary values. Here, we apply it in a $(1 + 1)$-geometry based on coordinates $(x, y)$ and choosing the second coordinate, namely $y$, to rotate. We note that the Tzitzeica–Monge–Ampère equation is invariant to the involution $x \leftrightarrow y$, hence the same results are obtained if the first coordinate $x$ is rotated.

(ii) The case of $\varphi_1 = ig_1$ yields the question if this situation is an exception. A simple computation yields that a Wick–Tzitzeica soliton is purely complex, i.e. $\varphi^c = ig(x, y)$, if and only if $g$ is a Tzitzeica graph since $g$ satisfies (2.2); the associated $\lambda^c$ is $-\text{Tzitzeica}(g)$. This fact yields new examples of Wick–Tzitzeica solitons:

\[
\varphi_2^c = \frac{i}{x^2 + y^2} = \frac{i}{z\bar{z}}, \quad \varphi_3^c = -\frac{3 + x\bar{y}}{x + y} = \frac{2(z^2 - \bar{z}^2 - 12i)}{(1 - i)z + (1 + i)\bar{z}},
\]

\[
\varphi_4^c = \frac{i\sqrt{1 + axy}}{2} = \frac{1}{2} \sqrt{-4 + a(x^2 - \bar{z}^2)}.
\]

(2.6)

Hence, we have another polynomial relation between two Wick–Tzitzeica solitons: $\varphi_1[(\varphi_2^c)^2 + 1] = -ai$. 

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(iii) More generally, let us start with a Monge–Ampère equation for real valued $g : D \subseteq \mathbb{R}^n \to \mathbb{R}$:

$$\det D^2 g = f(g, \nabla g)$$ (2.8)

and we associate the Wick–Monge–Ampère equation for $\varphi = ig : D \to \mathbb{C}$:

$$\det D^2 \varphi = i^n f\left(\frac{\varphi}{i}, \frac{\nabla \varphi}{i}\right).$$ (2.9)

Supposing that $f$ is $r$-homogeneous:

$$f(\lambda u, \lambda v) = \lambda^r f(u, v)$$ (2.10)

we get that (2.9) becomes the equation:

$$\det D^2 \varphi = i^{n-r} f(\varphi, \nabla \varphi).$$ (2.11)

Let us remark that the $f$ of Tzitzeica–Monge–Ampère equation (2.1) is homogeneous with $r = 4$. The case of $n - r$ being multiple of 4 can be considered as self-Wick; these solitons can be considered as pairs $(g, \varphi) : D \to (\mathbb{R}, i\mathbb{R})$ satisfying the same Monge–Ampère equation (2.8).

If, instead of complex deformation $g \in \mathbb{R} \to \varphi := ig \in i\mathbb{R}$, we consider the classical Wick rotation $g(x^1, \ldots, x^{n-1}, x^n) = \varphi(x^1, \ldots, x^{n-1}, ix^n)$ then (2.8) becomes

$$-\det D^2 \varphi = f(\varphi, \nabla \varphi)$$ (2.8 Wick)

which is an equation in $(x^1, \ldots, x^{n-1}, ix^n)$.

(iv) For the complex expression of a function $\varphi(z, \bar{z})$ above, we compute also the determinant:

$$\Delta(\varphi) := \begin{vmatrix} \frac{\partial^2 \varphi}{\partial z^2} & \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \\ \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} & \frac{\partial^2 \varphi}{\partial \bar{z}^2} \end{vmatrix}. $$

We obtain

$$\Delta(\varphi_1) = \frac{3}{4} \varphi_1^4, \quad \Delta(\varphi_2) = -3 \varphi_2^4,$$

$$\Delta(\varphi_3) = -16a^2 \varphi_a^{-4}, \quad \Delta(\varphi_3) = -48 \left(\frac{\partial^2 \varphi_3}{\partial z^2}\right)^4.$$ (2.12)

We remark the mild character of $\varphi_{1,2,a}$. Also, $\Delta(\varphi_2^\circ) = -5(\varphi_2^\circ)^4$, $\Delta(\varphi_a^\circ) = \frac{a^2}{4}(\varphi_a^\circ)^{-4}$.

(v) In [11] p. 195, the geometrical theory of Monge–Ampère general equation is presented:

$$A_{z_{xx}} + 2B_{z_{xy}} + C_{z_{yy}} + D + E(z_{xx}z_{yy} - z_{xy}^2) = 0$$ (2.13)

as well as their classification in elliptic/hyperbolic class according to the positivity/negativity of

$$\Delta := AC - B^2 - DE.$$
Wick–Tzitzeica solitons and their Monge–Ampère equation

Our (2.2) has $A = B = C = 0$, $D = \lambda(xz_x + yz_y - z)^4$ and $E = -1$. Then

$$\Delta(\lambda) = \lambda(xz_x + yz_y - z)^4$$  \hspace{1cm} (2.14)$$

and hence the Wick–Tzitzeica equation associated to $M_1$ is elliptic while the Wick–Tzitzeica equations associated to $M_2$ and $M_3$ are hyperbolic. The ellipticity of $M_1$ means the convexity of $g_1: \mathbb{R}^2_{+,+} \rightarrow \mathbb{R}$, $g_1(x,y) = \frac{1}{x^3}$. Recall, after [8, pp. 531 and 532], that a convex $C^2$-function $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ yields a Monge–Ampère measure $\mu_u$:

$$\mu_u(E) := \int_E \det D^2 u(x) dx$$  \hspace{1cm} (2.15)$$

for any Borel set $E \subset \Omega$. Since

$$\det D^2 g_1(x,y) = \frac{3}{x^4y^4},$$  \hspace{1cm} (2.16)$$

we get for its Monge–Ampère measure on products $E = (a,b) \times (c,d) \subset \mathbb{R}^2_{+,+}$:

$$\mu_{g_1}((a,b) \times (c,d)) = \frac{1}{3} \left( \frac{1}{a^3} - \frac{1}{b^3} \right) \left( \frac{1}{c^3} - \frac{1}{d^3} \right).$$  \hspace{1cm} (2.17)$$

3. The Homogeneous ODE Systems Associated to $g_2$ and $\varphi_2$

Let $u_1(x,y,z) = xyz$ be the natural 3D function associated to $M_1$, namely $M_1$ is the level set of $u_1$ and value 1. A very interesting appearance of $u_1$ is as inverse Jacobi multiplier for the 3D quadratic Lotka–Volterra systems, conform [10, pp. 5 and 6]. Also, in [15] p. 113], the gradient flow of $u_1$ is discussed:

$$\begin{align*}
\dot{x} &= yz \\
\dot{y} &= zx \\
\dot{z} &= xy.
\end{align*}$$  \hspace{1cm} (3.1)$$

The same system appears in [11] p. 128, [7] p. 30] under the name of Nahm system of static $SU(2)$-monopoles. In a similar manner, we have for $u_2(x,y,z) = z(x^2 + y^2)$ of $M_2$ the ODE system:

$$\begin{align*}
\dot{x} &= 2xz \\
\dot{y} &= 2yz \\
\dot{z} &= x^2 + y^2
\end{align*}$$  \hspace{1cm} (3.2)$$

while for Wick potential $u_2^\varphi(x,y,z) = z(x^2 - y^2)$, we get

$$\begin{align*}
\dot{x} &= 2xz \\
\dot{y} &= -2yz \\
\dot{z} &= x^2 - y^2.
\end{align*}$$  \hspace{1cm} (3.3)$$
Is easy to derive a pair of first integrals for these last two systems:

\[ F_1^{g_2} = \frac{y}{x}, \quad F_2^{g_2} = x^2 + y^2 - 2z^2 = \frac{1}{\varphi_2(x, z)} + \frac{1}{\varphi_2(y, z)}; \]
\[ F_1^{\phi_2} = xy = \frac{i}{\varphi_1}, \quad F_2^{\phi_2} = F_2^{g_2}. \] (3.4)

We remark that \( F_1^{\phi_2} \) and \( F_2^{\phi_2} \) are quadratic functions and using the formalism of [6], it follows that the system (3.3) admits a Nambu–Poisson description with \( F_1^{\phi_2} \) and \( F_2^{\phi_2} \) as Hamiltonians. The function \( g_2(x, y) = x^2 + y^2 \) is pointed out in [10, p. 9] as being the inverse Jacobi multiplier for the linear differential system:

\[
\begin{align*}
\dot{x} &= \mu x - y, \\
\dot{y} &= x + \mu y,
\end{align*}
\mu \neq 0 \leftrightarrow \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

which is not of Poisson type.

For completeness, we consider also \( u_3(x, y, z) = xy + yz + zx \) and \( M_3 \) appears as the level set of \( u_3 \) corresponding to the value \(-3\). The gradient system of \( u_3 \) is again homogeneous:

\[
\begin{align*}
\dot{x} &= y + z = \varphi_{1,1}(y, z) = 2\bar{x} \\
\dot{y} &= z + x = \varphi_{1,1}(x, z) = 2\bar{y} \\
\dot{z} &= x + y = \varphi_{1,1}(x, y) = 2\bar{z}
\end{align*}
\] (3.5)

and admits a time-dependent first integral:

\[
\mathcal{F} = e^{-2t}(x + y + z). \] (3.6)

A compact form of the system (3.5) is

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2T^{-1} \begin{pmatrix} x \\ y \end{pmatrix}. \] (3.7)

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References

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