Almost analytic Kähler forms with respect to a quadratic endomorphism with applications in Riemann-Finsler geometry

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Abstract

The almost analyticity with respect to a quadratic endomorphism T is introduced in an algebraic setting concerning a commutative and associative algebra $\mathcal{A}$. Two main properties are proved: the first concerns the simultaneous closedness for an almost analytic 1-form ω and $T\omega$ while the second regards the vanishing of the interior product of such a form with the Nijenhuis tensor of T. Also, we introduce an extension of the Frölicher-Nijenhuis formalism to this framework as well as a hermitian type property. When $\mathcal{A}$ is the algebra of smooth functions on a given (even dimensional) manifold we recover the classical notion of almost analytic 1-form. We study this analyticity and the hermitian type property for the Cartan 1-form of a Riemann-Finsler geometry. Also, we study the almost analytic functions on the tangent bundle of a Riemann-Finsler geometry with respect to the associated almost para-complex and almost complex structure of this geometry. We introduce two new types of Hessian and respectively Laplacian corresponding to these structures. Two types of gradient Ricci solitons are introduced in the tangent bundle.

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Introduction

The notion of almost analytic form was introduced a long time ago in the almost complex geometry and hence, it was treated in local coordinates, especially by Japanese geometers [25]. A global approach appeared in [22], unfortunately only in Romanian. Some of these global techniques were used in [20] and [21]; for example in the former paper a differential is introduced in the algebra of almost analytic forms and a corresponding Poincaré type lemma is proved.

In a recent work, namely [14], we propose a unifying setting of almost analyticity of forms which include both (almost) complex and (almost) para-complex structure. More precisely, given a quadratic endomorphism F on a smooth (even dimensional) manifold M let us consider its dual tensor F^* on differential forms. Then an almost F-analytic 1-form ω on M is closed if and only if its conjugate $F^*\omega$ is closed. Also, we extend this notion to arbitrary r-forms with r lower or equal half the dimension of M and in the paper [13] we extend all this techniques on Lie algebroids.

The aim of present note is to generalize this differential geometric topic into an algebraic framework using the setting of Kähler forms provided by Chapter 3 of [18]. More precisely, a (differential) 1-form ω is replaced here with an element from a commutative and associative algebra $\mathcal{A}$ (over a field $\mathbb{F}$) and a (smooth) vector field X is replaced with a derivation of $\mathcal{A}$. Having an (quadratic) endomorphism T of $\mathcal{A}$ we associate a conjugate of X with respect to T and define ω as being almost analytic with respect to T through a relation involving the differential $d\omega$. A main feature of such almost analytic forms is that ω and $T\omega$ are simultaneous closed or not, conform Proposition 1.2. Moreover, this result holds also with the differential d_T induced by T through the Frölicher-Nijenhuis type calculus. Another important property of almost analytic 1-forms which extends in this algebraic setting is the vanishing of the interior product of ω with the conjugate of Nijenhuis tensor of T; see Proposition 1.4. We add also to our study a property of hermitian type in this setting. The first section ends with the expression of the d_T^2 which vanishes when applied to almost T-analytic elements of $\mathcal{A}$. In the general case, d_T^2 admits a simplification if we add a dd_T-Lemma.

As application of this study we consider the Riemann-Finsler geometry (N, F) which is naturally endowed with an almost complex structure Ψ_F, an almost para-complex one P_F and a 1-form ω_F. We obtain that this 1-form behaves similar for P_F and Ψ_F: is not almost analytic but is hermitian. We study also the smooth functions on T_0N which are (weak) almost analytic with respect to P_F and Ψ_F. More precisely, in the para-complex case we obtain, in particular, all the smooth functions on the base N while for the complex case, we derive a very complicated equation. In the particular case of a Riemannian geometry and a search restricted to the base functions for the almost complex situation we obtain the totally geodesic functions i.e. the functions with vanishing Hessian.

This example yields the introduction of two new types of Hessian and Laplacian in Riemann-Finsler geometry; hence an almost analytic function with respect to P_F is almost para-harmonic and an almost analytic function with respect to Ψ_F is an almost complex-harmonic. On this way we extend the usual implication from the theory of complex functions: analyticity implies harmonicity. We give the value of these Laplacians on the energy F^2 as well as the expression on Euclidean and Berwald geometries. A global expression is obtained for the almost para-complex Laplacian in a Landsberg geometry by using the vertical gradient and the divergence with respect to the Cartan or Chern-Rund connections. Due to these Hessians we introduce two new types of gradient Ricci solitons in the tangent bundle of a Riemann-Finsler geometry (N, F) by means of the Jacobi endomorphism of (N, F). We point out that recently, the Ricci solitons are used in [10] to express the Zermelo navaigation problem for Randers metrics of constant flag curvature c and the class of our gradient Ricci solitons provided by Proposition 3.8 concerns exactly with this case of constant flag curvature.

The last section concerns with the study of Hessian and Laplacian in the tangent bundle of a Riemann-Finsler geometry. Since in literature appear already several "horizontal" and "vertical" Laplacians, see [17] and [27], we discuss both the Hessian and Laplacian associated to the Sasaki metric G_F respectively the Hessian and Laplacian generated by a Finslerian connection; note that the Levi-Civita connection of the Riemannian metric G_F is not a Finslerian one. As main tools we deal with the Cartan, Chern-Rund, Berwald and Hashiguchi connection and it follows that the almost-complex Laplacian introduced in the previous section is exactly the Berwald Laplacian.

1 The algebraic approach of almost analyticity

Let $\mathbb{F}$ be a field of characteristic zero and $\mathcal{A}$ a commutative and associative $\mathbb{F}$-algebra. After [18, p. 69], the module of *Kähler differentials* of $\mathcal{A}$ is the $\mathcal{A}$-module $\Omega^1(\mathcal{A})$, which is the free $\mathcal{A}$-module generated by the set $\{d(F)|F \in \mathcal{A}\}$, modulo the submodule generated by all elements of either one of the following three types: i) $d(F+G)-d(F)-d(G)$, ii) $d(FG)-Fd(G)-G(dF)$, iii) $d(a)$, where $a \in \mathbb{F}$ and $F, G \in \mathcal{A}$. It results the map $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ given by $d : F \to d(F)$.

Now we consider the $\mathcal{A}$-module $\Omega^p(\mathcal{A}) = \wedge^p \Omega^1(\mathcal{A})$ where $\wedge$ is the wedge product over $\mathcal{A}$ and $\Omega^0(\mathcal{A}) := \mathcal{A}$. Using the expression of [18, p. 70], the elements of $\Omega^p(\mathcal{A})$ with $p > 0$ are called *Kähler p-forms*. The $\mathbb{F}$-linear map d extends by functoriality of $\wedge$ to an $\mathbb{F}$-linear map $\wedge^\bullet d : \wedge^\bullet \mathcal{A} \to \Omega^\bullet(\mathcal{A})$:

$$\wedge^\bullet d(F_1 \wedge ... \wedge F_p) := d(F_1) \wedge ... \wedge d(F_p) \tag{1.1}$$

for any $F_1, ..., F_p \in \mathcal{A}$. Also, d extends to a graded $\mathbb{F}$-map $d : \Omega^\bullet(\mathcal{A}) \to \Omega^{\bullet+1}(\mathcal{A})$ called *de Rham differential* and given by:

$$d(GdF_1 \wedge ... \wedge dF_p) := dG \wedge dF_1 \wedge .... \wedge dF_p \tag{1.2}$$

for all $G, F_1, ..., F_p \in \mathcal{A}$. It is a graded derivation of degree 1 of $(\Omega^\bullet(\mathcal{A}), \wedge)$:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta \tag{1.3}$$

for $\omega \in \Omega^p(\mathcal{A})$ respectively $\eta \in \Omega^\bullet(\mathcal{A})$. As usual, $d \circ d = 0$ and if $d\omega = 0$ then ω is called *closed* and if $\omega = d\theta$ then ω is called *exact*, [18, p. 72].

The next $\mathcal{A}$-module we need is the set $\mathcal{X}^1(\mathcal{A})$ of *derivations* of $\mathcal{A}$ given by [18, p. 69]:

$$\mathcal{X}^1(\mathcal{A}) = Hom_{\mathcal{A}}(\Omega^1(\mathcal{A}), \mathcal{A}). \tag{1.4}$$

It results that any $X \in \mathcal{X}^1(\mathcal{A})$ can be considered as $\hat{X} : \mathcal{A} \to \mathcal{A}$ with:

$$\hat{X}(F) := X(dF) \in \mathcal{A} \tag{1.5}$$

for every $F \in \mathcal{A}$; but for the simplification of notations we give up to use the hat. From the definition of $\Omega^1(\mathcal{A})$ it results:

$$X(FG) = X(d(FG)) = X(FdG + GdF) = FX(dG) + GX(dF) = FX(G) + GX(F) \tag{1.6}$$

which means that X is indeed a derivation of $\mathcal{A}$. Also, every $X \in \mathcal{X}^1(\mathcal{A})$ yields a graded $\mathcal{A}$-linear map of degree -1:

$$i_X : \Omega^\bullet(\mathcal{A}) \to \Omega^{\bullet-1}(\mathcal{A}) \tag{1.7}$$

the express of it can be find in [18, p. 76]. Note that we have $i_X \circ i_X = 0$ and if $\omega \in \Omega^1(\mathcal{A})$ then $i_X\omega = X(\omega)$.

Fix now a $\mathbb{F}$-linear map $T : \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A})$ satisfying the nilpotence $T^2 = \varepsilon 1_{\Omega^1(\mathcal{A})}$ where $\varepsilon = \pm 1 \in \mathbb{F}$. Then to any derivation $X \in \mathcal{X}^1(\mathcal{A})$ we associate its *T-conjugate* $\overline{X} \in \mathcal{X}^1(\mathcal{A})$:

$$\overline{X}(\omega) := \frac{1}{\varepsilon} X(T\omega) = \varepsilon X \circ T(\omega) \tag{1.8}$$

for all $\omega \in \Omega^1(\mathcal{A})$. From $\overline{X} = \varepsilon X \circ T$ it results $\overline{\overline{X}} = \varepsilon X$.

Definition 1.1 $\omega \in \Omega^1(\mathcal{A})$ is called *almost T-analytic* if:

$$i_{\overline{X}} d\omega = \varepsilon i_X d(T\omega) \tag{1.9}$$

for all $X \in \mathcal{X}^1(\mathcal{A})$. Let $\Omega^1(\mathcal{A}, T)$ be the $\mathbb{F}$-module of these forms.

The first main properties of almost T-analytic Kähler forms are provided by:

Proposition 1.2 i) *If $\omega \in \Omega^1(\mathcal{A}, T)$ then ω is closed if and only if $T\omega$ is closed.*
ii) *Let $\omega \in \Omega^1(\mathcal{A})$. Then $\omega \in \Omega^1(\mathcal{A}, T)$ if and only if $T\omega \in \Omega^1(\mathcal{A}, T)$.*

Proof i) and ii) are direct consequences of (1.9) and the similar equation with $X \to \overline{X}$:

$$i_X d\omega = i_{\overline{X}} d(T\omega). \tag{1.10}$$

(1.9) means that: $\Omega^1(\mathcal{A}, T) = \cap_{X \in \mathcal{X}^1(\mathcal{A})} Ker\{i_{\overline{X}} \circ d - \varepsilon i_X \circ d \circ T\} = \cap_{X \in \mathcal{X}^1(\mathcal{A})} Ker\{i_{X \circ T} \circ d - i_X \circ d \circ T\}$. □

Let us remark that the definition means that for all $Y \in \mathcal{X}^1(\mathcal{A})$ we have:

$$Y(i_{\overline{X}} d\omega) = \varepsilon Y(i_X d(T\omega)). \tag{1.11}$$

Another main tool is the Lie derivative on $\mathcal{X}^1(\mathcal{A})$ given by the usual bracket of derivatives:

$$\mathcal{L}_X Y(F) := X(Y(F)) - Y(X(F)) \tag{1.12}$$

for all $F \in \mathcal{A}$ and $X, Y \in \mathcal{X}^1(\mathcal{A})$. Hence a direct computation gives the general formula for 1-forms:

$$Y(i_X d\omega) = X(i_Y \omega) - Y(i_X \omega) - i_{\mathcal{L}_X Y} \omega. \tag{1.13}$$

Remarks 1.3 i) The general formula of differential d is equation (3.28) of [18, p. 75].
ii) From (1.13) it follows that $\omega \in \Omega^1(\mathcal{A})$ is *closed* if and only if:

$$X(i_Y \omega) - Y(i_X \omega) = i_{\mathcal{L}_X Y} \omega \tag{1.14}$$

for any $X, Y \in \mathcal{X}^1(\mathcal{A})$. □

We have that (1.11) means:

$$\overline{X}(i_Y \omega) - Y(i_{\overline{X}} \omega) - i_{\mathcal{L}_{\overline{X}} Y} \omega = \varepsilon \left[X(i_Y T\omega) - Y(i_X T\omega) - i_{\mathcal{L}_X Y} T\omega \right]. \tag{1.15}$$

Since (1.8) means:

$$i_X T\omega = \varepsilon i_{\overline{X}} \omega \tag{1.16}$$

this relation can be written also as:

$$\overline{X}(i_Y \omega) - Y(i_{\overline{X}} \omega) - i_{\mathcal{L}_{\overline{X}} Y} \omega = X(i_{\overline{Y}} \omega) - Y(i_{\overline{X}} \omega) - i_{\overline{\mathcal{L}_X Y}} \omega \tag{1.17}$$

equivalently:

$$\overline{X}(i_Y \omega) - i_{\mathcal{L}_{\overline{X}} Y} \omega = X(i_{\overline{Y}} \omega) - i_{\overline{\mathcal{L}_X Y}} \omega. \tag{1.18}$$

Due to the Lie derivative we introduce *the Nijenhuis tensor* of T as $N_T : \mathcal{X}^1(\mathcal{A}) \times \mathcal{X}^1(\mathcal{A}) \to \mathcal{X}^1(\mathcal{A})$ given by:

$$N_T(X, Y) := \mathcal{L}_{\overline{X}} \overline{Y} - \overline{\mathcal{L}_{\overline{X}} Y} - \overline{\mathcal{L}_X \overline{Y}} + \varepsilon \mathcal{L}_X Y \tag{1.19}$$

which gives a third property of almost T-analytic Kähler forms:

Proposition 1.4 *If* $\omega \in \Omega^1(\mathcal{A}, T)$ *then*:

$$i_{\overline{N_T}}\omega = 0 (= i_{N_T}T\omega). \tag{1.20}$$

Proof We must prove that:

$$i_{\overline{\mathcal{L}_{\overline{X}}\overline{Y}} - \varepsilon\mathcal{L}_{\overline{X}}Y - \varepsilon\mathcal{L}_X\overline{Y} + \varepsilon\overline{\mathcal{L}_X Y}}\omega = 0 \tag{1.21}$$

for any $X, Y \in \mathcal{X}^1(\mathcal{A})$. But (1.18) means:

$$i_{\overline{\mathcal{L}_X Y} - \mathcal{L}_{\overline{X}}Y}\omega = X(i_{\overline{Y}}\omega) - \overline{X}(i_Y\omega). \tag{1.22}$$

With $X \to \overline{X}$ and $Y \to \overline{Y}$ this relation becomes:

$$i_{\overline{\mathcal{L}_{\overline{X}}\overline{Y}} - \varepsilon\mathcal{L}_{X\overline{Y}}}\omega = \varepsilon\overline{X}(i_Y\omega) - \varepsilon X(i_{\overline{Y}}\omega). \tag{1.23}$$

Then, the sum of (1.23) with the multiplication of (1.22) with ε yields (1.21). From $i_{\overline{N_T}}\omega = \varepsilon N_T \circ T(\omega)$ and the first part it result the second part of (1.20). $\square$

It is natural to consider the cases below:

Definition 1.5 i) T is called *integrable* if $N_T = 0$. ii) $\omega \in \Omega^1(\mathcal{A})$ is called T-*analytic* if $\omega \in \Omega^1(\mathcal{A}, T)$ and T is integrable.

Hence ω is T-analytic if and only if $T\omega$ is so. In order to provide examples of elements from $\Omega^1(\mathcal{A}, T)$ we introduce:

Definition 1.6 $F \in \mathcal{A}$ is called *almost T-analytic* if there exists $G \in \mathcal{A}$ such that *the almost ε-CR equation* holds:

$$T(dF) = dG. \tag{1.24}$$

Let $\Omega^0(\mathcal{A}, T)$ be the set of these elements and let us say that G is *the potential* of $F \in \Omega^0(\mathcal{A}, T)$.

Examples 1.7 i) If $F \in \Omega^0(\mathcal{A}, T)$ then $\omega := dF \in \Omega^1(\mathcal{A}, T)$ since the both hand sides of (1.9) are zero. This Kähler form is closed and hence we have $d : \Omega^0(\mathcal{A}, T) \to \Omega^1(\mathcal{A}, T)$. In fact, (1.24) means that $F \in \Omega^0(\mathcal{A}, T)$ if and only if the 1-form $T(dF)$ is exact.
Moreover, due to the quadratic property of T it result the following duality: if $F \in \Omega^0(\mathcal{A}, T)$ with the potential G then $G \in \Omega^0(\mathcal{A}, T)$ with the potential εF.
ii) Suppose that $\mathbb{F} = \mathbb{R}$ and $\mathcal{A}$ is the algebra $C^\infty(M)$ of smooth functions on an even-dimensional manifold M^{2n}; hence $\Omega^1(\mathcal{A})$ is the usual $C^\infty(M)$-module of 1-forms on M and $\mathcal{X}^1(\mathcal{A})$ is the Lie algebra of vector fields on M. Let F be an almost complex or almost para-complex structure on M according to $\varepsilon = -1$ respectively $\varepsilon = +1$; recall that an almost para-complex structure is an almost product one with equal dimension for its $\pm$-eigenbundles. Then T is the dual endomorphism corresponding to F:

$$T\omega(X) := \omega(FX) \tag{1.25}$$

for all $\omega \in \Omega^1(M)$ and $X \in \mathcal{X}^1(M)$. Hence the definition 1 is exactly the definition 1.2 of [14, p. 323] and thus we generalize the classical notion of almost analytic 1-form of [23]- [26].
iii) Returning to the general case let ω_1 and ω_2 which are cohomological equivalent i.e. $\omega_2 = \omega_1 + dF$. Since $d\omega_2 = d\omega_1$ and $d(T\omega_2) = d(T\omega_1) + d(T(dF))$ it results that for $\omega_1 \in \Omega^1(\mathcal{A}, T)$ we

have $\omega_2 \in \Omega^1(\mathcal{A}, T)$ if and only if $d \circ T \circ d(F) = 0$. In particular, if $F \in \Omega^0(\mathcal{A}, T)$ it results that ω_1 and ω_2 are simultaneous in $\Omega^1(\mathcal{A}, T)$ or not. $\Box$

In the following we develop an analogue of the Frölicher-Nijenhuis calculus. The first step is the *T-interior product* $i_T : \Omega^\bullet(\mathcal{A}) \to \Omega^\bullet(\mathcal{A})$ given on 0-forms, 1-forms and exact 2-forms by the rules:
i) $i_T F = 0$, $i_T \omega := T\omega$ respectively,
ii) $Y(i_X(i_T d\omega)) := Y(i_{\overline{X}} d\omega) + \overline{Y}(i_X d\omega)$.
The second step is the T-differential $d_T : \Omega^\bullet(\mathcal{A}) \to \Omega^{\bullet+1}(\mathcal{A})$:

$$d_T = i_T \circ d - \varepsilon d \circ i_T. \tag{1.26}$$

The i) above yields:

$$i_T \circ i_T = \varepsilon 1_{\Omega^1(\mathcal{A})} \tag{1.27}$$

i.e. $i_T|_{\Omega^1(\mathcal{A})}$ has the same nature as T, while (1.11) with $\omega \to T\omega$ gives the following expression of ii):

$$Y(i_X(i_T d\omega)) = \varepsilon Y(i_X d(T\omega)) + \overline{Y}(i_{\overline{X}} d(T\omega)). \tag{1.28}$$

These tools yield a stronger version of ii) of Proposition 1.2:

Proposition 1.8 I) *For* $\omega \in \Omega^1(\mathcal{A}, T)$ *the following sentences are equivalent*:
i) ω *is closed*, ii) $T\omega$ *is closed*, iii) ω *is* T*-closed*: $d_T\omega = 0$, iv) $T\omega$ *is* T*-closed.*
II) *If* $\omega \in \Omega^1(\mathcal{A}, T)$ *then* ω *is almost* d_T*-analytic i.e.* (1.9) *holds also with* d *replaced with* d_T:

$$i_{\overline{X}} d_T \omega = \varepsilon i_X d_T(T\omega). \tag{1.29}$$

Proof I) Fix $X, Y \in \mathcal{X}^1(\mathcal{A})$. We have:

$$i_X d_T \omega = i_X(i_T d\omega) - \varepsilon i_X d(T\omega) \tag{1.30}$$

and then, with (1.10):

$$Y(i_X d_T \omega) = Y(i_{\overline{X}} d\omega) + \overline{Y}(i_X d\omega) - Y(i_{\overline{X}} d\omega) = \overline{Y}(i_X d\omega). \tag{1.31}$$

It results the equivalence of i) and iii) and the conclusion follows.
II) The relation (1.29) means:

$$i_{\overline{X}}(i_T d\omega) - \varepsilon i_{\overline{X}}(d(i_T\omega)) = \varepsilon \imath_X[i_T d(T\omega) - \varepsilon d(i_T T\omega)].$$

equivalently:

$$i_{\overline{X}}(i_T d\omega) - \varepsilon i_{\overline{X}} d(T\omega) = \varepsilon i_X(i_T d(T\omega)) - \varepsilon i_X d\omega.$$

This last equation holds term with term. $\Box$

A simplification in the definition of i_T on 2-forms is inspired by the Hermitian geometry (M^{2n}, g, J) where the fundamental 2-form $\Omega := g(J\cdot, \cdot)$ satisfies:

$$\Omega(JX, Y) = -\Omega(X, JY) \tag{1.32}$$

for any vector fields X, Y on the manifold M. Hence, for our setting we introduce:

Definition 1.9 The form $\omega \in \Omega^1(\mathcal{A})$ is called *T-hermitian* if:

$$Y(i_{\overline{X}}d\omega) = \varepsilon\overline{Y}(i_X d\omega) \tag{1.33}$$

for any $X, Y \in \mathcal{X}^1(\mathcal{A})$. Let $\Omega^1(\mathcal{A}, Th)$ the $\mathbb{F}$-module of these 1-forms.

Remarks 1.10 i) If $\omega \in \Omega^1(\mathcal{A}, Th) \cap \Omega^1(\mathcal{A}, T)$ then, after a straightforward calculus, we derive:

$$i_X(i_T d\omega) = (1+\varepsilon)i_{\overline{X}}d\omega \tag{1.34}$$

which for the complex case means: $i_T d\omega = 0$ and $d_T\omega = d(T\omega)$.
ii) Let $\omega \in \Omega^1(\mathcal{A}, T)$. Then $\omega \in \Omega^1(\mathcal{A}, Th)$ if and only if $T\omega \in \Omega^1(\mathcal{A}, Th)$. Indeed, if $\omega \in \Omega^1(\mathcal{A}, Th)$ then (1.33) with the terms replaced from (1.9) + (1.10) means:

$$\varepsilon Y(i_X d(T\omega)) = \varepsilon\overline{Y}(i_{\overline{X}}d(T\omega))$$

and hence:

$$Y(i_X d(T\omega)) = \overline{Y}(i_{\overline{X}}d(T\omega)). \tag{1.35}$$

With $X \to \overline{X}$ we obtain:

$$Y(i_{\overline{X}}d(T\omega)) = \varepsilon\overline{Y}(i_X d(T\omega)) \tag{1.36}$$

which yields the claimed sentence. □

In the following we study the expression of d_T^2 for the diagram:

$$\Omega^0(\mathcal{A}) \xrightarrow{d_T} \Omega^1(\mathcal{A}) \xrightarrow{d_T} \Omega^2(\mathcal{A}) \tag{1.37}$$

which means:

$$\Omega^0(\mathcal{A}) \ni F \to d_T F := \omega_F \to d_T\omega_F := \rho_F. \tag{1.38}$$

We have:

$$\omega_T = i_T(dF) = T(dF), \quad \rho_F = i_T(d\omega_F) - \varepsilon d(i_T\omega_F) = i_T(dd_T F). \tag{1.39}$$

For $F \in \Omega^0(\mathcal{A}, T)$ we obtain: $d_T^2 F = 0$ while for the general case of F is necessary to consider the 2-form:

$$\mu_F = dd_T F \tag{1.40}$$

which yields:

$$d_T^2 F = i_T\mu_F. \tag{1.41}$$

Inspired by the dd^c-lemma of complex geometry, see for example [1, p. 75], we introduce:

Definition 1.11 The pair $(\mathcal{A}, T)$ satisfies the dd_T-Lemma if:

$$ker d \cap im d_T = im dd_T. \tag{1.42}$$

Hence, if $(\mathcal{A}, T)$ satisfies the dd_T-Lemma there exists a closed 1-form η_F such that:

$$\mu_F = d_T\eta_F \tag{1.43}$$

and we get the final expression of d_T^2:

$$d_T^2 F = i_T d_T\eta_F = -\varepsilon i_T \circ d(T\eta_F). \tag{1.44}$$

2 Applications to Riemann-Finsler geometry

Let N be now a smooth n-dimensional manifold with $n \geq 2$ and $\pi : TN \to N$ its tangent bundle. Let $x = (x^i) = (x^1, ..., x^n)$ be local coordinates on N and $(x, y) = (x^i, y^i) = (x^1, ..., x^n, y^1, ..., y^n)$ be the induced local coordinates on TN. Denote by O the null-section of π. A main structure of TM is the *tangent structure* $J = \frac{\partial}{\partial y^i} \otimes dx^i$ which satisfies also a quadratic equation: $J^2 = 0$.

Recall after [3] or [5] that a *Finsler fundamental function* on N is a map $F : TN \to \mathbb{R}_+$ with the following properties:
F1) F is smooth on the slit tangent bundle $T_0N := TN \setminus O$ and continuous on O,
F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$,
F3) the matrix $(g_{ij}) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is invertible and its associated quadratic form is positive definite.
The tensor field $g = \{g_{ij}(x, y); 1 \leq i, j \leq n\}$ is called *the Finsler metric* and the homogeneity of F implies:

$$F^2(x, y) = g_{ij} y^i y^j = y_i y^i, \tag{2.1}$$

where $y_i = g_{ij} y^j$. The pair (N, F) is called *Finsler manifold.* In particular, if g does not depend on y, we recover the Riemannian geometry.

On $M := T_0N$ we have two distributions:
i) $V(T_0N) := \ker \pi_*$, called *the vertical distribution* and not depending of F; let v be the associated projector. It is integrable and has the basis $\{\frac{\partial}{\partial y^i}; 1 \leq i \leq n\}$. A remarkable section of it is *the Liouville vector field* $\mathbb{C} = y^i \frac{\partial}{\partial y^i}$.
ii) $H(T_0N)$ with the basis $\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}\}$, where:

$$N^i_j = \frac{1}{2}\frac{\partial \gamma^i_{00}}{\partial y^j} \tag{2.2}$$

with $\gamma^i_{00} = \gamma^i_{jk} y^j y^k$ built from the usual Christoffel symbols:

$$\gamma^i_{jk} = \frac{1}{2} g^{ia} \left(\frac{\partial g_{ak}}{\partial x^j} + \frac{\partial g_{ja}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^a}\right). \tag{2.3}$$

A simple notation for γ^i_{00} is G^i. $H(T_0N)$ is often called the *Cartan* (or canonical) *nonlinear connection* of the geometry (N, F) and a remarkable section of it is *the geodesic spray*:

$$S_F = y^i \frac{\delta}{\delta x^i}. \tag{2.4}$$

Let h be the associated projector. The dual basis of the above local basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ of $\Gamma(T_0N)$ is $(dx^i, \delta y^i = dy^i + N^i_j dx^j)$. On T_0N we have a Riemannian metric of Sasaki type:

$$G_F = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j. \tag{2.5}$$

Another Finslerian objects are two tensor fields of $(1, 1)$-type:
I) [15] $P_F : \Gamma(T_0N) \to \Gamma(T_0N)$:

$$P_F\left(\frac{\delta}{\delta x^i}\right) = \frac{\delta}{\delta x^i}, \quad P_F\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial y^i}. \tag{2.6}$$

Let us remark that P_F is a global geometric object although it is defined using a fixed coordinate chart; in fact is exactly $h - v$ and with equations (2.7) of [8, p. 14] we have:

$$P_F \circ J = -J, \quad J \circ P_F = J. \tag{2.7}$$

It results that P_F is an almost para-complex structure and the pair (P_F, G_F) is an almost para-Kähler structure on T_0N.

II) [16] $\Psi_F : \Gamma(T_0N) \to \Gamma(T_0N)$:

$$\Psi_F\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial y^i}, \quad \Psi_F\left(\frac{\partial}{\partial y^i}\right) = -\frac{\delta}{\delta x^i}. \tag{2.8}$$

It results that Ψ_F is an almost complex structure and the pair (Ψ_F, G_F) is an almost Kähler structure on T_0N.

We have the following brackets:

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = \frac{\partial N^k_i}{\partial y^j}\frac{\partial}{\partial y^k}, \quad \left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^i_{jk}\frac{\partial}{\partial y^i} \tag{2.9}$$

where the curvature R is:

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}. \tag{2.10}$$

The geometry (N, F) is called *R-flat* if $R^{\cdot}_{\cdot\cdot} = 0$ which means the integrability of the distribution $H(T_0N)$; in the Riemannian case this means that the Riemannian metric g is flat. The tensors P_F, Ψ_F are not homogeneous and their homogeneous variant is studied in [6]. It is well-known that these ε-structures are integrable if and only if (N, F) is R-flat. From (2.9) it follows also that: $J \circ [\cdot, \cdot] = 0$ on T_0N.

The main 1-form of the Riemann-Finsler geometry is the Cartan 1-form [4, p. 410]:

$$\omega_F = d_J F^2 \tag{2.11}$$

which is not closed; its differential $d\omega_F := \theta_F$ is the symplectic form which yields the Hamiltonian nature of the geodesic spray with respect to the regular Lagrangian F^2. The 1-form: $\omega^*_F := \omega_F \circ \Psi_F$ was considered in [11] under the name of Ψ_F-*adjoint* of ω_F. The almost analyticity and the hermitian properties of ω_F are given by:

Proposition 2.1 i) *ω_F is not almost analytic with respect to P_F but is a P_F-hermitian form.* ii) *ω_F is not almost analytic with respect to Ψ_F but is a Ψ_F-hermitian form.*

Proof i) ω_F is almost analytic with respect to P_F if and only if:

$$\theta_F(P_F X, Y) = d(\omega_F \circ P_F)(X, Y) \tag{2.12}$$

for all $X, Y \in \Gamma(T_0N)$ which means:

$$P_F X(JY(F^2)) - J([P_F X, Y])(F^2) = X(J \circ P_F Y(F^2)) - J \circ P_F([X, Y])(F^2) \tag{2.13}$$

or equivalently, with (2.7):

$$J([vX, Y])(F^2)(= 0) = vX(JY(F^2)). \tag{2.14}$$

The only non-null case of vX and JY is provided by: $X = \frac{\partial}{\partial y^j}$, $Y = \frac{\delta}{\delta x^i}$ for which (2.14) means $0 = 2g_{ij}$ impossible. The P_F-hermitian property means:

$$P_F X(JY(F^2)) - Y(J \circ P_F X(F^2)) - J([P_F X, Y])(F^2) + X(J \circ P_F Y(F^2)) - P_F Y(JX(F^2)) -$$
$$-J([X, P_F Y])(F^2) = 0 \tag{2.15}$$

for all $X, Y \in \Gamma(T_0 N)$; or equivalently, after a straightforward computation as before:

$$2hX(JY(F^2)) - 2hY(JX(F^2)) = J([P_F X, Y] + [X, P_F Y])(F^2) = 0. \tag{2.16}$$

The only non-null case of hX, JX, hY, JY is provided by: $X = \frac{\delta}{\delta x^i}$, $Y = \frac{\delta}{\delta x^j}$ for which (2.16) reads:

$$\frac{\delta}{\delta x^j}\left(\frac{\partial F^2}{\partial y^i}\right) = \frac{\delta}{\delta x^i}\left(\frac{\partial F^2}{\partial y^j}\right) \tag{2.17}$$

which, written again globally, means $d_h \omega_F = 0$. Equivalently, if $X = hX$ and $Y = hY$ then (2.16) means:

$$hX(JhY(F^2)) = hY(JhX(F^2)). \tag{2.18}$$

ii) ω_F is almost analytic with respect to Ψ_F if and only if:

$$\theta_F(\Psi_F X, Y) = -d(\omega \circ \Psi_F)(X, Y) \tag{2.19}$$

for all $X, Y \in \Gamma(T_0 N)$; equivalently:

$$\Psi_F X(JY(F^2)) - J([\Psi_F X, Y])(F^2) = X(J \circ \Psi_F Y(F^2)) - J \circ \Psi_F([X, Y])(F^2). \tag{2.20}$$

Since:

$$J \circ \Psi_F = -v \tag{2.21}$$

the last equation is:

$$\Psi_F X(JY(F^2)) = v([X, Y])(F^2) - X(vY(F^2)). \tag{2.22}$$

Due to the presence of both JY and vY we must discuss the both cases I: $Y \in H(T_0 N)$ and II: $Y \in V(T_0 N)$.
I) let $Y = \frac{\delta}{\delta x^j}$. Then (2.22) is:

$$\Psi_F X\left(\frac{\partial F^2}{\partial y^j}\right) = v\left([X, \frac{\delta}{\delta x^j}]\right)(F^2) = [X, \frac{\delta}{\delta x^j}](F^2) \tag{2.23}$$

With $X = \frac{\delta}{\delta x^i}$ the equation (2.23) reads: $2g_{ij} = R^k_{ij}\frac{\partial F^2}{\partial y^k} = R^k_{ij} y_k$, which is impossible since the left hand side is 0-homogeneous while the right hand side is 2-homogeneous.
II) let $Y = \frac{\partial}{\partial y^j}$. Hence (2.22) is:

$$0 = v\left([X, \frac{\partial}{\partial y^j}]\right)(F^2) - X\left(\frac{\partial F^2}{\partial y^j}\right) \tag{2.24}$$

which means:

$$X\left(\frac{\partial F^2}{\partial y^j}\right) = [X, \frac{\partial}{\partial y^j}](F^2). \tag{2.25}$$

With $X = \frac{\partial}{\partial y^i}$ the equation (2.25) reads: $2g_{ij} = 0$ again impossible.
The Ψ_F-hermitian property means:

$$\Psi_F X(JY(F^2)) - Y(J \circ \Psi_F X(F^2)) - J([\Psi_F X, Y])(F^2) + X(J \circ \Psi_F Y(F^2)) - \Psi_F Y(JX(F^2)) -$$
$$-J([X, \Psi_F Y])(F^2) = 0 \tag{2.26}$$

for all $X, Y \in \Gamma(T_0N)$ or equivalently:

$$\Psi_F X(JY(F^2)) + Y(vX(F^2)) = \Psi_F Y(JX(F^2)) + X(vY(F^2)). \tag{2.27}$$

I) let $Y = \frac{\delta}{\delta x^j}$. Then (2.27) is:

$$\Psi_F X(\frac{\partial F^2}{\partial y^j}) + \frac{\delta}{\delta x^j}(vX(F^2)) = \frac{\partial}{\partial y^j}(JX(F^2)). \tag{2.28}$$

For $X = \frac{\delta}{\delta x^i}$ we have the true equality $2g_{ij} = 2g_{ij}$ while for $X = \frac{\partial}{\partial y^i}$ we have the true equality (2.17).
II) let $Y = \frac{\partial}{\partial y^j}$. Hence (2.27) is:

$$\frac{\partial}{\partial y^j}(vX(F^2)) = -\frac{\delta}{\delta x^j}(JX(F^2)) + X\left(\frac{\partial F^2}{\partial y^j}\right). \tag{2.29}$$

For $X = \frac{\delta}{\delta x^i}$ we have the true equality (2.17) while for $X = \frac{\partial}{\partial y^i}$ we have the true equality $2g_{ij} = 2g_{ij}$. □

The result of Proposition 2.1 means that from the point of view of both analyticity and the hermitian properties the endomorphisms P_F and Ψ_F are similar although, as the computations reveal, the para-complex structure is more easy to handle.

3 Almost para-CR and almost CR equations in a Riemann-Finsler geometry

Now, we study the almost analytic functions in Riemann-Finsler geometry:

Proposition 3.1 *Let (N, F) be a Riemann-Finsler geometry such that the* 1*-de Rham cohomology space of tangent bundle is zero, $H^1(TN) = 0$. Then:*
i) *$f \in C^\infty(T_0N)$ is almost analytic with respect to P_F if and only if the* 1*-form $d_{P_F} f$ is closed,*
ii) *$f \in C^\infty(T_0N)$ is almost analytic with respect to Ψ_F if and only if the* 1*-form $d_{\Psi_F} f$ is closed.*

Proof Due to the hypothesis regarding cohomology we have that the almost para-CR equation $d\hat{f} = df \circ P_F$ is equivalent with $d(d_{P_F} f) = 0$ while the almost CR-equation $d\hat{f} = df \circ \Psi_F$ is equivalent with $d(d_{\Psi F} f) = 0$. Both differentials d_{P_F}, d_{Ψ_F} work on scalar fields from T_0N. □

Let us remark that in local coordinates:
i) *the almost para-CR equation* $d\hat{f} = df \circ P_F$ means:

$$\frac{\partial \hat{f}}{\partial y^j} = -\frac{\partial f}{\partial y^j}, \quad \frac{\delta \hat{f}}{\delta x^i} = \frac{\delta f}{\delta x^i}. \tag{3.1}$$

ii) *the almost CR-equation* $d\hat{f} = df \circ \Psi_F$ means:

$$\frac{\partial \hat{f}}{\partial y^j} = -\frac{\delta f}{\delta x^j}, \quad \frac{\delta \hat{f}}{\delta x^i} = \frac{\partial f}{\partial y^i}. \tag{3.2}$$

and inspired by Proposition 3.1 we introduce a weaker variant of almost analyticity through:

Definition 3.2 For an arbitrary Riemann-Finsler geometry (N, F) and $f \in C^\infty(T_0N)$:
i) f is called *weak almost P_F-analytic function* if $d(d_{P_F} f) = 0$, ii) f is called *weak almost Ψ_F-analytic function* if $d(d_{\Psi_F} f) = 0$.

We have:

$$\begin{cases} d(d_{P_F} f)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0, \quad d(d_{P_F} f)(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = 2R^a_{ij} \frac{\partial f}{\partial y^a}, \\ d(d_{P_F} f)(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i}) = 2\frac{\partial}{\partial y^j}\left(\frac{\delta f}{\delta x^i}\right) \end{cases} \tag{3.3}$$

$$\begin{cases} d(d_{\Psi_F} f)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = -\frac{\partial}{\partial y^i}\left(\frac{\delta f}{\delta x^j}\right) + \frac{\partial}{\partial y^j}\left(\frac{\delta f}{\delta x^i}\right), \\ d(d_{\Psi_F} f)(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = \frac{\delta}{\delta x^i}\left(\frac{\partial f}{\partial y^j}\right) - \frac{\delta}{\delta x^j}\left(\frac{\partial f}{\partial y^i}\right) + R^a_{ij} \frac{\delta f}{\delta x^a}, \\ d(d_{\Psi_F} f)(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i}) = \frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^j \partial y^i} - \frac{\partial^2 G^a}{\partial y^j \partial y^i} \frac{\delta f}{\delta x^a} \end{cases} \tag{3.4}$$

and hence we derive:

Proposition 3.3 i) *The function $f \in C^\infty(T_0N)$ is weak almost P_F-analytic function if and only if the functions $\frac{\delta f}{\delta x^i}$ depends only on x i.e. are functions on the base N and:*

$$R^a_{ij} \frac{\partial f}{\partial y^a} = 0. \tag{3.5}$$

In particular, if $H^1(T_0N) = 0$ then: i1) *any $f \in C^\infty(N)$ is almost analytic with respect to P_F with $\hat{f}_C = f + C \in C^\infty(N)$ for an arbitrary constant C; hence the 1-form df is an $+1$-eigenvector for P^*_F.*
i2) *Moreover, if (N, F) is R-flat then $f = \varphi(F)$ with an arbitrary smooth $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is analytic with respect to P_F with*:

$$\hat{f}(x, y) = -\int_{t=0}^{1} \varphi'(F(x, ty))F(x, ty)dt. \tag{3.6}$$

ii) *The function $f \in C^\infty(T_0N)$ is weak almost Ψ_F-analytic function if and only if*:

$$\begin{cases} \frac{\partial}{\partial y^i}\left(\frac{\delta f}{\delta x^j}\right) = \frac{\partial}{\partial y^j}\left(\frac{\delta f}{\delta x^i}\right), \\ \frac{\delta}{\delta x^i}\left(\frac{\partial f}{\partial y^j}\right) - \frac{\delta}{\delta x^j}\left(\frac{\partial f}{\partial y^i}\right) + R^a_{ij} \frac{\delta f}{\delta x^a} = 0, \\ \frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^j \partial y^i} - \frac{\partial^2 G^a}{\partial y^j \partial y^i} \frac{\delta f}{\delta x^a} = 0. \end{cases} \tag{3.7}$$

Example 3.4 (Riemannian geometry) Suppose that (N, g) is a Riemannian manifold. Then (3.5) is:

$$R^a_{ijk}(x) y^k \frac{\partial f}{\partial y^a}(x, y) = 0 \tag{3.8}$$

with $R^{\cdot}_{\cdots}$ the Riemannian curvature tensor field of g. Also, (3.7_3) becomes:

$$\frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^j \partial y^i} = \gamma^a_{ji}(x) \frac{\delta f}{\delta x^a} \tag{3.9}$$

and if we search for functions f on the base N we obtain:

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \gamma^a_{ij}\frac{\partial f}{\partial x^a} \tag{3.10}$$

which means that the Hessian of f with respect to g vanishes i.e. f are *the totally geodesic functions* of the geometry (N, g). It results immediately the expression of $\hat{f}$ from (3.2):

$$\hat{f}(x, y) = -\frac{\partial f}{\partial x^i}(x)y^i = -f^C(x, y) \tag{3.11}$$

where f^C means the complete lift of f to TN. The condition (3.8) or (3.7_2) means for a base function $f = f(x)$:

$$R^a_{ijk}(x)\frac{\partial f}{\partial x^a}(x) = 0 \tag{3.12}$$

and we remark that in the flat case this condition is satisfied.

We can provide an interpretation for (3.12) as follows: for any vector fields X, Y, Z on N the application of the Riemannian curvature $R(\cdot,\cdot)\cdot$ yields the vector field $R(X, Y)Z$; hence (3.12) means that f is a first integral for all the vector fields $R(X, Y)Z$. In conclusion, if for any $x \in N$ the curvature $R_x(\cdot,\cdot)\cdot$ spans the whole tangent space T_xN then h must be a constant; this is the case of non-flat space-forms. The same formula (3.12) is globally expressed as $dh \circ R = 0$ and it is obtained in [7, p. 175] as the characterization condition which assures that the vertical lift of a totally geodesic function h is totally geodesic with respect to G_F. □

Inspired by this example we introduce two new types of Hessian for $f \in C^\infty(T_0N)$ in a Riemann-Finsler geometry:
a) *the almost para-complex Hessian*:

$$PCH^f_{ij} := 2\frac{\partial}{\partial y^i}\left(\frac{\delta f}{\delta x^j}\right) = 2d(d_hf)\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) \tag{3.13}$$

with d_h the horizontal component of the total differential d of T_0N i.e. the differential with respect to the horizontal projector h.
b) *the almost complex Hessian*:

$$CH^f_{ij} := \frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^i \partial y^j} - \frac{\partial^2 G^a}{\partial y^i \partial y^j}\frac{\delta f}{\delta x^a}. \tag{3.14}$$

Remarks 3.5 i) Let us point out that another expression for these operators is:

$$PCH_{ij} = \frac{\partial}{\partial y^i} \circ P_F \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \circ P_F \frac{\partial}{\partial y^i}, \quad CH_{ij} = \frac{\partial}{\partial y^i} \circ \Psi_F \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \circ \Psi_F \frac{\partial}{\partial y^i} \tag{3.15}$$

and an unpleasant characteristic of them is the non-symmetry; the Hessian PCH^f is symmetric if and only if (3.7_1) holds while the Hessian CH is symmetric if and only if the geometry (N, F) is R-flat.

Let us point out that (3.7_1) holds if f is the complete lift h^C of a smooth base function $h \in C^\infty(N)$ since in this case we have:

$$\frac{\partial}{\partial y^i}\left(\frac{\delta f}{\delta x^j}\right) = \frac{\partial^2 h}{\partial x^i \partial x^j} - \frac{\partial^2 G^a}{\partial y^i \partial y^j}\frac{\partial h}{\partial x^a} = \frac{\partial}{\partial y^j}\left(\frac{\delta f}{\delta x^i}\right) \tag{3.16}$$

and again, in the Riemannian case, the middle term is the Hessian of h.
ii) The contraction of these Hessians with y^i and y^j yields the following scalar fields:

$$PCH^f_{00} := 2y^iy^jPCH^f_{ij} = 2S_F(\mathbb{C}(f)), CH^f_{00} := y^iy^jCH^f_{ij} = S_F(S_F(f)) + \mathbb{C}(\mathbb{C}(f)) - \mathbb{C}(f) \quad (3.17)$$

where we use the 2-homogeneity of S_F, $[\mathbb{C}, S_F] = S_F$, and that of G^a: $N^a_u y^u = 2G^a$. In particular, if f is r-homogeneous with respect to y i.e. $\mathbb{C}(f) = rf$ then:

$$PCH^f_{00} = 2rS_F(f), \quad CH^f_{00} = S_F(S_F(f)) + (r^2 - r)f \quad (3.18)$$

and if f is a Rayleigh dissipation function of S_F, namely $S_F(f) < 0$ according to [9, p. 1558], then PCH^f_{00} has the opposite sign of r.
iii) Concerning with the second part of (3.13) let us remark that for any $f \in C^\infty(T_0N)$:

$$d(d_{P_F}f) = 2d(d_hf) \quad (3.19)$$

and hence we obtain another characterization: $f \in C^\infty(T_0N)$ is weak almost P_F-analytic if and only if $d(d_hf) = 0$.

It follows that if f is *horizontally constant* i.e. $d_hf = 0$ then f is weak almost P_F-analytic. For example F is horizontally constant and if the geodesic spray S_F is Ricci-constant conform [4, p. 407] then the Ricci scalar function $R \in C^\infty(T_0N)$, defined in the cited paper as the trace of the Jacobi tensor (see below), is also horizontally constant. □

It follows also new types of Laplacians as trace of these Hessians:
c) *the almost para-complex Laplacian*:

$$PC\Delta(f) := Tr_gPCH^f = g^{ij}PCH^f_{ij} = 2g^{ij}\frac{\partial}{\partial y^i}\left(\frac{\delta f}{\delta x^j}\right) \quad (3.20)$$

d) *the almost complex Laplacian*:

$$C\Delta(f) := Tr_gCH^f = g^{ij}CH^f_{ij} = g^{ij}\left(\frac{\delta^2 f}{\delta x^j\delta x^i} + \frac{\partial^2 f}{\partial y^i\partial y^j} - \frac{\partial^2 G^a}{\partial y^i\partial y^j}\frac{\delta f}{\delta x^a}\right). \quad (3.21)$$

In [17, p. 68] is affirmed that this is exactly the Riemannian Laplacian with respect to the Riemannian metric G_F. We discuss this statement in the following section.

Examples 3.6 i) We have $PC\Delta(F^2) = 0$ and $C\Delta(F^2) = 2n$ since $CH^{F^2}_{ij} = 2g_{ij}$; so F^2 is an almost para-complex-harmonic function and we "hear" the dimension on T_0N. These equalities are similar to the Proposition 3.3. of [27, p. 134] where for a horizontal respectively vertical Laplacian on (N, F) is obtained $\Delta_hF^2 = 0$ and $\Delta_vF^2 = 2n$. Moreover, every horizontally constant function is almost para-complex-harmonic.
ii) (Euclidean geometry) Let $(N, g) = (\mathbb{R}^n, can)$ be the n-dimensional Euclidean space and $f \in C^\infty(T\mathbb{R}^n) = C^\infty(\mathbb{R}^{2n})$. Then:

$$PC\Delta(f) = 2\sum_{i=1}^n \frac{\partial^2 f}{\partial x^i\partial y^i}, \quad C\Delta(f) = \sum_{i=1}^n\left(\frac{\partial^2 f}{\partial(x^i)^2} + \frac{\partial^2 f}{\partial(y^i)^2}\right) = \Delta^{(\mathbb{R}^{2n}, can)}f \quad (3.22)$$

and hence:
ii1) the para-complex-harmonic functions on $T(\mathbb{R}^n, can) = (\mathbb{R}^{2n}, can)$ are: $f(x, y) = a(x) + b(y)$

with smooth a and b,
ii2) the complex-harmonic functions on $T(\mathbb{R}^n, can) = (\mathbb{R}^{2n}, can)$ are exactly the Euclidean harmonic functions.
Let $< \cdot, \cdot >_n$ be the Euclidean inner product on the base $\mathbb{R}^n$ and two fixed vectors $\alpha, \beta \in \mathbb{R}^n$. Then the function:

$$f_{\alpha,\beta}(x, y) := \exp(< \alpha, x >_n + < \beta, y >_n)$$

is an eigenfunction of $PC\Delta$ with the eigenvalue $< \alpha, \beta >_n$ and for $C\Delta$ with the eigenvalue $\|\alpha\|_n^2 + \|\beta\|_n^2$.
iii) (Berwald and Landsberg geometries) Recall after [5, p. 38] that a Finsler geometry is a Berwald one if $G^i = \Gamma^i_{uv}(x)y^u y^v$ and this is the more closed to Riemannian geometry from several reasons one of them being that Riemannian metrics are Berwald; but there exist non-Riemannian Berwald manifolds. Then in such a geometry we have:

$$\begin{cases} PC\Delta(f) = 2g^{ij}\left[\frac{\partial^2 f}{\partial y^i \partial x^j} - \Gamma^a_{ij}(x)\frac{\partial f}{\partial y^a} - \Gamma^a_{ju}(x)y^u \frac{\partial^2 f}{\partial y^a \partial y^i}\right], \\ C\Delta(f) = g^{ij}\left[\frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^i \partial y^j} - \Gamma^a_{ij}(x)\frac{\delta f}{\delta x^a}\right]. \end{cases} \tag{3.23}$$

On the other hand let us denote, following [19], with ∇^{HC} the horizontal covariant derivative for the Cartan or Chern-Rund connection of (N, F). Then on page 6 of the cited preprint is computed the divergence of a vertical gradient $Z = (Z^i = \frac{\partial f}{\partial y^i}) := grad_v f$ as follows:

$$\nabla^{HC} \cdot Z = g^{ij}\left[\frac{\partial}{\partial y^j}\left(\frac{\delta f}{\delta x^i}\right) + L^a_{ij}\frac{\partial f}{\partial y^a}\right] \tag{3.24}$$

where L is *the Landsberg tensor field* of (N, F). The geometry (N, F) is a *Landsberg* one if $L = 0$ and by Proposition 2.1.3 of [5, p. 39] every Berwald geometry is Landsberg. In conclusion, in a Landsberg, particularly Berwald (more particularly Riemann), geometry we have a global formula for $PC\Delta$:

$$PC\Delta(f) = \nabla^{HC} \cdot grad_v(2f). \tag{3.25}$$

From this formula we can introduce the weighted case of $PC\Delta$ following [12]. Let $\mu \in C^\infty(T_0 N)$ with $\mu > 0$. *The weighted divergence* of Z is:

$$\nabla^{HC}_\mu \cdot Z := \frac{1}{\mu}\nabla^{HC} \cdot (\mu Z)$$

and hence *the weighted almost para-complex Laplacian* is:

$$PC\Delta_\mu(f) := \nabla^{HC}_\mu \cdot grad_v(2f) = \frac{1}{\mu}\nabla^{HC} \cdot \mu grad_v(2f).$$

In particular, if $\mu = \mu(x)$ then:

$$PC\Delta_\mu(f) = \frac{1}{\mu}\nabla^{HC} \cdot grad_v(2\mu f) = \frac{1}{\mu}PC\Delta(\mu f) = PC\Delta(f) + \frac{2}{\mu}g^{ij}\frac{\partial \mu}{\partial x^j}\frac{\partial f}{\partial y^i}.$$

For example: $PC\Delta_\mu(F^2) = \frac{4}{\mu}\mu^C$.
iv) (Eigenvalue problems) Let us search $f = \varphi(F^2)$ as eigenfunction for $C\Delta$ with a smooth $\varphi : \mathbb{R}^*_+ \to \mathbb{R}$. Since:

$$C\Delta(f) = 4\varphi'' F^2 + 2n\varphi' \tag{3.26}$$

it follows that $\varphi = \varphi(t)$ solves the eigenvalue problem:

$$4t\varphi'' + 2n\varphi' = \lambda\varphi \tag{3.27}$$

for $\lambda \in \mathbb{R}$. Multiplying with $t^{\frac{n}{2}-1}$ we arrive at:

$$(t^{\frac{n}{2}}\varphi')' = \frac{\lambda}{4}t^{\frac{n}{2}-1}\varphi \tag{3.28}$$

and for strictly negative λ's we have the general solution $\varphi(t) = C_1 J_0(\sqrt{-\lambda t}) + C_2 Y_0(\sqrt{-\lambda t})$ where C_1, C_2 are constants and J_0, Y_0 are the Bessel functions of the first and second kind respectively. For the eigenvalue $\lambda = 0$ it follows the eigenfunction:

$$\varphi_{n\geq 3}(t) = \frac{2C}{2-n}t^{1-\frac{n}{2}}, \quad \varphi_{n=2}(t) = C\ln t, \tag{3.29}$$

with C an arbitrary constant. In conclusion, the function $f = \frac{2C}{2-n}F^{2-n}$, respectively $f = 2C\ln F$, is both almost para-complex-harmonic and almost complex-harmonic. Let us remark that $\lambda = 0$ in (3.27) together with $n = 1$ means $2t\varphi'' + \varphi' = 0$ which, after [2, p. 216], means that $L = \varphi(F^2)$ is a singular Lagrangian on TN; this is a reason for our choice $n \geq 2$ of Section 2.
v) Closely related to the harmonicity is the notion of *Dirichlet energy density*: in a Riemannian geometry (N, g) the energy density of $h \in C^\infty(N)$ is:

$$e(f) := \frac{1}{2}|df|_g = \frac{1}{2}g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \tag{3.30}$$

and the harmonic functions are the critical points of this functional. Hence in a Riemann-Finsler geometry (N, F) for $f \in C^\infty(T_0N)$ its energy density is:

$$e_F(f) := \frac{1}{2}|df|_{G_F} = \frac{1}{2}g^{ij}\left(\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + \frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}\right). \tag{3.31}$$

For example $e_F(F^2) = 2F^2$. Suppose now that $(N, F = F_g)$ is the Riemannian geometry of a Riemannian metric g on N and $f = h^C$. A straightforward computation yields:

$$e_{F_g}(h^C) = \frac{1}{2}g^{ij}\left(\frac{\partial h}{\partial x^i}\frac{\partial h}{\partial x^j} + H^h_{iu}H^h_{jv}y^u y^v\right) \tag{3.32}$$

with H^h the Hessian of h with respect to g. One obtains that if h is totally geodesic with respect to g then:

$$e_{F_g}(h^C) = e_g(h). \tag{3.33}$$

Returning to the general case of (N, F) we point that $e(f)$ arises naturally from the relation:

$$C\Delta(f^2) = 2fC\Delta(f) + 4e(f). \tag{3.34}$$

□

We introduce now two types of gradient Ricci solitons in the tangent bundle of a Riemann-Finsler geometry. There are necessary some notations. More precisely, the Jacobi endomorphism of (N, F) is defined in [4, p. 407] as:

$$\Phi = v \circ \mathcal{L}_{S_F}h = \mathcal{L}_{S_F}h \circ h \tag{3.35}$$

with local expression:

$$\Phi = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j = \left[2\frac{\partial G^i}{\partial x^j} - S_F \left(\frac{\partial G^i}{\partial y^j} \right) - \frac{\partial G^i}{\partial y^a} \frac{\partial G^a}{\partial y^j} \right] \frac{\partial}{\partial y^i} \otimes dx^j \tag{3.36}$$

and then *the Ricci scalar function* is: $Ric = Tr(\Phi) = R^i_i$. Finally, the Ricci tensor field of (N, F) is:

$$Ric_{ij} = \frac{1}{2} \frac{\partial^2 (F^2 Ric)}{\partial y^i \partial y^j} \tag{3.37}$$

and hence we introduce:

Definition 3.7 1) The *symmetric almost para-complex Hessian* is:

$$PCsH_{ij} := \frac{1}{2}(PCH_{ij} + PCH_{ji}) = \frac{\partial}{\partial y^i} \left(\frac{\delta}{\delta x^j} \right) + \frac{\partial}{\partial y^j} \left(\frac{\delta}{\delta x^i} \right). \tag{3.38}$$

The *symmetric almost complex Hessian* is:

$$CsH_{ij} := \frac{1}{2}(CH_{ij} + CH_{ji}) =$$

$$= \frac{1}{2} \left[\frac{\delta}{\delta x^i} \left(\frac{\delta}{\delta x^j} \right) + \frac{\delta}{\delta x^j} \left(\frac{\delta}{\delta x^i} \right) \right] + \frac{\partial^2}{\partial y^i \partial y^j} - \frac{\partial^2 G^k}{\partial y^i \partial y^j} \frac{\delta}{\delta x^k}. \tag{3.39}$$

2) Let $f \in C^\infty(T_0 N)$ and $\lambda \in \mathbb{R}$. Then:
i) the pair (f, λ) is *a weak almost para-complex gradient Ricci soliton* for (N, F) if:

$$PCsH^f_{ij} + Ric_{ij} + \lambda g_{ij} = 0, \tag{3.40}$$

ii) the pair (f, λ) is *a weak almost complex gradient Ricci soliton* for (N, F) if:

$$CsH^f_{ij} + Ric_{ij} + \lambda g_{ij} = 0. \tag{3.41}$$

A contraction with y^i, y^j yields, via (3.17):

$$2S_F(\mathbb{C}(f)) + F^2(Ric + \lambda n) = 0, \quad S_F(S_F(f)) - \mathbb{C}(\mathbb{C}(f)) - \mathbb{C}(f) + F^2(Ric + \lambda n) = 0 \tag{3.42}$$

and we finish this section with a class of such Ricci solitons provided by the constancy of the flag curvature:

Proposition 3.8 *Let (N, F) be a Riemann-Finsler geometry of constant flag curvature c and suppose that f is a weak almost para-complex (respectively weak almost-complex) totally geodesic function. Then the pair $\left(f, \lambda = -\frac{n-1}{n}c\right)$ is a weak almost para-complex (respectively complex) gradient Ricci soliton. In particular, $\left(\frac{2}{2-n}F^{2-n}, \frac{1-n}{n}c\right)$ for $n \geq 3$ or $(2 \ln F, -\frac{c}{2})$ for $n = 2$ is both weak almost para-complex and weak almost complex gradient Ricci soliton.*

Proof If (N, F) has the constant flag curvature then:

$$Ric = (n-1)c, \quad Ric_{ij} = (n-1)cg_{ij} \tag{3.43}$$

and the conclusion follows directly. The last part is a consequence of iv) of Examples 3.6. □

4 Hessian and Laplacian on the tangent bundle of a Riemann-Finsler geometry

Let us denote by $F\nabla$ the Levi-Civita connection of the Riemannian metric G_F and by FH^f the corresponding Hessian of $f \in C^\infty(T_0N)$. Hence:

$$FH^f(X,Y) = X(Y(f)) - F\nabla_X Y(f) \tag{4.1}$$

for every vector fields X, Y on T_0N and in particular we have the horizontal and vertical components:

$$hFH^f_{ij} := FH^f\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right), \quad vFH^f_{ij} := FH^f\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right). \tag{4.2}$$

It follows:

$$hFH^f_{ji} = \frac{\delta^2 f}{\delta x^j \delta x^i} - F\nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i}(f), \quad vFH^f_{ji} = \frac{\partial^2 f}{\partial y^j \delta y^i} - F\nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i}(f) \tag{4.3}$$

and hence the Riemann-Finsler Hessian of f is:

$$F\Delta(f) = g^{ij}(hFH^f_{ji} + vFH^f_{ji}) = g^{ij}\left[\frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^i \partial y^j} - F\nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i}(f) - F\nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i}(f)\right] \tag{4.4}$$

and the expression of $F\nabla$ appears in [3, p. 228]. Comparing with our (3.21) it results a very different expression; for example $F\Delta(F^2) = 2n + g^{ij}R^a_{ij}g_{ak}y^k$.

But we can express our Hessian CH^f using the theory of Finslerian connections which are triples $\Gamma = (N^k_i, F^k_{ij}(x,y), C^k_{ij}(x,y))$ where $F^{\cdot}_{\cdot\cdot}$ behave like the coefficients of a linear connection and C is a tensor field on T_0N. Such a Finslerian connection yields the linear connection $\Gamma\Delta$ on T_0N given by:

$$\begin{cases} \Gamma\Delta_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} := F^k_{ij} \frac{\delta}{\delta x^k}, & \Gamma\Delta_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} := F^k_{ij} \frac{\partial}{\partial y^k} \\ \Gamma\Delta_{\frac{\partial}{\partial y^j}} \frac{\delta}{\delta x^i} := C^k_{ij} \frac{\delta}{\delta x^k}, & \Gamma\Delta_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} := C^k_{ij} \frac{\partial}{\partial y^k}. \end{cases} \tag{4.5}$$

It follows the corresponding Hessian again with horizontal-horizontal and vertical-vertical components:

$$\begin{cases} h\Gamma H^f_{ji} := \Gamma H^f\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = \frac{\delta^2 f}{\delta x^j \delta x^i} - F^k_{ij} \frac{\delta f}{\delta x^k}, \\ v\Gamma H^f_{ji} := \Gamma H^f\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = \frac{\partial^2 f}{\partial y^j \partial y^i} - C^k_{ij} \frac{\partial f}{\partial y^k} \end{cases} \tag{4.6}$$

which is symmetric if and only if (N, F) is flat and $\Gamma\Delta$ is h- and v-symmetrical i.e. $F^k_{ij} = F^k_{ji}$ respectively $C^k_{ij} = C^k_{ji}$. The vertical-horizontal component of the Hessian is:

$$vh\Gamma H^f_{ij} := \Gamma H^f\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = \frac{\partial}{\partial y^i}\left(\frac{\delta f}{\delta x^j}\right) - C^k_{ij} \frac{\delta f}{\delta x^k}. \tag{4.7}$$

We derive also the corresponding Laplacians:

$$\Gamma\Delta(f) = g^{ij}\left(h\Gamma H^f_{ji} + v\Gamma H^f_{ji}\right), \quad \Gamma\Delta_{vh}(f) = g^{ij} vh\Gamma H^f_{ij}. \tag{4.8}$$

There are four remarkable Finslerian connections, [3, p. 227]:
-Cartan $Ca = (N^k_i, F^k_{ij}, C^k_{ij})$,

-Chern-Rund $CR = (N_i^k, F_{ij}^k, 0)$,
-Berwald $B = (N_i^k, G_{ij}^k, 0)$,
-Hashiguchi $H = (N_i^k, G_{ij}^k, C_{ij}^k)$,
where:

$$F_{ij}^k = \frac{1}{2} g^{ka} \left(\frac{\delta g_{aj}}{\delta x^i} + \frac{\delta g_{ia}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^a} \right), \quad C_{ij}^k = \frac{1}{2} g^{ka} \frac{\partial g_{ij}}{\partial y^a}, \quad G_{ij}^k = \frac{\partial^2 G^k}{\partial y^i \partial y^j}. \tag{4.9}$$

In conclusion we have six remarkable Laplacians in addition to the Riemann-Finsler Laplacian $F\Delta$:

$$\begin{cases} Ca\Delta(f) = g^{ij} \left(\frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^j \partial y^i} - F_{ij}^k \frac{\delta f}{\delta x^k} - \frac{1}{2} g^{ka} \frac{\partial g_{ij}}{\partial y^a} \frac{\partial f}{\partial y^k} \right) \\ CR\Delta(f) = g^{ij} \left(\frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^j \partial y^i} - F_{ij}^k \frac{\delta f}{\delta x^k} \right) \\ B\Delta(f) = g^{ij} \left(\frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^j \partial y^i} - \frac{\partial G^k}{\partial y^i \partial y^j} \frac{\delta f}{\delta x^k} \right) \\ H\Delta(f) = g^{ij} \left(\frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^j \partial y^i} - \frac{\partial G^k}{\partial y^i \partial y^j} \frac{\delta f}{\delta x^k} - \frac{1}{2} g^{ka} \frac{\partial g_{ij}}{\partial y^a} \frac{\partial f}{\partial y^k} \right) \\ Ca\Delta_{vh}(f) = H\Delta_{vh}(f) = g^{ij} \left[\frac{\partial}{\partial y^i} \left(\frac{\delta f}{\delta x^j} \right) - C_{ij}^k \frac{\delta f}{\delta x^k} \right] \\ CR\Delta_{vh}(f) = B\Delta_{vh}(f) = g^{ij} \frac{\partial}{\partial y^i} \left(\frac{\delta f}{\delta x^j} \right). \end{cases} \tag{4.10}$$

It results that our almost para-complex Laplacian is $2CR\Delta_{vh} = 2B\Delta_{vh}$ and the almost complex Laplacian $C\Delta$ is exactly the Berwald Laplacian $B\Delta$.

These Laplacians have the same value on F^2:

$$\begin{cases} PC\Delta(F^2) = Ca\Delta_{vh}(F^2) = CR\Delta_{vh} = B\Delta_{vh}(F^2) = H\Delta_{vh}(F^2) = 0, \\ Ca\Delta(F^2) = CR\Delta(F^2) = B\Delta(F^2) = H\Delta(F^2) = 2n \end{cases} \tag{4.11}$$

and since the characterization of Landsberg geometry is $F_{ij}^k = G_{ij}^k$, conform [3, p. 230], it follows that:

$$\textit{Landsberg}: \quad Ca\Delta = H\Delta, \quad CR\Delta = B\Delta. \tag{4.12}$$

In the particular case of Riemannian geometry we have $F_{ij}^k = G_{ij}^k = \gamma_{ij}^k(x)$ and then:

$$\textit{Riemannian}: Ca\Delta(f) = CR\Delta(f) = B\Delta(f) = H\Delta(f) = g^{ij} \left(\frac{\delta^2 f}{\delta x^j \delta x^i} + \frac{\partial^2 f}{\partial y^i \partial y^j} - \gamma_{ij}^a \frac{\delta f}{\delta x^a} \right). \tag{4.13}$$

For example let $f = h^C$ and denotes H^h the Hessian of $h \in C^\infty(N)$ with respect to the Riemannian metric $g(x)$. Then:

$$\cdot\Delta(h^C) = g^{ij} y^a \left(\frac{\partial H_{ia}^h}{\partial x^j} - \gamma_{ja}^k H_{ik}^h - \gamma_{ij}^k H_{ka}^h \right) \tag{4.14}$$

and we recognize in the right-hand-side the Christoffel process of the metric g. Hence if H^h is parallel with respect to g, in particular h is totally geodesic with respect to g or H^h is multiple of g, then h^C is harmonic with respect to $\cdot\Delta$.

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References

[1] D. Angella, *Cohomological aspects in complex non-Kähler geometry*, Lecture Notes in Mathematics, 2095, Springer, Cham, 2014. MR3137419 (Cited on p. 7.)

[2] P. L. Antonelli; D. Hrimiuc, *φ-Lagrange Laplacians*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), 43(1997), no. 2, 215-221 (1998). MR1679419 (2001a:58046) (Cited on p. 15.)

[3] A. Bejancu; H. R. Farran, *Foliations and geometric structures*, Mathematics and Its Applications (Springer), 580, Springer, Dordrecht, 2006. MR2190039 (2006j:53034) (Cited on p. 8, 18.)

[4] I. Bucataru; Z. Muzsnay, *Sprays metrizable by Finsler functions of constant flag curvature*, Differential Geom. Appl., 31(2013), no. 3, 405-415. MR3049635 (Cited on p. 9, 14, 16.)

[5] S.-S. Chern, Z. Shen, *Riemann-Finsler geometry*, Nankai Tracts in Mathematics, 6, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. MR2169595 (2006d:53094) (Cited on p. 8, 15.)

[6] M. Crampin, *Kähler and para-Kähler structures associated with Finsler spaces of non-zero constant flag curvature*, Houston J. Math., 34(2008), no. 1, 99-114. MR2383698 (2009c:53104) (Cited on p. 9.)

[7] M. Crasmareanu, *Killing potentials*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), 45(1999), no. 1, 169-176 (2000). MR1813279 (2001j:53044) (Cited on p. 13.)

[8] M. Crasmareanu, *Nonlinear connections and semisprays on tangent manifolds*, Novi Sad J. Math., 33(2003), no. 2, 11-22. MR2046158 (2004m:53030) (Cited on p. 9.)

[9] M. Crasmareanu, *Rayleigh dissipation from the general recurrence of metrics in path spaces*, Nonlinear Anal. Real World Appl., 13(2012), no. 4, 1551-1561. MR2890992 (Cited on p. 14.)

[10] M. Crasmareanu, *New tools in Finsler geometry: stretch and Ricci solitons*, Math. Rep. (Bucur.), 16(66)(2014), no. 1, 83-93. MR3304401 (Cited on p. 2.)

[11] M. Crasmareanu, *Semi-basic 1-forms and Courant structure for metrizability problems*, Publ. Inst. Math. (Beograd) (N.S.), 98(112)(2015), 153-163. (Cited on p. 9.)

[12] M. Crasmareanu, *Last multipliers on weighted manifolds and the weighted Liouville equation*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 77(2015), no. 3, 53-58. (Cited on p. 15.)

[13] M. Crasmareanu; C. Ida, *Almost analyticity on almost (para) complex Lie algebroids*, Results Math., 67(2015), no. 3-4, 495-519. MR3343858 (Cited on p. 2.)

[14] M. Crasmareanu; C. Ida, *Almost analytic forms with respect to a quadratic endomorphism and their cohomology*, Turkish J. Math., 39(2015), no. 3, 322-334. MR3343487 (Cited on p. 1, 5.)

[15] M. Crasmareanu; L.-I. Piscoran, *Para-CR structures of codimension 2 on tangent bundles in Riemann-Finsler geometry*, Acta Math. Sin. (Engl. Ser.), 30(2014), no. 11, 1877-1884. MR3266784 (Cited on p. 8.)

[16] M. Crasmareanu; L.-I. Piscoran, *CR structures of codimension* 2 *on tangent bundles in Riemann-Finsler geometry*, Period. Math. Hungar., 73(2016), no. 2, 240-250. (Cited on p. 9.)

[17] S. Dragomir; B. Larato, *Harmonic functions on Finsler spaces*, Istanbul Üniv. Fen Fak. Mat. Derg., 48(1987/89), 6776, (1991). MR1119678 (92g:53014) (Cited on p. 2, 14.)

[18] C. Laurent-Gengoux; A. Pichereau; P. Vanhaecke, *Poisson structures*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 347. Springer, Heidelberg, 2013. MR2906391 (Cited on p. 2, 3, 4.)

[19] E. Minguzzi, *A divergence theorem for pseudo-Finsler spaces*, arXiv: 1508.06053. (Cited on p. 15.)

[20] Gh. Pitis; D. Smaranda, *Formes presque analytiques sur une variété presque complexe*, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 21(1991), no. 2, 23-29. MR1295614 (95h:58001) (Cited on p. 1.)

[21] C. Simionescu; Gh. Pitis, *Some properties of almost analytic forms*, Bul. Univ. Brasov Ser. C, 31(1989), 75-80. MR1048202 (91b:53085) (Cited on p. 1.)

[22] D. Smaranda, *The algebra of almost analytic forms and criteries of almost analyticity*, in: Special chapters in differential geometry, University of Bucharest, 1981, 67-87. (in Romanian) (Cited on p. 1.)

[23] S. Tachibana, *Analytic tensor and its generalization*, Tôhoku Math. J. (2), 12(1960), 208-221. MR0158346 (28 #1571) (Cited on p. 5.)

[24] S. Tachibana; S. Kotô, *On almost-analytic functions, tensors and invariant subspaces*, Tôhoku Math. J. (2), 14(1962), 177-186. MR0141062 (25 #4475) (Cited on p. 5.)

[25] K. Yano, *Differential geometry on complex and almost complex spaces*, International Series of Monographs in Pure and Applied Mathematics, vol. 49, Pergamon Press, New York, 1965. MR0187181 (32 #4635) (Cited on p. 1, 5.)

[26] K. Yano; M. Ako, *On certain operators associated with tensor fields*, Kodai Math. Sem. Rep., 20(1968), 414-436. MR0234370 (38 #2687) (Cited on p. 5.)

[27] C. Zhong; T. Zhong, *Horizontal Laplace operator in real Finsler vector bundles*, Acta Math. Sci. Ser. B Engl. Ed., 28(2008), no. 1, 128-140. MR2389588 (2009a:58049) (Cited on p. 2, 14.)

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