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From rotations of conics to a class of Riemann-Finslerian flows

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Dedicated to Academician Radu Miron on the occasion of his 91'th birthday

Abstract. The aim of this paper is to produce new examples of (semi-) Riemannian and Finsler structures in dimension two having as model a scalar deformation of conics which generalizes the rotation with a right angle. It continues [6] and [8] from the point of view of relationship between quadratic polynomials (which provide equations of conics in dimension 2) and Finsler geometries. A type of two-dimensional Finslerian flow is introduced, based on the previous deformation and we completely solve the corresponding particular case of Riemannian flow.

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Introduction

Two recent papers [6] and [8], devoted to Finsler geometry, start with a deformation of a conic Γ obtained by deforming the gradient vector field for the quadratic form defining Γ. These deformation are inspired by the scaling (linear) transformation of Computer Graphics: $(x, y) \in \mathbb{R}^2 \to (\lambda_x \cdot x, \lambda_y \cdot y) \in \mathbb{R}^2$, following [13, p. 136]. The well-known invariants from the Euclidean geometry of conics are computed for these new conics which depend on two scalars denoted α and β.

In this following note we present another type of deformation based on the well-known rotation of the plane. More precisely, we consider the consider the linear transformation $(x, y) \to (-\alpha y, \beta x)$, which for $\alpha = \beta = 1$ is the trigonometric rotation with the right angles. We call (α, β)-*rotated* the new conic and the diagonal case $\alpha = \beta$ is particularly analyzed, with a special view towards the trigonometric case $\alpha = \beta = 1$. Moreover, we treat this deformation in terms of complex numbers.

In the next section we move to the Riemann-Finslerian framework of dimension two and consider the deformation inspired by the previous section. We finish this paper with a type of Finslerian flows which can be the starting point of future studies following the way opened by the famous Ricci flow of Riemannian geometry, [4]. Due to the complex form of Finslerian deformation even in the Randers case, we can solve completely only the corresponding particular case of Riemannian flows. The solution is a time-dependent metric and a case of decreasing area is pointed out. We remark that in dimension four some recent bi-metric approaches of spacetime geometries appear in [1]-[2] and [3] while a geometrical study in arbitrary dimension is the very old paper [10].

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1. The generalized rotation of conics

In the two-dimensional Euclidean space $\mathbb{R}^2$ let us consider the conic Γ implicitly defined by $f \in C^{\infty}\left(\mathbb{R}^2\right)$ as: $\Gamma = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$ where f is a quadratic function of the form $f(x,y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$ with $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$.

Definition 1.1 Fix the scalars α, β with $\alpha\beta \neq 0$. The (α,β)-*rotation* of Γ is the conic:

$$\begin{cases} \Gamma^r = \Gamma^r_{\alpha,\beta} : f^r(x,y) := f(-\alpha y, \beta x) = 0, \\ f^r(x,y) = (\beta^2 r_{22})x^2 + 2(-\alpha\beta r_{12})xy + (\alpha^2 r_{11})y^2 + 2(\beta r_{20})x + 2(-\alpha r_{10})y + r_{00}. \end{cases} \tag{1.1}$$

Examples 1.2: i) Fix other non-vanishing scalars a, b. The ellipse $E(a,b)$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ and the hyperbola $H(a,b)$: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ have the following (α,β)-rotation:

$$E^r_{\alpha,\beta} : \frac{\beta^2 x^2}{b^2} + \frac{\alpha^2 y^2}{a^2} - 1 = 0, \quad H^r_{\alpha,\beta} : \frac{\beta^2 x^2}{b^2} - \frac{\alpha^2 y^2}{a^2} + 1 = 0. \tag{1.2}$$

Hence E^r is also an ellipse and H^r is a hyperbola. The equilateral hyperbola $\Gamma : xy = C = constant$ has the (α,β)-rotation:

$$\Gamma^r : \alpha\beta xy = -C \tag{1.3}$$

which is also an equilateral hyperbola.
ii) For $p > 0$ let the parabola $P(p) : y^2 - 2px = 0$. Its (α,β)-rotation is:

$$P^r_{\alpha,\beta} : x^2 + 2\frac{\alpha p}{\beta^2} y = 0 \tag{1.4}$$

which is also a parabola.
iii) Consider again the ellipse $E(a,b)$ with $a > b > 0$. The family of all *confocal* conics with $E(a,b)$ is given by:

$$\Gamma_\lambda : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} - 1 = 0 \tag{1.5}$$

for $\lambda \in \mathbb{R} \setminus \{a,b\}$. The (α,β)-rotation of Γ_λ is:

$$(\Gamma_\lambda)^r_{\alpha,\beta} : \frac{\beta^2 x^2}{b-\lambda} + \frac{\alpha^2 y^2}{a-\lambda} - 1 = 0. \tag{1.6}$$

□

In order to study the (α,β)-rotations we recall the algebraic invariants associated to Γ:

$$\Delta = \begin{vmatrix} r_{11} & r_{12} & r_{10} \\ r_{12} & r_{22} & r_{20} \\ r_{10} & r_{20} & r_{00} \end{vmatrix}, \quad D = \delta + I r_{00} - r_{10}^2 - r_{20}^2, \quad I = r_{11} + r_{22}, \quad \delta = r_{11}r_{22} - r_{12}^2. \tag{1.7}$$

More precisely, the main result of this Section follows directly:

Theorem 1.3 *The new conic* $\Gamma^r_{\alpha,\beta}$ *has the following invariants:*

$$I^r = \alpha^2 r_{11} + \beta^2 r_{22}, \delta^r = (\alpha\beta)^2\delta, D^r = (\alpha\beta)^2\delta + \alpha^2(r_{11}r_{00} - r_{10}^2) + \beta^2(r_{22}r_{00} - r_{20}^2),$$
$$\Delta^r = (\alpha\beta)^2\Delta. \tag{1.8}$$

Then the initial conic Γ *and* Γ^r *have the same nature.*

A special attention deserves the diagonal case $\alpha = \beta$ for which we have:

$$I^r = \alpha^2 I, \quad \delta^r = \alpha^4 \delta, \quad D^r = (\alpha^4 - \alpha^2)\delta + \alpha^2 D, \quad \Delta^r = \alpha^4 \Delta. \tag{1.9}$$

By performing a second rotation for this last case we obtain:

$$\left(\Gamma^r_{\alpha,\alpha}\right)^r_{\alpha,\alpha} : \alpha^4(r_{11}x^2 + 2r_{12}xy + r_{22}y^2) - \alpha^2(2r_{10}x + 2r_{20}y) + r_{00} = 0. \tag{1.10}$$

The trigonometric case $\alpha = \beta = 1$ gives for the last two equations:

$$I^r = I, \quad \delta^r = \delta, \quad D^r = D, \quad \Delta^r = \Delta,$$

$$\left(\Gamma^r_{1,1}\right)^r_{1,1} : r_{11}x^2 + 2r_{12}xy + r_{22}y^2 - 2r_{10}x - 2r_{20}y + r_{00} = 0. \tag{1.11}$$

Returning to the general case of α and β we treat the mixed deformation with complex numbers following the model of [7]; a classification of conics written in the complex plane appears in [9, p. 640]. More precisely, with the usual notation $z = x + iy \in \mathbb{C}$ we derive the complex expression of Γ:

$$\Gamma : F(z, \bar{z}) := Az^2 + Bz\bar{z} + \bar{A}\bar{z}^2 + Cz + \bar{C}\bar{z} + r_{00} = 0 \tag{1.12}$$

with:

$$A = \frac{r_{11} - r_{22}}{4} - \frac{r_{12}}{2} i \in \mathbb{C}, \quad 2B = r_{11} + r_{22} = I \in \mathbb{R}, \quad C = r_{10} - r_{20} i \in \mathbb{C}. \tag{1.13}$$

It follows that the usual rotation performed with the angle φ to eliminate the mixed term xy has the meaning to reduce/rotate A in the real line while the translation which eliminates the term y has a similar meaning with respect to C. The inverse relationship between f and F is:

$$r_{11} = B + 2\Re A, \quad r_{22} = B - 2\Re A, \quad r_{12} = -2\Im A, \quad r_{10} = \Re C, \quad r_{20} = -\Im C \tag{1.14}$$

with $\Re$ and $\Im$ respectively the real and imaginary part. Hence the angle φ is provided by the formula:

$$\tan 2\varphi := \frac{2r_{12}}{r_{11} - r_{22}} = -\frac{\Im A}{\Re A} = -\tan \arg A \to 2\varphi = -\arg A. \tag{1.15}$$

The expression of the invariants of Γ in terms of A, B, C is:

$$I = 2B, \quad \delta = B^2 - 4|A|^2, \quad D = \delta + 2r_{00}I - |C|^2 \tag{1.16$_1$}$$

$$\Delta = r_{00}(B^2 - 4|A|^2) - B|C|^2 + 2\Re C(\Re A \Re C + \Im A \Im C) + 2\Im C(\Re C \Im A - \Re A \Im C). \tag{1.16$_2$}$$

The transformation of the complex coefficients under the (α, β)-rotation is:

$$A^r = \frac{\beta^2 - \alpha^2}{4} B - \frac{\alpha^2 + \beta^2}{2} \Re A - \alpha\beta \Im A i, B^r = \frac{\alpha^2 + \beta^2}{2} B + (\alpha^2 - \beta^2)\Re A,$$

$$\tilde{C} = -\beta \Im C + \alpha \Re C i. \tag{1.17}$$

For the considered particular case $\alpha = \beta$ we obtain:

$$A^r = -\alpha^2 A, \quad B^r = \alpha^2 B, \quad \tilde{C} = -\alpha C i \tag{1.18}$$

while the trigonometric case $\alpha = \beta = 1$ yields:

$$A^r = -A, \quad B^r = B, \quad \tilde{C} = -Ci \tag{1.19}$$

Returning to the general complex formalism above, in the case of a non-degenerate Γ, which means $\Delta \neq 0$, we can also express *the eccentricity* e by:

$$e^2 := 2 - \frac{I}{\lambda} = 1 - \frac{\delta}{\lambda^2}, \quad \lambda^2 - I\lambda + \delta = 0. \tag{1.20}$$

It follows that μ and e are provided by:

$$\lambda_{\pm} := B \pm 2|A| \to e^2 = \frac{\pm 4|A|}{B \pm 2|A|} \tag{1.21}$$

and hence the eccentricity is preserved by a diagonal rotation $\alpha = \beta$ since we use (1.18).

We finish this section by discussing the commutation of a rotation with the previous two gradient deformations of conics:
1) the (α, β)-*deformation* of Γ is the conic, [6, p. 87]:

$$\tilde{\Gamma} = \Gamma_{\alpha,\beta} : \alpha \left[\frac{1}{2} f_x\right]^2 + \beta \left[\frac{1}{2} f_y\right]^2 = 0. \tag{1.22}$$

2) the (α, β)-*mixed deformation* of Γ is the conic, [8]:

$$\Gamma^m = \Gamma^m_{\alpha,\beta} : f^m(x, y) := g_{\alpha,\beta}(In(x, y), \frac{1}{2}\nabla f(x, y)) = \alpha y \left[\frac{1}{2} f_x\right] + \beta x \left[\frac{1}{2} f_y\right] = 0. \tag{1.23}$$

A straightforward computation gives:
1+rotation)

$$(\tilde{\Gamma})^r : \beta^2(\alpha r_{12}^2 + \beta r_{22}^2)x^2 - 2\alpha\beta r_{12}(\alpha r_{11} + \beta r_{22})xy + \alpha^2(\alpha r_{11}^2 + \beta r_{12}^2)y^2 + 2\beta(\alpha r_{12}r_{10} + \beta r_{22}r_{20})x - 2\alpha(\alpha r_{11}r_{10} + \beta r_{12}r_{20})y + \alpha r_{10}^2 + \beta r_{20}^2 = 0, \tag{1.24}$$

$$\widetilde{\Gamma^r} : \beta^2(\alpha r_{12}^2 + \beta r_{22}^2)x^2 - 2\alpha\beta r_{12}(\alpha r_{11} + \beta r_{22})xy + \alpha^2(\alpha r_{11}^2 + \beta r_{12}^2)y^2 + 2\beta(\alpha r_{12}r_{10} + \beta r_{22}r_{20})x - 2\alpha(\alpha r_{11}r_{10} + \beta r_{12}r_{20})y = 0, \tag{1.25}$$

and then $(\tilde{\Gamma})^r = \widetilde{\Gamma^r}$ if and only if $\alpha r_{10}^2 + \beta r_{20}^2 = 0$ which for the diagonal case means $r_{10} = r_{20} = 0$. Hence, in this diagonal case the new conic is:

$$(\tilde{\Gamma})^r = \widetilde{\Gamma^r} : (r_{12}^2 + r_{22}^2)x^2 - 2Ir_{12}xy + (r_{11}^2 + r_{12}^2)y^2 = 0. \tag{1.26}$$

2+rotation)

$$(\Gamma^m)^r : \beta r_{12}x^2 - 2(\alpha r_{11} + \beta r_{22})xy + \alpha r_{12}y^2 - 2r_{10}x - 2r_{20}y = 0, \tag{1.27}$$

$$(\Gamma^r)^m : \beta r_{12}x^2 - (\alpha r_{11} + \beta r_{22})xy + \alpha r_{12}y^2 + r_{10}x - r_{20}y = 0, \tag{1.28}$$

and then $(\Gamma^m)^r = (\Gamma^r)^m$ if and only if $r_{10} = r_{20} = \alpha r_{11} + \beta r_{22} = 0$ which means that the new conic is:

$$(\Gamma^m)^r = (\Gamma^r)^m : r_{12}(\beta x^2 + \alpha y^2) = 0. \tag{1.29}$$

Let us remark that the origin belongs to both conics (1.26) and (1.29).

For the completeness of the subject we include here the iterations of gradient transformations 1 and 2:

$$\widetilde{\Gamma^m} : [r_{12}^2 + \frac{(\alpha r_{11} + \beta r_{22})^2}{\alpha\beta}](\beta x^2 + \alpha y^2) + 4r_{12}(\alpha r_{11} + \beta r_{22})xy + 2[\beta r_{12}r_{20} + (\alpha r_{11} + \beta r_{22})r_{10}]x + 2[(\alpha r_{11} + \beta r_{22})r_{20} + \alpha r_{12}r_{10}]y + \alpha r_{10}^2 + \beta r_{20}^2 = 0, \tag{1.30}$$

$$\tilde{\Gamma}^m : (\alpha r_{11} + \beta r_{12})r_{12}(\beta x^2 + \alpha y^2) + (\alpha^2 r_{11}^2 + 2\alpha\beta r_{12}^2 + \beta r_{22}^2)xy + \beta(\alpha r_{12}r_{10} + \beta r_{22}r_{20})x + \alpha(\alpha r_{11}r_{10} + \beta r_{12}r_{20})y = 0. \tag{1.31}$$

In particular, the diagonal condition $\alpha = \beta$ gives:

$$\widetilde{\Gamma^m} : (r_{12}^2 + I^2)(x^2 + y^2) + 4r_{12}Ixy + 2(r_{12}r_{20} + r_{10}I)x + 2(r_{20}I + r_{12}r_{10})y + r_{10}^2 + r_{20}^2 = 0, \tag{1.32}$$

$$\tilde{\Gamma}^m : r_{12}I(x^2 + y^2) + (r_{11}^2 + 2r_{12}^2 + r_{22}^2)xy + (r_{12}r_{10} + r_{22}r_{20})x + (r_{11}r_{10} + r_{12}r_{20})y = 0. \tag{1.33}$$

2. The rotation of two-dimensional Finsler structures

Let M be an open subset of $\mathbb{R}^m$ considered as a smooth m-dimensional manifold with $m \geq 2$ and $\pi : TM \to M$ its tangent bundle. Let $x = (x^i) = (x^1, ..., x^m)$ be the coordinates on M and $(x, y) = (x^i, y^i) = (x^1, ..., x^m, y^1, ..., y^m)$ the induced coordinates on TM. Denote by O the null-section of π.

Recall after [12] that a *Finsler fundamental function* on M is a map $F : TM \to \mathbb{R}_+$ with the following properties:
F1) F is smooth on the slit tangent bundle $T_0M := TM \setminus O$ and continuous on O,
F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$,
F3) the matrix $(g_{ij}) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is invertible and its associated quadratic form is of constant rank.
The tensor field $g = \{g_{ij}(x, y); 1 \leq i, j \leq m\}$ is called *the Finsler metric* and the homogeneity of F implies:

$$F^2(x, y) = g_{ij}y^iy^j = y_iy^i \tag{2.1}$$

where $y_i = g_{ij}y^j$. The pair (M, F) is called *Finsler manifold*. We point out the possibility of singular Finsler metrics as in [11].

Fix now the dimension $m = 2$ for which we change the notation: $(x^1, x^2) \to (x, y)$, $(y^1, y^2) \to (\dot{x}, \dot{y})$. Fix also the vector $\bar{\alpha} = (\alpha, \beta) \in \mathbb{R}^2_{+,+}$ with all strictly positive components although there are cases when some of them can be null or even negative. Inspired by the previous Section we introduce:

Definition 2.1 The *$\bar{\alpha}$-rotation* of F is $F^r = F^r_{\bar{\alpha}} : TM \to \mathbb{R}$ given by:

$$F^r(x, y, \dot{x}, \dot{y}) = F(x, y, -\alpha\dot{y}, \beta\dot{x}). \tag{2.2}$$

From (2.1) due to homogeneity it results a basic equation of Finsler geometry:

$$\frac{1}{2}(F^2)_{y^i} = g_{ij}y^j \tag{2.3}$$

This new Fislerian fundamental function yields a new Finslerian metric $g^r = g^{\bar{\alpha}}$ which we call *the $\bar{\alpha}$-rotation* of g. A straightforward computation yields:

$$g^r_{11} = \beta^2 g_{22}, \quad g^r_{22} = \alpha^2 g_{11}, \quad g^r_{12} = -\alpha\beta g_{12}. \tag{2.4}$$

Example 2.2 (Euclidean geometry) The Euclidean metric g_{can} is transformed into the Riemannian metric: $g^r_{can} = diag(\beta^2, \alpha^2)$. Applying a second rotation we get $(g^r_{can})^r = \alpha^2\beta^2 g_{can}$ which is a homothetical transformation. Hence, if $\alpha\beta = 1$ we get an involution on the positive cone of conformal Euclidean metrics $ConfEuclidean = \{\lambda g_{can}; \lambda \in (0, +\infty)\}$. □

Example 2.3 (Randers geometry) Let F be a Randers fundamental function of Minkowski type:

$$F_b(x, y, \dot{x}, \dot{y}) = F_b(\dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} + b\dot{x} \tag{2.5}$$

with $0 < b < 1$. The corresponding Finsler metric is:

$$g^b_{11} = 1 + b^2 + b\frac{2\dot{x}^3 + 3\dot{x}\dot{y}^2}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}, \quad g^b_{12} = \frac{b\dot{y}^3}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}, \quad g^b_{22} = 1 + \frac{b\dot{x}^3}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}. \tag{2.6}$$

The new Finslerian metric with $\alpha = \beta = 1$ is:

$$g^r_{11} = g^b_{22}, \quad g^r_{12} = -g^b_{12}, \quad g^r_{22} = g^b_{11}. \tag{2.7}$$

This proves that the new Finslerian structure F^r defines a completely new Finsler geometry on M. □

Example 2.4 (Spherically symmetric Finsler functions) Let $I \subseteq \mathbb{R}_+$ be an interval and $A, B : I \to \mathbb{R}$ two smooth functions. We define the orthogonally invariant Finsler function:

$$F(x, y, \dot{x}, \dot{y}) = \sqrt{A(x^2 + y^2)(\dot{x}^2 + \dot{y}^2) + B(x^2 + y^2) < (x, y), (\dot{x}, \dot{y}) >^2_{can}}. \tag{2.8}$$

Its Finsler metric is a non-diagonal Riemannian one:

$$g_{11} = A + Bx^2, \quad g_{12} = Bxy, \quad g_{22} = A + By^2. \tag{2.9}$$

The new Finslerian fundamental function is:

$$F^m(x, y, \dot{x}, \dot{y}) = \sqrt{A(\beta^2\dot{x}^2 + \alpha^2\dot{y}^2) + B(-2\alpha\beta xy\dot{x}\dot{y} + \beta^2 y^2\dot{x}^2 + \alpha^2 x^2\dot{y}^2)} \tag{2.10}$$

and hence the new Finslerian metric is again a non-diagonal Riemannian metric:

$$g^m_{11} = \beta^2(A + By^2), \quad g^m_{12} = -\alpha\beta Bxy, \quad g^m_{22} = \alpha^2(A + Bx^2). \tag{2.11}$$

□

3. Finslerian flows

For the given manifold M let $Finsler(M \times \mathbb{R})$ be the infinite space of all possible time-dependent Finslerian metrics on M as well as $T^s_2(TM \times \mathbb{R})$ the space of all time-dependent symmetric tensor fields of $(0, 2)$-type on TM. Following the theory of geometric (more precisely Riemannian) flows we introduce:

Definition 3.1 A *Finslerian flow* on M is a dynamical system modeled by the partial differential equations:

$$\partial_t g_t = \mathcal{F}(g_t) \tag{3.1}$$

where $\mathcal{F} : Finsler(M \times \mathbb{R}) \to T^s_2(TM \times \mathbb{R})$ is a given map and g_t is a family of Finslerian metrics depending on the parameter t belonging to the interval $I \subseteq \mathbb{R}$.

Examples 3.2 i) (Special Riemannian flows) If we restrict the functional $\mathcal{F}$ to $Riemann(M \times \mathbb{R})$ to be the (-2)Ricci curvature then we obtain the famous Ricci flow provided the proof of two outstanding conjectures: Poincaré Conjecture and Thurston Geometrization Conjecture. For a relationship between Randers metrics and Ricci solitons via the Zermelo navigation problem see [5].
ii) Other famous Riemannian flows are: the Calabi flow and the Yamabe flow.
iii) Time-dependent Randers metrics are recently used in the study of causal relationships on space-time manifolds in [14]. □

Returning to the general Finslerian framework and vector $\bar{\alpha}$ of previous Section we consider:

Definition 3.3 The *Finslerian $\bar{\alpha}$-rotation flow* is that given by:

$$\mathcal{F}(g) = g^r = g^{\bar{\alpha}}. \tag{3.2}$$

Inspired by [6, p. 96] we introduce the corresponding *aria variation* as the smooth function
$A : TM \times \mathbb{R} \to TM \times \mathbb{R}$ given by:

$$\partial_t A(x, y, \dot{x}, \dot{y}, t) = \sum_{i,j=1}^{2} g^r_{ij} g^{ij} \tag{3.3}$$

where, as usual, g^{ij} are the components of inverse g^{-1}.

Example 3.4 (Riemannian $\bar{\alpha}$-flow) With the computations of (2.4) we have:

$$\partial_t a_{11} = \beta^2 a_{22}, \quad \partial_t a_{12} = -\alpha\beta a_{12}, \quad \partial_t a_{22} = \alpha^2 a_{11}. \tag{3.4}$$

Then $a_{12}(t) = e^{-\alpha\beta t}$ on $I = \mathbb{R}$, $a_{11}(t) = u(x,y)\cosh(\alpha\beta t) + v(x,y)\sinh(\alpha\beta t)$ and $a_{22}(t) = \frac{\alpha}{\beta}[v(x,y)\cosh(\alpha\beta t) + u(x,y)\sinh(\alpha\beta t)$.

We have immediately that:

$$\partial_t A(x,y,\dot{x},\dot{y},t) = \alpha^2 a_{11}^2 + 2\alpha\beta a_{12}^2 + \beta^2 a_{22}^2 \tag{3.5}$$

and hence we have:

$$\partial_t A = \alpha^2[(u^2+v^2)\cosh(2\alpha\beta t) + uv\sinh(2\alpha\beta t)] + 2\alpha\beta e^{-2\alpha\beta t}. \tag{3.6}$$

It results:

$$A = \frac{\alpha}{2\beta}[(u^2+v^2)\sinh(2\alpha\beta t) + uv\cosh(2\alpha\beta t)] - e^{-2\alpha\beta t}. \tag{3.7}$$

For $\alpha\beta < 0$ and $uv > 0$ it results a negative A which means an area-decreasing flow. □

We finish with the following remark: in the reference [15], from two Finsler functions F_+, F_-, it is obtained a *bi-metric*:

$$F = \sqrt{F_+ \cdot F_-}. \tag{3.8}$$

The negative result of [15] concerning the physical implications of this metric as well as the considerations of our Section 1 suggests other two deformations:

$$F_{2,\alpha,\beta} = \sqrt{\alpha F_+^2 + \beta F_-^2}, \quad F_{m,\alpha,\beta} = \sqrt[m]{\alpha F_+^m + \beta F_-^m}, \quad m \in \mathbb{N}^* \tag{3.9}$$

which will be studied in a future work.

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