A ROSEN TYPE BI-METRIC UNIVERSE AND ITS PHYSICAL PROPERTIES

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Dedicated to the one century of Schwarzschild's metric

Abstract. The paper studies a spacetime endowed with two stationary metrics. The first one is a Riemannian one, called the R-Schwarzschild metric. It satisfies Einstein vacuum field equations, describes correctly the slow down of clocks in the gravitational field, the orbits of the planets and the perihelion drift. The R-Schwarzschild metric can be seen as the basic texture of the spacetime. All objects having mass are ruled by this Riemannian metric. The second metric, the light-adapted one, is deduced both taking into account the Rosen type bi-metric compatibility condition and by the preservation of the axiom of the speed of the light limit. This second metric offers the texture of the "light-like" objects. The main "normal" surprise is that this metric can be only the classical Schwarzschild metric. So, a Rosen type bi-metric universe exists and its properties are in accordance with the experimental physical evidences.

1. Historical preliminaries

The well known Einstein's equations of the gravitational field with the cosmological modification are:

$$R_{ij} - \frac{1}{2}g_{ij}R - g_{ij}\Lambda = \frac{8\pi G}{c^4}T_{ij}.$$

They link the curvature of the spacetime to the matter it contains, which means that they describe how matter and energy determine the curvature of the spacetime.

It is not very easy to determine these equations. In simple words, Einstein connected the divergence of the Ricci tensor via Bianchi's identity to the conservation of the Energy-Momentum tensor. Technically speaking, Einstein succeeded to find a divergence free mixed tensor $R^h_k - \frac{1}{2}\delta^h_k R$, which covariant form $R_{ij} - \frac{1}{2}g_{ij}R$ differs to the energy-momentum tensor T_{ij} by a multiplicative constant. Here $R := R^h_h = g^{hs}R_{sh}$ is the scalar curvature. Then, since the Newtonian field equation $\nabla^2\phi = 4\pi G\rho$ can be replaced by the new field equation $\nabla^2\phi - \Lambda\phi = 4\pi G\rho$, Einstein proposed the zero-divergence tensor $R_{ij} - \frac{1}{2}g_{ij}R - g_{ij}\Lambda$ as a left member of the gravitational field equation.

In the particular case when the gravitational field is produced by a single massive body that is concentrated in a small region of space, Einstein established the relativistic vacuum field equations:

$$R_{ij} = 0.$$

These equations can be obtained using a special coordinate system which eliminates the gravitational field at least along a single worldcurve. The Fermi coordinates express the point of view of a freely falling observer who can have the right to interpret his state as "at rest". So, the tidal acceleration law:

$$\frac{d^2}{dt^2}\frac{\partial \boldsymbol{x}}{\partial q} = -d^2\phi_x \cdot \frac{\partial \boldsymbol{x}}{\partial q}$$

can be seen as a separation of geodesic with respect to a metric $g = (g_{ij})$ in the form:

$$\frac{D^2}{d\tau^2}\frac{\partial x^h}{\partial q} = -K^h_k \cdot \frac{\partial x^k}{\partial q}, \quad K^h_k = R^h_{ikj}\frac{dz^i}{d\tau}\frac{dz^j}{d\tau}.$$

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The encapsulated formula $\nabla^2\phi = 0$ for the trace of $d^2\phi_x$ becomes $R_{ij} = 0$. In the case of a "week gravitational field", for "slow" particles moving under a rule created by a Newtonian gravitational potential ϕ, the classical field equations emerge from their relativistic counterparts:

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} \longrightarrow \frac{d^2x^\alpha}{dt^2} = -\frac{\partial\phi}{\partial x^\alpha},$$
$$R_{ij} - \frac{1}{2}g_{ij}R - g_{ij}\Lambda = \frac{8\pi G}{c^4}T_{ij} \longrightarrow \nabla^2\phi = 4\pi\rho,$$
$$R_{ij} = 0 \longrightarrow \nabla^2\phi = 0.$$

A very complete treatment of all these things can be seen in [2]. The first exact solution of Einstein's vacuum field equations was discovered by Karl Schwarzschild in 1915 and published in 1916 in [20]. The same exact solution was discovered by Johannes Droste in 1916 and published in [6]. When we are talking about *exact solutions of Einstein's field equations* it is impossible to neglect [21], "the Bible" of this domain published by H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt at Cambridge University Press. Consequences of vacuum field equations solutions as bending of light and the perihelion drift are also analyzed by textbooks as [1], [2], [4], [5], [16].

Let us point out that there exists the possibility to use two metrics instead of one to describe the gravity. N. Rosen, M. Milgrom, C. de Rham, G. Gabazade, A. Tolley, F. Hassan, S. Hawking, G.W. Gibbons, T. Eguchi, A.J. Hanson, P.B. Kronheimer use this idea to represent the permanent gravitational field induced by the first tensor together with another gravitational field in which the test particles are moving. See [19], [7], [8], [13], [14]. [9], [10], [15]. Another bi-metric approach of gravity can be seen in Palatini's work [18]. The $f(R)$-gravity is a family of modified gravity theories which use the difference of two connections and a function f depending of the scalar curvature R. When the second connection is null and the function coincides with R we obtain the general relativity. This arbitrary function allows models which explain the structure formation of the universe. The $f(R)$ Palatini's formalism used in [3] and [17] allows applications in cosmology and astrophysics. There is a paper of J. Elhers, F.A. E. Pirani, A. Schild, [11], where the authors present an axiomatic method of deriving a spacetime from compatible conformal and projective structures on a four dimensional manifold. To achieve this, using the difference between the components of the conformal and the projective connections, $\Delta^a_b c := \Pi^a_b c - K^a_b c$, they describe projective geodesics using conformal null vectors. A Lorentz geometry in which the projective null geodesics are identical with the conformal null geodesic appears. The signature of the semi-Riemannian spacetimes in [11] is $(-+++)$. However, many statements can be modified so as to hold this physically motivated restriction. In the same [11] we found that the spacetime metrics can, and have to be deduced from the qualitative assumption on the mobility of the rigid bodies.

In the Rosen type bi-metric system we imagined, which is of course different from the Rosen original one, there are two metrics instead of two structures as above. The conformal and the projective connections above are replaced by the Christoffel symbols Γ^k_{ij} and γ^k_{ij} of the two metrics. The compatibility condition starts from the difference of Christoffel symbols $\Delta^k_{ij} = \Gamma^k_{ij} - \gamma^k_{ij}$. We choose the first metric as a Riemannian one. The computations lead to results which satisfy the experimental physical evidences. Having in mind the qualitative assumption on the mobility of the rigid bodies we accept this $(++++)$ choice. The hypothesis of the null geodesics is essential in a spacetime construction. In our work the second metric fulfills this property. The coexistence of the two metrics through the compatibility condition creates our geometric space-time structure.

A sketch of the definition of a Rosen type bi-metric universe must highlight a manifold endowed with two (semi-) Riemannian metrics satisfying a list of properties:
1) the two metrics have different time-time component and the same space-space ones,
2) γ, the speed of light in the gravitational medium created by the second metric is smaller or equal to c, the speed of light in vacuum,
3) the first metric satisfies the vacuum field equations, i.e. is Ricci flat,
4) the Rosen compatibility condition is fulfilled.

Can we obtain for a Rosen type bi-metric system accurate-real results both in the bending of light and the perihelion drift cases? The answer is yes.

The two metrics are deduced combining mathematical techniques with results from physics. In fact

it is a continuing struggle to put together:
1) the Ricci flat condition,
2) the Rosen compatibility condition,
3) the $\gamma \leq c$ condition,
to "match" with the physical value deflection of light denoted by $T(D)$.
The results of these seven axioms lead to a final result: the two metrics are the R-Schwarzschild metric together with the "light-adapted" classical Schwarzschild metric. Massive bodies move along the geodesic generated by the first one while photons move along geodesics generated by the second.

This paper is the only one where the Rosen condition is presented as a possibility to put together a Riemannian metric with a semi-Riemannian one to pass the solar system tests.

2. Riemannian exact solutions for vacuum field equations

Let c be the speed of light in vacuum and let us impose that the speed of light is greater or equal to any other speed v: $v \leq c$.
If we write formally a expression Q as a Taylor series in powers of $\frac{1}{c}$:

$$Q = a_0 + a_1 \cdot \frac{1}{c} + a_2 \cdot \frac{1}{c^2} + a_3 \cdot \frac{1}{c^3} + ... + a_k \cdot \frac{1}{c^k} + ...$$

we say that the order of Q is $O(\frac{1}{c^m})$ if $a_0 = a_1 = ... = a_{m-1} = 0$ and $a_m \neq 0$. How is working this formal definition in a given physical context? We write each relativistic expression (components of the gravitational field, metric tensor, equations) as a Taylor series in powers of $\frac{1}{c}$, do our computations with these series and then we truncate the result at the point that is appropriate for our physical context.
In this paper, in our attempt to find the light-adapted metric we compute the speed of light (denoted by γ) in the gravitational medium created by the metric in each case. Let us see an example. Suppose $\mathbb{X}(t) = (ct, x^1(t), x^2(t), x^3(t))$ describes the worldcurve parametrized by the time t of an spatial object in the Minkowski frame. Then, its relativistic speed is

$$\left|\left|\frac{d\mathbb{X}}{dt}\right|\right|^2 = c^2 - \left(\frac{dx^1}{dt}\right)^2 - \left(\frac{dx^2}{dt}\right)^2 - \left(\frac{dx^3}{dt}\right)^2 = c^2 - v^2$$

where

$$v = \sqrt{\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2}$$

is the ordinary velocity of the object. If the object is a photon, then $\left|\left|\frac{d\mathbb{X}}{dt}\right|\right|^2 = c^2 - c^2 = 0$ as we expect.
If we consider the same worldcurve $\mathbb{X}(t) = (ct, x^1(t), x^2(t), x^3(t))$ in the metric

$$ds^2 = g_{00}(dx^0)^2 + g_{\alpha\beta}dx^\alpha dx^\beta,$$

where α, β are spatial indexes according to the formalism above, we have

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + \sum_{\alpha\neq\beta=1}^{3} g_{\alpha\beta}dx^\alpha dx^\beta.$$

Then

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (g_{00} - 1)(dx^0)^2 + (g_{11} + 1)(dx^1)^2 +$$
$$+ (g_{22} + 1)(dx^2)^2 + (g_{33} + 1)(dx^3)^2 + \sum_{\alpha\neq\beta=1}^{3} g_{\alpha\beta}dx^\alpha dx^\beta,$$

i.e.

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (g_{00} - 1)(dx^0)^2 + \sum_{\alpha,\beta=1}^{3} \bar{g}_{\alpha\beta}dx^\alpha dx^\beta$$

where $\bar{g}_{\alpha\beta} = g_{\alpha\beta}$ if $\alpha \neq \beta$ and $\bar{g}_{\alpha\beta} = 1 + g_{\alpha\beta}$ if $\alpha = \beta$. Or, this means that for a photon we obtain

$$0 = \left(\frac{d\mathbb{X}}{dt}\right)^2 = c^2 - \gamma^2 + (g_{00} - 1)\left(\frac{dx^0}{dt}\right)^2 + \sum_{\alpha,\beta=1}^{3} \bar{g}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}$$

where

$$\gamma = \sqrt{\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2}.$$

The formalism of this paper is fixed by:

Definition 2.1. i) The quantity γ given by the formula

$$\gamma^2 = c^2 + (g_{00} - 1)\left(\frac{dx^0}{dt}\right)^2 + \sum_{\alpha,\beta=1}^{3} \bar{g}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}$$

is called *the speed of the light* in the gravitational medium created by the metric above.
ii) A four dimensional manifold M endowed with two (semi-) Riemannian metrics

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

$$ds^2 = g'_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

will be called *Rosen type bi-metric universe* if:
1) $g_{00} \neq g'_{00}$,
2) the first metric satisfies the vacuum field equations: $R_{ij} = 0$, i.e. is Ricci flat,
3) the second one satisfies $\gamma \leq c$, that is the speed of light in the gravitational medium created by the second metric is less than or equal c, the speed of light in vacuum.
4) (The compatibility condition) Consider $\Delta^k_{ij} = \Gamma^k_{ij} - \gamma^k_{ij}$ where Γ^k_{ij} and γ^k_{ij} are the Christoffel symbols of the first and the second metric respectively and *the Rosen type Riemannian curvature tensor*:

$$\bar{R}^h_{ijk} = \Delta^h_{ik,j} - \Delta^h_{ij,k} + \Delta^h_{mj}\Delta^m_{ik} - \Delta^h_{mk}\Delta^m_{ij}.$$

Then the Ricci tensor attached $\bar{R}_{ij} := \bar{R}^h_{ihj}$ satisfies *the Rosen compatibility condition* $\bar{R}_{ij} = 0$.

An exemple: Let us observe that the two metrics

$$ds^2 = c^2dt^2 + dx^2 + dy^2 + dz^2$$

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

where the second one is the Minkowski metric satisfy the Rosen compatibility condition. Furthermore, the speed of light in the second metric is c. The Christoffel symbols are all 0 and the equations for geodesics are the same for both metrics. In fact straight lines are geodesics for both metrics.

Theorem 2.1: *Consider the vacuum field equations $R_{ij} = 0$. Then*

$$ds^2 = c^2\left(1 + \frac{B}{r}\right)dt^2 + \frac{1}{1 + \frac{B}{r}}dr^2 + r^2d\varphi^2 + r^2\sin^2\varphi d\theta^2 \tag{2.1}$$

is a Schwarzschild type Riemannian solution for an arbitrary constant B.

Proof: We proceed as in the pseudo-Riemannian case, where Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations outside of a spherical, non-rotating, gravitational body must be the Schwarzschild metric. In fact we prove a Riemannian equivalent of Birkhoff's theorem using the ideas seen in [2] and [12]. Let $T = T(r)$, $Q = Q(r)$ be two real functions such that the metric:

$$ds^2 = c^2 \cdot e^T dt^2 + e^Q dr^2 + r^2d\varphi^2 + r^2\sin^2\varphi d\theta^2$$

verifies $R_{ij} = 0$. If we denote $x^0 := ct$ i.e. $dx^0 = cdt$, then the coefficients of the metric are

$$g_{00} = e^T, \quad g_{11} = e^Q, \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\varphi,$$

and

$$g^{00} = e^{-T}, \qquad g^{11} = e^{-Q}, \quad g^{22} = \frac{1}{r^2}, \quad g^{33} = \frac{1}{r^2\sin^2\varphi}.$$

We have

$$\frac{\partial g_{ij}}{\partial x^0}=0,\quad i,j=\overline{0,3},\quad |ij,k|=\Gamma^i_{jk}=0,\quad i\neq j\neq k.$$

The non-zero Christoffel symbols are

$$\Gamma^0_{01}=\Gamma^0_{10}=\frac{T'}{2},\quad \Gamma^1_{00}=-\frac{T'}{2}e^{T-Q},\quad \Gamma^1_{11}=\frac{Q'}{2},\quad \Gamma^1_{22}=-re^{-Q},\quad \Gamma^1_{33}=-re^{-Q}\sin^2\varphi,\quad \Gamma^2_{21}=\Gamma^2_{12}=\frac{1}{r},$$

$$\Gamma^2_{33}=-\sin\varphi\cos\varphi,\quad \Gamma^3_{31}=\Gamma^3_{13}=\frac{1}{r},\quad \Gamma^3_{23}=\Gamma^3_{32}=\cot\varphi.$$

The only non-zero components of the Ricci tensor are

$$R_{00}=e^{T-Q}\left(\frac{T''}{2}+\frac{(T')^2}{4}-\frac{T'Q'}{4}+\frac{T'}{r}\right),\quad R_{11}=-\left(\frac{T''}{2}+\frac{(T')^2}{4}-\frac{T'Q'}{4}-\frac{Q'}{r}\right)$$

$$R_{22}=1-e^Q+re^{-Q}\left(\frac{Q'}{2}-\frac{T'}{2}\right),\quad R_{33}=\sin^2\varphi R_{22}.$$

The conditions $R_{00}=0$ and $R_{11}=0$ determine both T and Q. Indeed, $R_{00}=0$ and $R_{11}=0$ imply $T'+Q'=0$, that is $T+Q$=constant=k. Thus $e^T=e^{-Q}e^k$, i.e. the metric is

$$ds^2=c^2\cdot e^{-Q}e^k dt^2+e^Q dr^2+r^2d\varphi^2+r^2\sin^2\varphi d\theta^2.$$

If we let $t=e^{k/2}u$, then $dt^2=e^k du^2$ and the metric becomes

$$ds^2=c^2\cdot e^{-Q}du^2+e^Q dr^2+r^2d\varphi^2+r^2\sin^2\varphi d\theta^2.$$

Therefore we may choose $T+Q=0$, i.e. $Q=-T$. Replacing in the second equation we have

$$rT''+r(T')^2+2T'=\left(re^T\right)''=0.$$

Then $e^T=A+\frac{B}{r}$. We impose that $e^T\to 1$ as $r\to\infty$; it results $A=1$. Therefore $e^T=1+\frac{B}{r}$ and $e^Q=e^{-T}=\frac{1}{1+\frac{B}{r}}$. Let us observe that for T and Q so far determined, $R_{22}=R_{33}=0$. The obtained Schwarzschild type Riemannian metrics is exactly (2.1). $\square$

3. The orbit of a planet in the Schwarzschild type Riemannian metric

Consider the differential equation which describes the gravitational attraction between a planet and a star

$$\ddot{\vec{a}}=-\frac{GM}{r^3}\cdot\vec{a},$$

where $\vec{a}$ is the position vector, $G=6,67\cdot 10^{-11}m^3/Kg\cdot s^2$ is the gravitational constant, M is the mass of the star and $r=\parallel\vec{a}\parallel$. Let also J be the magnitude of the angular moment of the planet. If we consider polar coordinates and $r=r(\theta)=\dfrac{1}{u(\theta)}$, the previous equation becomes

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{J^2},\quad \mu=GM.$$

The classical solution is

$$u(\theta)=\frac{\mu}{J^2}+A\cos(\theta-\alpha),$$

where A is an arbitrary constant which can be obtained from the initial condition and α is a phase shift. Since the phase shift alters the position of the planet at time $t=0$ and we are interested only in the orbit itself, we may consider $\alpha=0$. Denoting the eccentricity e by $e:=\dfrac{AJ^2}{\mu}$, the orbit is the conic

$$u(\theta)=\frac{\mu}{J^2}(1+e\cos\theta)\quad\text{or equivalently,}\quad r(\theta)=\frac{\frac{J^2}{\mu}}{1+e\cos\theta}.$$

The next result provides the differential equation which predicts the orbit of a planet in its movement around the Sun in the new context of the metric above.

Theorem 3.2: *The orbit of a planet in the Schwarzschild type Riemannian metric is described by the equation*

$$\frac{d^2u}{d\theta^2} + u = \frac{c^2B}{2J^2} - \frac{3B}{2}u^2. \tag{3.1}$$

Proof: In the same way as before we denote $x^0 := ct$. The worldcurve of the planet is the geodesic $\zeta(\tau) := (t(\tau), r(\theta), \varphi(\tau), \theta(\tau))$ of the Schwarzschild type Riemannian metric. We are looking for a solution in the (x, y) plane, that is $\varphi = \frac{\pi}{2}$. The reduced metric is

$$ds^2 = \left(1 + \frac{B}{r}\right)(dx^0)^2 + \frac{1}{1 + \frac{B}{r}}dr^2 + r^2 d\theta.$$

Since $\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{r}$ and $\Gamma^3_{ij} = 0$ in the other cases, the equation corresponding to the variable θ is

$$\ddot{\theta}(\tau) + \frac{2}{r(\tau)} \cdot \dot{r}(\tau) \cdot \dot{\theta}(\tau) = \frac{1}{r^2}\left(r^2(\tau)\,\dot{\theta}(\tau)\right)' = 0.$$

We denote $r^2\,\dot{\theta} = J$ and this J describes the magnitude of the angular momentum of the planet exactly as in the classical case. We cancel τ in the next computations.

We continue with the geodesic equation corresponding to the variable x^0. Since only $\Gamma^0_{01} = \Gamma^0_{10} = \dfrac{1}{1+\dfrac{B}{r}} \cdot \dfrac{-B}{2r^2}$ the equation in x^0 is

$$\ddot{x}^0 - \frac{B}{r^2} \cdot \frac{1}{1+\dfrac{B}{r}} \cdot \dot{x}^0 \cdot \dot{r} = 0.$$

But from $x^0 = ct$ it results

$$\ddot{t} - \frac{B}{r^2} \cdot \frac{1}{1+\frac{B}{r}} \cdot \dot{t} \cdot \dot{r} = 0 \quad \text{i.e.} \quad \left(\left(1 + \frac{B}{r}\right) \cdot \dot{t}\right)' = 0,$$

that is

$$\dot{t} = \frac{E}{1+\frac{B}{r}},$$

where E is a constant. In the case of the equation corresponding to the variable r we use directly the metric condition. Taking into account that $ds^2 = c^2 d\tau^2$ we obtain

$$c^2 = c^2\left(1 + \frac{B}{r}\right)\left(\dot{t}\right)^2 + \frac{1}{1+\frac{B}{r}}\left(\dot{r}\right)^2 + r^2\left(\dot{\theta}\right)^2.$$

Let us replace $\dot{t}$ and $\dot{\theta}$ in the previous equation. We have

$$c^2\left(1 - E^2\right) + c^2 \cdot B \cdot \frac{1}{r} = \left(\dot{r}\right)^2 + \frac{J^2}{r^2} + \frac{B \cdot J^2}{r^3}.$$

Consider $r = r(\theta)$. It results

$$\dot{r} = \frac{dr}{d\theta} \cdot \dot{\theta} = \frac{dr}{d\theta} \cdot \frac{J}{r^2}.$$

If $r := \dfrac{1}{u}$ then $\dfrac{dr}{d\theta} = -\dfrac{1}{u^2}\dfrac{du}{d\theta}$ i.e. $\dot{r} = J^2 \cdot \left(\dfrac{du}{d\theta}\right)^2$. The previous equation becomes

$$c^2(1 - E^2) + c^2 B u = J^2\left(\frac{du}{d\theta}\right)^2 + J^2u^2 + BJ^2u^3.$$

If we differentiate with respect to θ and then we divide by $\dfrac{du}{d\theta}$ we obtain the equation (3.1). □

4. Solving the equation

At this moment we need to clarify the form of the Schwarzschild type Riemannian metric. We have accepted that as $r \to \infty$ the metric approaches the ordinary Euclidean metric. Let us continue saying that when r is large, the Schwarzschild type metric is a *weak metric*, that is

$$g_{ij} = a_0 + \frac{a_2}{c^2} + \frac{a_3}{c^3} + \dots.$$

Let us continue considering $g_{00} = c^2\left(1 + \frac{B}{r}\right)$ and calculating $\Gamma^1_{00} = \frac{c^2 B}{2r^2}$ which is the only nonzero Γ^i_{00}. So, the r component of the geodesic equation is $\frac{d^2 r}{d\tau^2} = \Gamma^1_{00}$ that is

$$\frac{d^2 r}{d\tau^2} = \frac{c^2 B}{2r^2}.$$

As r approaches the infinity, $d\tau$ becomes dt and the previous equation is the original Newton equation $\frac{d^2 r}{dt^2} = \frac{-GM}{r^2}$ if and only if $B = \frac{-2GM}{c^2}$. It results that

$$1 + \frac{B}{r} = 1 - \frac{2GM}{c^2} \cdot \frac{1}{r}.$$

This way the gravitational Newtonian potential $\phi(x,y,z) = -\frac{GM}{r} = -\frac{\mu}{r}$ is involved in the coefficients of the metric and $\frac{1}{c^2}$ highlights the weak gravitational field we need, see also [2]. The quantity $r_M := \frac{2GM}{c^2} = -\frac{\mu}{r}$ has the dimension of a length and it is called *the gravitational radius*, or *the Schwarzschild radius*, corresponding to the mass M.

Theorem 4.3: *The gravitational medium induced by the Schwarzschild type Riemannian metric*

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \frac{1}{1 - \frac{2GM}{c^2 r}}dr^2 + r^2 d\varphi^2 + r^2 \sin^2\varphi d\theta^2 \tag{4.1}$$

causes the slow down of clocks.

Proof: We consider the Schwarzschild type Riemannian metric (4.1) in the frame S of an observer O_1 who is far from source and a second observer O_2 who is close to the source. Suppose both observers motionless with respect to frame S. For both observers $dr = d\varphi = d\theta = 0$ so their proper time t and τ respectively, satisfy the equation:

$$c^2 d\tau^2 = ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2, \quad \text{i.e.} \quad d\tau = \sqrt{1 - \frac{2GM}{c^2 r}}dt < dt. \tag{4.2}$$

So, the time interval of $\Delta\tau$ seconds of O_2's clock will appear to be less than Δt seconds on O_1's clock. □

Definition 4.1 The Schwarzschild type Riemannian metric above will be called *the R-Schwarzschild metric.*

Theorem 4.4: *In the setting of the R-Schwarzschild metric, the orbit of a planet is described by the equation*

$$\frac{d^2 v}{d\theta^2} + v = \frac{\mu}{J^2} + \frac{3\mu}{c^2}\left(v - \frac{2\mu}{J^2}\right)^2 \tag{4.3}$$

which has the solution

$$v(\theta) = \frac{\mu}{J^2}\left(1 + e\cos(\theta + F\theta)\right) + O\left(\frac{1}{c^2}\right), \tag{4.4}$$

for $F := \frac{3\mu^2}{c^2 J^2}$.

Proof: Let us start from the equation (3.1) of the orbit of the planet in which we replace B by $-\frac{2GM}{c^2}$. The new equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{-\mu}{J^2} + \frac{3\mu}{c^2} \cdot u^2.$$

Replacing $u := v - \frac{2\mu}{J^2}$ we obtain

$$\frac{d^2v}{d\theta^2} + v = \frac{\mu}{J^2} + \frac{3\mu}{c^2}\left(v - \frac{2\mu}{J^2}\right)^2.$$

This equation differs from the classical one by $\frac{3\mu}{c^2}\left(v - \frac{2\mu}{J^2}\right)^2$. This "correction" of the classical orbit is due to the gravitation encapsulated in our Schwarzschild type metric. It is natural to search the solution as

$$v(\theta) = a(\theta) + \frac{w(\theta)}{c^2},$$

where $a(\theta) = \dfrac{\mu}{J^2}(1 + e\cos\theta)$ is the classical solution of the classical orbit equation

$$\frac{d^2a}{d\theta^2} + a = \frac{\mu}{J^2}.$$

We have

$$\frac{d^2a}{d\theta^2} + a + \frac{1}{c^2}\left(\frac{d^2w}{d\theta^2} + w\right) = \frac{\mu}{J^2} + \frac{3\mu}{c^2}\left(\frac{\mu}{J^2}(-1 + e\cos\theta) + \frac{w}{c^2}\right)^2,$$

or equivalently

$$\frac{1}{c^2}\left(\frac{d^2w}{d\theta^2} + w\right) = \frac{3\mu^3}{c^2J^4}(-1 + e\cos\theta)^2 + O\left(\frac{1}{c^4}\right).$$

The term $O\left(\frac{1}{c^4}\right)$ has a small influence on w. It remains to solve

$$\frac{d^2w}{d\theta^2} + w = \frac{3\mu^3}{J^4}\left(1 + \frac{e^2}{2} - 2e\cos\theta + \frac{e^2}{2}\cos 2\theta\right).$$

The solutions of the following three equations

$$\frac{d^2w_1}{d\theta^2} + w_1 = \frac{3\mu^3}{J^4}\left(1 + \frac{e^2}{2}\right), \frac{d^2w_2}{d\theta^2} + w_2 = -\frac{6e\mu^3}{J^4}\cos\theta, \frac{d^2w_3}{d\theta^2} + w_3 = \frac{3\mu^3e^2}{2J^4}\cos 2\theta$$

are

$$w_1 = \frac{3\mu^3}{J^4}\left(1 + \frac{e^2}{2}\right), \quad w_2 = -\frac{3e\mu^3}{J^4}\theta\cos\theta, \quad w_3 = -\frac{3\mu^3e^2}{2J^4}\cos 2\theta,$$

and therefore we have

$$v(\theta) = \frac{\mu}{J^2}(1 + e\cos\theta) + \frac{3\mu^3}{J^4c^2}\left(1 + \frac{e^2}{2} - e\theta\sin\theta - \frac{e^2}{2}\cos 2\theta\right).$$

Einstein's idea was to use only the non-periodic term in the classical solution. Then

$$v(\theta) = \frac{\mu}{J^2}\left(1 + e\cos\theta - \frac{3\mu^2e}{c^2J^2}\theta\sin\theta\right) + O\left(\frac{1}{c^2}\right)$$

can be thought as

$$v(\theta) = \frac{\mu}{J^2}(1 + e\cos(\theta + F\theta)) + O\left(\frac{1}{c^2}\right),$$

where $F := \frac{3\mu^2}{c^2J^2}$. Neglecting the term $O\left(\frac{1}{c^2}\right)$ which adds only a small contribution, the trajectory is still the old conic. $\square$

5. The perihelion drift

The correction of the classical trajectory described by the Schwarzschild type orbit reaches the perihelion when $\cos(\theta + F\theta) = 1$, therefore $\theta = \theta_n = \frac{2n\pi}{1+F}$ for an integer n.
It results $\theta \approx 2n\pi\left(1 - F + O\left(F^2\right)\right)$. That is, $2\pi F$ is the perihelion drift at each revolution.

If N is the number of orbits for a given period of time T then the perihelion drift P_d is

$$P_d = \frac{6\pi\mu^2}{c^2 J^2} \cdot N.$$

For Mercury we obtain 43 seconds of arc per century.

6. A consequence of the violation of the speed of the light limit

Let us return to the R-Schwarzschild metric written now as

$$ds^2 = g_\mu = c^2\left(1 - \frac{2\mu}{c^2 r}\right)dt^2 + \frac{1}{1 - \frac{2\mu}{c^2 r}}dr^2 + r^2 d\varphi^2 + r^2\sin^2\varphi d\theta. \tag{6.1}$$

It remains to find out what is happening with the light in this Riemannian universe. First at all we need to compute the speed γ of light in the gravitational field induced by the metric above. The computation is done in such a way then we can make one important conclusion after. To do this, we treat a general case by considering $g_{00}^\varepsilon = c^2(1 - \dfrac{2\varepsilon\mu}{c^2 r})$, where ε can be 1 or -1.

Theorem 6.5: *Consider the general metric*

$$ds^2 = g_\mu^\varepsilon = c^2\left(1 - \frac{2\varepsilon\mu}{c^2 r}\right)dt^2 + \frac{1}{1 - \frac{2\mu}{c^2 r}}dr^2 + r^2 d\varphi^2 + r^2\sin^2\varphi d\theta^2. \tag{6.2}$$

i) *In Carthesian coordinates the metric above has the form*

$$ds^2 = \sum_{i=0}^{3}\left(dx^i\right)^2 - \frac{2\mu}{c^2 r}\left(\varepsilon(dx^0)^2 - \frac{1}{1 - \frac{2\mu}{c^2 r}}\sum_{\alpha,\beta=1}^{3}\frac{x^\alpha x^\beta}{r^2}dx^\alpha dx^\beta\right). \tag{6.3}$$

ii) *A deflected photon in the (x^1, x^2) plane which comes from the undeflected photon $X(t) = (ct, h, ct, 0)$ has the speed*

$$\gamma = c + \frac{\varepsilon\mu}{cr} - \frac{\mu}{cr}\cdot\frac{(x^2)^2}{r^2}\cdot\frac{1}{1 - \frac{2\mu}{c^2 r}}. \tag{6.4}$$

iii) *The deflected photon doesn't violate the speed of light limit c if and only if $\varepsilon = -1$. In this case*

$$\gamma = c - \frac{\mu}{cr} - \frac{\mu(x^2)^2}{cr^3} + O\left(\frac{1}{c^3}\right).$$

Proof: i) Let $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ and $r^2 = x^2 + y^2 + z^2$. It results $rdr = xdx + ydy + zdz$ which gives:

$$dr = \sum_{\alpha=1}^{3}\frac{x^\alpha}{r}dx^\alpha, \quad dr^2 = \sum_{\alpha,\beta=1}^{3}\frac{x^\alpha x^\beta}{r^2}dx^\alpha dx^\beta.$$

Taking into account that

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\varphi^2 + r^2\sin^2\varphi d\theta^2 = \sum_{\alpha=1}^{3}(dx^\alpha)^2$$

it results (6.3).
ii) Suppose that $X(t) = \left(ct, x^1(t), x^2(t), x^3(t)\right)$ describes the worldcurve of an object parametrized by the time t. In the Euclidean metric we have

$$\left(\frac{ds}{dt}\right)^2 = \|\dot{X}(t)\|^2 = c^2 + \left((\dot{x}^1(t))^2 + (\dot{x}^2(t))^2 + (\dot{x}^3(t))^2\right) = c^2 + v^2,$$

where v is the usual spatial speed of the object. If the object is a photon, then $\left(\frac{ds}{dt}\right)^2 = c^2+c^2 = 2c^2$ and so

$$2c^2 = \left(\frac{ds}{dt}\right)^2 = c^2 + \gamma^2 - \frac{2\mu}{c^2 r}\left(\varepsilon c^2 - \frac{1}{1-\frac{2\mu}{c^2 r}}\sum_{\alpha,\beta=1}^{3}\frac{x^\alpha x^\beta}{r^2}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right)$$

where $\gamma(t) = \sqrt{(\dot{x}^1(t))^2 + (\dot{x}^2(t))^2 + (\dot{x}^3(t))^2}$ is the speed of the photon in the gravitational medium created by the metric above; recall Definition 2.1.

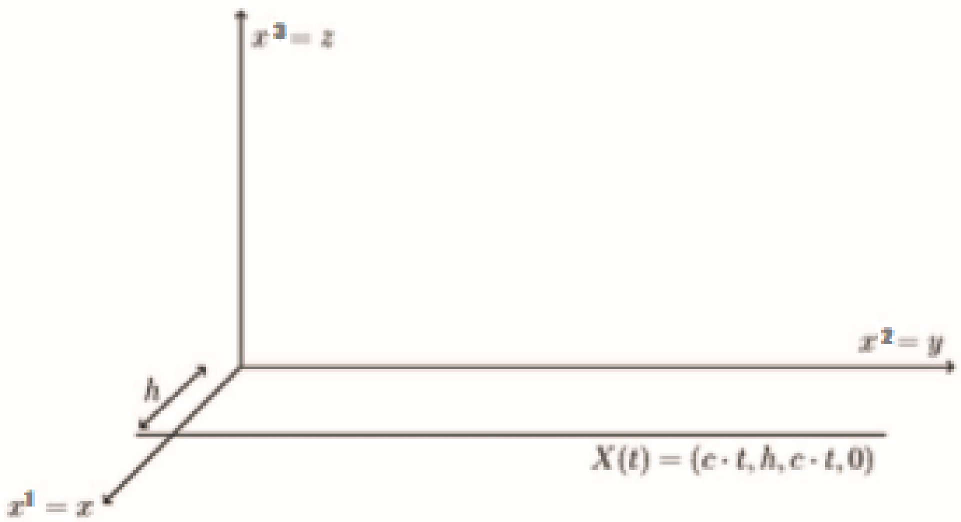

FIGURE 1.

We determine γ along the worldcurve X of an undeflected photon in the (x^1, x^2) plane at the fixed distance h from x^2-axis. The undeflected worldcurve of the photon becomes $X(t) := (ct, h, ct, 0)$. According to the general theory the deflected photon has a worldcurve parametrized by

$$X_d(t) := \left(ct, h + O\left(\frac{1}{c}\right), ct + O(1), 0\right).$$

Since $\dot{X}_d(t) := \left(c, O\left(\frac{1}{c}\right), c + O(1), 0\right)$, that is

$$\frac{dx^1}{dt} = O\left(\frac{1}{c}\right), \quad \frac{dx^2}{dt} = c + O(1), \quad \frac{dx^3}{dt} = 0,$$

it results that

$$\gamma^2 = c^2 + \frac{2\mu}{c^2 r}\left(\varepsilon c^2 - \frac{1}{1-\frac{2\mu}{c^2 r}}\frac{(x^2)^2}{r^2}\cdot c^2\right),$$

i.e.

$$\gamma^2 = c^2\left(1 + \frac{2\varepsilon\mu}{c^2 r} - \frac{2\mu (x^2)^2}{c^2 r^3}\frac{1}{1-\frac{2\mu}{c^2 r}}\right).$$

Taking into account that $\sqrt{1+2A} \approx 1 + A$, the result is (6.4).
iii) Let us write the previous formula in the form

$$\gamma = c + \frac{\mu}{cr^3}\left(\varepsilon r^2\left(1 - \frac{2\mu}{c^2 r}\right) - \left(x^2\right)^2\right)\frac{1}{1-\frac{2\mu}{c^2 r}}.$$

If $\varepsilon = -1$, it results $\gamma = c - u$ with $u > 0$. Therefore $\gamma < c$. If $\varepsilon = 1$ then the quantity $\frac{\mu}{r^3 c}\frac{1}{1-\frac{2\mu}{c^2 r}}\left(\left(x^2\right)^2 - \frac{2\mu r}{c^2}\right)$ allows positive and negative values. It means that it exists for $r < \frac{c^2 (x^2)^2}{2\mu}$

a region where the speed γ of light in the gravitational medium induced by the metric $g_\mu^{\varepsilon=1}$ violates the light speed limit, that is $\gamma > c$.

Since $\dfrac{1}{1-\dfrac{2\mu}{c^2r}} \approx 1 + \dfrac{2\mu}{c^2r} + O\left(\dfrac{1}{c^3}\right)$ it results the formula of γ from iii). $\square$

The proof shows that R-Schwarzschild metric $g_\mu = g_\mu^{\varepsilon=1}$ is inadequate if we are interested to study the bending of light. The new Riemannian metric corresponding to $\varepsilon = -1$ is the adequate one. We call it *the Schwarzschild light-adapted metric.*

Theorem 6.6: *Consider the Schwarzschild light-adapted metric*

$$ds^2 = g_\mu^{\varepsilon=-1} = c^2\left(1 + \frac{2\mu}{c^2r}\right)dt^2 + \frac{1}{1-\frac{2\mu}{c^2r}}dr^2 + r^2d\varphi^2 + r^2\sin^2\varphi d\theta^2. \tag{6.5}$$

The total deflection of the trajectory of a deflected photon is $T(D) = \frac{4GM}{c^2h}$.

Proof: The previous theorem proved that the speed of a deflected photon is

$$\gamma = c - \frac{\mu}{cr} - \frac{\mu(x^2)^2}{cr^3} + O\left(\frac{1}{c^3}\right).$$

According to the theory presented in [2] the total deflection is

$$-\frac{1}{c}\int_{-\infty}^{\infty}\frac{\partial\gamma}{\partial x^1}dx^2.$$

Then, canceling the $O\left(\frac{1}{c^4}\right)$ term we have:

$$\left.\frac{\partial\gamma}{\partial x^1}\right|_{X_d} = \left.\frac{GMh}{cr^3} + \frac{3GMh(x^2)^2}{cr^5}\right|_{X_d} = \frac{GMh}{c(h^2+(x^2)^2)^{\frac{3}{2}}} + \frac{3GMh(x^2)^2}{c(h^2+(x^2)^2)^{\frac{5}{2}}}$$

and elementary computations leads to

$$-\frac{1}{c}\int_{-\infty}^{\infty}\frac{\partial\gamma}{\partial x^1}dx^2 = -\frac{GMh}{c^2}\left(\int_{-\infty}^{\infty}\frac{dx^2}{(h^2+(x^2)^2)^{\frac{3}{2}}} - \int_{-\infty}^{\infty}\frac{3(x^2)^2dx^2}{(h^2+(x^2)^2)^{\frac{5}{2}}}\right) = -\left(\frac{2}{h^2} + \frac{2}{h^2}\right)\frac{GMh}{c^2}.$$

The total deflection becomes $T(D) = \frac{4GM}{c^2h}$. $\square$

At the surface of the sun we have:

$$h = \text{radius of the sun} = 7\times10^8 m;\ G = 6,67\times10^{-11}m^3/Kg\cdot s^2,$$
$$M = \text{mass of the sun} = 2\times10^{30}kg;\ c = 3\times10^8 m/s^2.$$

It results for $T(D_{sun}) = \frac{4GM}{c^2h} = 2\cdot 4,2328\times10^{-6}$ radians $\approx 0,873'' \times 2 = 1,75''$. This value is close to Einstein's 1916 computation when he established $1,66''$.

We may ask if the light-adapted metric $g_\mu^{\varepsilon=-1}$ is a metric we can add to g_μ to produce a Rosen type bi-metric universe. We may check the compatibility condition. But we prefer another way. The answer is no, because of the Theorem 6.7 bellow.
To understand this and for obtaining a Rosen type bi-metric universe, we are looking for possible g_{11} for our light-adapted metric. It is easier to look for a smooth function $a(r)$ such that $g_{00} = c^2\left(1 + \dfrac{2\mu}{c^2r} + \dfrac{2a(r)}{c^k}\right)$ participates at the creation of a $\gamma \le c$. The coefficient g_{00} is chosen exactly of this form to preserve the weak gravitational field idea we have accepted for these metrics.

Theorem 6.7: *Consider a general light-adapted metric*

$$ds^2 = g_{\mu,a}^{\varepsilon=-1} = c^2\left(1 + \frac{2\mu}{c^2r} + \frac{2a(r)}{c^k}\right)dt^2 + \frac{1}{1-\frac{2\mu}{c^2r}}dr^2 + r^2d\varphi^2 + r^2\sin^2\varphi d\theta^2. \tag{6.6}$$

i) *In Carthesian coordinates the metric before has the form*

$$ds^2 = \sum_{i=0}^{3}\left(dx^i\right)^2 + \frac{2\mu}{c^2r}\left((1 + \frac{ra(r)}{\mu c^{k-2}})(dx^0)^2 + \frac{1}{1-\frac{2\mu}{c^2r}}\sum_{\alpha,\beta=1}^{3}\frac{x^\alpha x^\beta}{r^2}dx^\alpha dx^\beta\right).$$

ii) *A deflected photon in the* (x^1, x^2) *plane which comes from the undeflected photon* $X(t) = (ct, h, ct, 0)$ *has the speed*

$$\gamma = c - \frac{\mu}{cr} - \frac{a(r)}{c^{k-1}} - \frac{\mu(x^2)^2}{cr^3} \cdot \frac{1}{1 - \frac{2\mu}{c^2 r}}. \tag{6.7}$$

iii) *Consider the bi-metric system formed by the Schwarzschild type Riemannian metric*

$$ds^2 = g_\mu^{\varepsilon=1} = c^2 \left(1 - \frac{2\mu}{c^2 r}\right) dt^2 + \frac{1}{1 - \frac{2\mu}{c^2 r}} dr^2 + r^2 d\varphi^2 + r^2 \sin^2 \varphi d\theta$$

and the general light-adapted metric

$$ds^2 = g_{\mu,a}^{\varepsilon=-1} = c^2 \left(1 + \frac{2\mu}{c^2 r} + \frac{2a(r)}{c^k}\right) dt^2 + \frac{1}{1 - \frac{2\mu}{c^2 r}} dr^2 + r^2 d\varphi^2 + r^2 \sin^2 \varphi d\theta^2.$$

The bi-metric system is a Rosen type bi-metric universe if and only if $a(r) = -c^k$. *In this case the second metric is the Schwarzschild classical metric.*

Proof: For i) and ii) the computations are exactly as in Theorem 6.5.
iii) If we denote by Γ_{ij}^k the Christoffel symbols of the R-Schwarschild metric $g_\mu^{\varepsilon=1}$ the only nonzero are

$$\Gamma_{01}^0 = \Gamma_{10}^0 = \frac{-B}{2r^2(1 + \frac{B}{r})}, \quad \Gamma_{00}^1 = \frac{B}{2r^2}(1 + \frac{B}{r}), \quad \Gamma_{11}^1 = \frac{B}{2r^2(1 + \frac{B}{r})}, \quad \Gamma_{22}^1 = -r(1 + \frac{B}{r}),$$

$$\Gamma_{33}^1 = -r(1+\frac{B}{r}) \sin^2 \varphi, \quad \Gamma_{21}^2 = \Gamma_{12}^2 = \frac{1}{r}, \quad \Gamma_{33}^2 = -\sin\varphi \cos\varphi, \quad \Gamma_{31}^3 = \Gamma_{13}^3 = \frac{1}{r}, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \cot\varphi,$$

where $B = \dfrac{-2\mu}{c^2}$. In the same way, if we denote by γ_{ij}^k the Christoffel symbols of the metric $g_{\mu,a}^{\varepsilon=-1}$ from (6.6) the only nonzero are

$$\gamma_{01}^0 = \gamma_{10}^0 = \frac{\frac{\dot{a}(r)}{c^k} - \frac{\mu}{c^2 r^2}}{1 + \frac{2\mu}{c^2 r} + \frac{2a(r)}{c^k}} = \frac{\frac{B}{2r^2} + \frac{\dot{a}(r)}{c^k}}{1 - \frac{B}{r} + \frac{2a(r)}{c^k}},$$

$$\gamma_{00}^1 = (\frac{\mu}{c^2 r^2} - \frac{\dot{a}\ (r)}{c^k})(1 - \frac{2\mu}{c^2 r}) = -\left(1 + \frac{B}{r}\right)\left(\frac{B}{2r^2} + \frac{\dot{a}\ (r)}{c^k}\right), \quad \gamma_{11}^1 = \frac{B}{2r^2(1 + \frac{B}{r})}, \quad \gamma_{22}^1 = -r(1+\frac{B}{r}),$$

$$\gamma_{33}^1 = -r(1 + \frac{B}{r}) \sin^2 \varphi, \quad \gamma_{21}^2 = \gamma_{12}^2 = \frac{1}{r}, \quad \gamma_{33}^2 = -\sin\varphi \cos\varphi, \quad \gamma_{31}^3 = \gamma_{13}^3 = \frac{1}{r}, \quad \gamma_{23}^3 = \gamma_{32}^3 = \cot\varphi.$$

According to the Rosen bigravity model we introduce $\Delta_{ij}^k = \Gamma_{ij}^k - \gamma_{ij}^k$. It results

$$\Delta_{01}^0 = \Delta_{10}^0 = \frac{\frac{-B}{2r^2}}{1 + \frac{B}{r}} - \frac{\frac{B}{2r^2} + \frac{\dot{a}(r)}{c^k}}{1 - \frac{B}{r} + \frac{2a(r)}{c^k}}, \quad \Delta_{00}^1 = \left(1 + \frac{B}{r}\right)\left(\frac{B}{r^2} + \frac{\dot{a}\ (r)}{c^k}\right)$$

and all the others $\Delta_{ij}^k = 0$. Let us consider the Riemannian curvature tensor

$$\bar{R}_{ijk}^h = \Delta_{ik,j}^h - \Delta_{ij,k}^h + \Delta_{mj}^h \Delta_{ik}^m - \Delta_{mk}^h \Delta_{ij}^m.$$

If we compute the Ricci tensor $\bar{R}_{ij} := \bar{R}_{ijh}^h$ for the modified gravitational field Δ_{ij}^k we have: $\bar{R}_{00} = \Delta_{00,1}^1 - \Delta_{01}^0 \Delta_{00}^1$, $\bar{R}_{11} = -\Delta_{01,1}^0 - (\Delta_{01}^0)^1$ and all others $\bar{R}_{ij} = 0$. We impose $\bar{R}_{00} = \bar{R}_{11} = 0$ and it results the system:

$$\Delta_{00,1}^1 - \Delta_{01}^0 \Delta_{00}^1 = 0; \quad \Delta_{01,1}^0 + (\Delta_{01}^0)^1 = 0.$$

Let us multiply the first equation by Δ_{01}^0 and the second equation by Δ_{00}^1. We obtain

$$\Delta_{00,1}^1 \Delta_{01}^0 - (\Delta_{01}^0)^1 \Delta_{00}^1 = 0; \quad \Delta_{01,1}^0 \Delta_{00}^1 + (\Delta_{01}^0)^1 \Delta_{00}^1 = 0,$$

that is

$$\Delta_{00,1}^1 \Delta_{01}^0 + \Delta_{01,1}^1 \Delta_{00}^1 = (\Delta_{00}^1 \Delta_{01}^0)_{,1} = 0.$$

Let us study the case

$$\Delta_{00}^1 \Delta_{01}^0 = 0$$

which leads to the differential equation

$$\left(\frac{\frac{B}{2r^2}}{1+\frac{B}{r}}+\frac{\frac{B}{2r^2}+\frac{\dot{a}(r)}{c^k}}{1-\frac{B}{r}+\frac{2a(r)}{c^k}}\right)\left(1+\frac{B}{r}\right)\left(\frac{B}{r^2}+\frac{\dot{a}\ (r)}{c^k}\right)=0.$$

By solving the differential equation $\dfrac{\dot{a}\ (r)}{c^k}=\dfrac{-B}{r^2}$ we obtain the solution $\dfrac{a(r)}{c^k}=\frac{B}{r}$, which, replaced in the formula of the light-adapted metric $g^{\varepsilon=-1}_{\mu,a}$, leads to the formula of the R-Schwarzschild metric $g^{\varepsilon=1}_{\mu}$. But we already proved that this one can not be considered as a light adapted metric in Theorem 6.5(iii). It remains to solve

$$\frac{\frac{B}{2r^2}}{1+\frac{B}{r}}+\frac{\frac{B}{2r^2}+\frac{\dot{a}(r)}{c^k}}{1-\frac{B}{r}+\frac{2a(r)}{c^k}}=0,$$

or equivalently

$$\dot{a}\ (r)=-\frac{B}{r^2+Br}a(r)-\frac{B}{r^2+Br}c^k.$$

According to the general theory, the solution is $a(r)=K\left(1+\frac{B}{r}\right)-c^k$, where K is a real constant. $K=0$ implies $a(r)=-c^k$. When we replace in the formula (6.6) of the general light-adapted metric $g^{\varepsilon=-1}_{\mu,a}$, it results the Schwarzschild's classical metric

$$ds^2=g^{\varepsilon=-1}_{\mu,-c^k}=-c^2\left(1-\frac{2\mu}{c^2r}\right)dt^2+\frac{1}{1-\frac{2\mu}{c^2r}}dr^2+r^2d\varphi^2+r^2\sin^2\varphi d\theta^2$$

and in this case the bending of light is part of the classical general relativity.

If $K\neq 0$, in order to preserve g_{11} dimensionless, we take $K=bc^k$. The condition $a(r)>0$ implies $b\geq 1$. For $b>1$ the formula (6.7) leads to a speed having the form:

$$\gamma=c(2-b)-\frac{\mu(1-2b)}{cr}-\frac{\mu(x^2)^2}{cr^3}+O\left(\frac{1}{c^3}\right).$$

Let us determine $b>1$ computing the total deflection and after, for this determined b we can see if it holds $\gamma<c$. The computation leads to

$$\left.\frac{\partial\gamma}{\partial x^1}\right|_{X_d}=(1-2b)\left.\frac{GMh}{cr^3}\right|_{X_d}+\left.\frac{3GMh(x^2)^2}{cr^5}\right|_{X_d}=(1-2b)\frac{GMh}{c(h^2+(x^2)^2)^{\frac{3}{2}}}+\frac{3GMh(x^2)^2}{c(h^2+(x^2)^2)^{\frac{5}{2}}}.$$

Then, the total deflection is

$$-\frac{1}{c}\int_{-\infty}^{\infty}\frac{\partial\gamma}{\partial x^1}dx^2=-\frac{GMh}{c^2}\left((1-2b)\int_{-\infty}^{\infty}\frac{dx^3}{(h^2+(x^2)^2)^{\frac{3}{2}}}+\int_{-\infty}^{\infty}\frac{3(x^2)^2dx^3}{(h^2+(x^2)^2)^{\frac{5}{2}}}\right)=$$

$$=-\left((1-2b)\frac{2}{h^2}+\frac{2}{h^2}\right)\frac{GMh}{c^2}=-(1-b)\frac{4GM}{hc^2}.$$

Since the total deflection is the previous formula, then $T(D)=-\frac{4GM}{hc^2}$ if and only if $b=0$, in collision with the fact that we are looking for $b>1$. As a conclusion, the only generalized light-adapted metric in this case is the original Schwarzschild metric.

Let us consider the other possible case

$$\Delta^1_{00}\Delta^0_{01}=-2k,$$

where $k\neq 0$. Replacing in the equation

$$\Delta^1_{00,1}-\Delta^0_{01}\Delta^1_{00}=0,$$

we obtain $\Delta^1_{00,1}=2k$ and hence

$$\Delta^1_{00}=2kr+l.$$

It results the differential equation

$$\left(1+\frac{B}{r}\right)\left(\frac{B}{r^2}+\frac{\dot{a}\ (r)}{c^k}\right)=2kr+l,$$

equivalent to

$$\frac{\dot{a}\,(r)}{c^k} = \frac{2kr+l}{1+\frac{B}{r}} - \frac{B}{r^2} = 2kr+l+O\left(\frac{1}{c^2}\right).$$

We replace directly in $\left.\frac{\partial \gamma}{\partial x^1}\right|_{X_d}$ and it results

$$\left.\frac{\partial \gamma}{\partial x^1}\right|_{X_d} = \left.\frac{GMh}{cr^3}\right|_{X_d} - \left.\frac{\dot{a}\,(r)h}{c^{k-1}r}\right|_{X_d} + \left.\frac{3GMh(x^2)^2}{cr^5}\right|_{X_d} =$$

$$= \frac{GMh}{c(h^2+(x^2)^2)^{\frac{3}{2}}} - 2khc - \frac{lhc}{(h^2+(x^2)^2)^{\frac{1}{2}}} + \frac{3GMh(x^2)^2}{c(h^2+(x^2)^2)^{\frac{5}{2}}}.$$

That is the formula

$$-\frac{1}{c}\int_{-\infty}^{\infty}\frac{\partial \gamma}{\partial x^1}dx^2 = -\frac{GMh}{c^2}\left(\int_{-\infty}^{\infty}\frac{dx^2}{(h^2+(x^2)^2)^{\frac{3}{2}}} + \int_{-\infty}^{\infty}\frac{3(x^2)^2dx^2}{(h^2+(x^2)^2)^{\frac{5}{2}}}\right) +$$

$$+h\left(2k\int_{-\infty}^{\infty}dx^2 + l\int_{-\infty}^{\infty}\frac{dx^2}{(h^2+(x^2)^2)^{\frac{1}{2}}}\right)$$

has to have the value $T(D) = -\frac{4GM}{hc^2}$. Therefore we need

$$\int_{-\infty}^{\infty}\left(2k + \frac{l}{(h^2+(x^2)^2)^{\frac{1}{2}}}\right)dx^2 = 0.$$

We prove using convergent integral techniques that $k=0$ and $l=0$. Indeed,

$$\lim_{x^2\to\infty}\left(2k + \frac{l}{(h^2+(x^2)^2)^{\frac{1}{2}}}\right) = 2k$$

implies that it exists α such that for any $x^2 > \alpha$ the values of $f(x^2) = 2k + \frac{l}{(h^2+(x^2)^2)^{\frac{1}{2}}}$ are in a small neighborhood of $2k$. Or this means that

$$\int_{\alpha}^{\infty}\left(2k + \frac{l}{(h^2+(x^2)^2)^{\frac{1}{2}}}\right)dx^2 = \infty,$$

in contradiction with our initial assumption. Therefore $k=0$ and it follows $l=0$, too. According to the previous case the only generalized light-adapted metric is the classical Schwarzschild metric.

So, the Rosen bimetric model we have created contains a Riemannian metric, namely the R-Schwarzschild metric, together with a semi-Riemannian metric, the Schwarzschild classical metric.

7. Conclusions

What is happening when we wish to shape the universe using a spacetime endowed with two metrics? If we consider this problem starting from the Einstein vacuum field equations $R_{ij} = 0$, we can prove that there is a Riemannian Schwarzschild-like stationary metric which satisfies these equations, who describes correctly the orbits of planets and the perihelion drift and predicts correctly the slowdown of clocks in the gravitational field. This remarkable metric is the R-Schwarzschild metric (6.1):

$$ds^2 = g_{\mu}^{\varepsilon=1} = c^2\cdot\left(1-\frac{2\mu}{c^2r}\right)dt^2 + \frac{1}{1-\frac{2\mu}{c^2r}}dr^2 + r^2d\varphi^2 + r^2\sin^2\varphi d\theta^2.$$

If we intend to describe the deflection of light, then we need to discuss the movement of a photon in the gravitational medium created by a metric. Theorem 6.6 shows that the axiom of the light speed limit is violated unless we change the first coefficient of the R-Schwarzschild metric. The new metric which arises is called Schwarzschild light-adapted metric, is denoted $g_{\mu}^{\varepsilon=-1}$ and has the form (6.5)

$$ds^2 = g_{\mu}^{\varepsilon=-1} = c^2\left(1+\frac{2\mu}{c^2 r}\right)dt^2 + \frac{1}{1-\frac{2\mu}{c^2 r}}dr^2 + r^2 d\varphi^2 + r^2\sin^2\varphi d\theta.$$

The bending of light using this metric leads to the same results for deflection as those computed with the Einstein metric from the general relativity.

But the Rosen compatibility condition is not satisfied. Even if the metric (6.5) is not the second metric of our bi-metric Rosen type universe, it is important because it helps us to find the right one. Theorem 6.7 shows that the only Rosen type bi-metric system which fulfills all the physical properties of our classical relativistic universe is described by the R-Schwarzschild metric

$$ds^2 = g_{\mu}^{\varepsilon=1} = c^2\left(1-\frac{2\mu}{c^2 r}\right)dt^2 + \frac{1}{1-\frac{2\mu}{c^2 r}}dr^2 + r^2 d\varphi^2 + r^2\sin^2\varphi d\theta,$$

together with the general light-adapted metric

$$ds^2 = g_{\mu,-c^k}^{\varepsilon=-1} = -c^2\left(1-\frac{2\mu}{c^2 r}\right)dt^2 + \frac{1}{1-\frac{2\mu}{c^2 r}}dr^2 + r^2 d\varphi^2 + r^2\sin^2\varphi d\theta^2.$$

The second metric is the original Schwarzschild metric. So, the Rosen type bi-metric universe is a mixed one. The first metric is a Riemannian metric. This R-Schwarzschild metric can be seen as its basic texture since this Riemannian metric rules over all objects having mass. The Schwarzschild light-adapted metric is a consequence of how this universe is reorganizing in order both to fulfill the Rosen compatibility condition and to preserve the speed of light limit axiom. This semi-Riemannian metric offers the texture of the "light-like" objects. The spatial part of both textures is the same. In this spatial part, the observers can interact with all the objects which work according to the fact that they have mass or they are "light-like".

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