A MIXED GRADIENT-TYPE DEFORMATION OF CONICS AND A CLASS OF FINSLERIAN-RIEMANNIAN FLOWS

M. CRASMAREANU

Dedicated to Academician Radu Miron on the occasion of his 91'th birthday

ABSTRACT. The aim of this paper is to produce new examples of (semi-) Riemannian and Finsler structures in dimension two having as model a scalar deformation of conics which involves a gradient vector field. It continues [6] from the point of view of relationship between quadratic polynomials (which provide equations of conics in dimension 2) and Finsler geometries. A type of Finslerian flow is introduced, based on the previous deformation and we completely solve the corresponding particular case of Riemannian flow.

INTRODUCTION

The recent paper [6], devoted to Finsler geometry, starts with a deformation of a conic Γ obtained by deforming the gradient vector field for the quadratic form defining Γ. This deformation is inspired by the scaling (linear) transformation of Computer Graphics: $(x,y) \in \mathbb{R}^2 \to (\lambda_x \cdot x, \lambda_y \cdot y) \in \mathbb{R}^2$, following [11, p. 136]. Using the well-known invariants from the Euclidean geometry of conics we give the classifications of the new conics which depends on two scalars denoted α and β. Since the new conic, denoted $\tilde{\Gamma}$, is a degenerate one we could interpret the map $\Gamma \to \tilde{\Gamma}$ as a "curve shortening" transformation.

In this following note we present another type of deformation based on the well-known involution of the plane. More precisely, we consider the (semi-) Riemannian metric $g_{\alpha,\beta}$ provided by the given scalars and the function defining the new conic is the result of applying $g_{\alpha,\beta}$ to the involution and the gradient vector field. We call *mixed deformation* the new conic and the diagonal case $\alpha = \beta$ is particularly analyzed. Moreover, we treat this deformation in terms of complex numbers.

In the next section we move to the Riemannian-Finslerian framework of dimension two and consider the deformation inspired by the previous section. We finish this paper with a type of Finslerian flows which can be the starting point of future studies following the way opened by the famous Ricci flow of Riemannian geometry, [4]. Due to the complex form of Finslerian deformation even in the Randers case, we can solve completely only the corresponding particular case of Riemannian flows. The solution is a time-dependent metric preserving the area. Let us point out that choosing two particular values of time, say t_0 and T, we get an initial metric $g_0 = g(t_0)$ and a final metric $g = g(T)$ and this setting is usually called *bi-metric study* on an arbitrary manifold. In dimension four some recent bi-metric approaches of spacetime geometries appear in [1]-[2] and [3] while a geometrical study in arbitrary dimension is the very old paper [8].

1. THE MIXED GRADIENT-TYPE DEFORMATION OF CONICS

In the two-dimensional Euclidean space $\mathbb{R}^2$ let us consider the conic Γ implicitly defined by $f \in C^\infty\left(\mathbb{R}^2\right)$ as: $\Gamma = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$ where f is a quadratic function of the form $f(x,y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$ with $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$.

It is well-known that the gradient vector field of f, namely $\nabla f = \left(f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}\right)$, gives important properties of Γ; for example, the centers of Γ are exactly the critical points of ∇f. Inspired by this fact we introduced in [6, p. 86]:

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Definition 1.1 Fix the scalars α, β with $\alpha\beta \neq 0$. The (α,β)-*deformation* of Γ is the conic:

$$\tilde{\Gamma} = \Gamma_{\alpha,\beta} : \alpha\left[\frac{1}{2}f_x\right]^2 + \beta\left[\frac{1}{2}f_y\right]^2 = 0. \tag{1.1}$$

If instead of the usual Euclidean metric g_{can} of the plane we consider the (semi-) Riemannian metric $g_{\alpha,\beta} = diag(\alpha,\beta)$ then the function defining $\tilde{\Gamma}$, namely $\alpha f_x^2 + \beta f_y^2$, is the square norm of the gradient field ∇f with respect to $g_{\alpha,\beta}$.

Now, we propose another deformation which involves the standard *involution of the plane*, $In : \mathbb{R}^2 \to \mathbb{R}^2$:

$$In(x,y) := (y,x). \tag{1.2}$$

This remarkable map yields the following notion:

Definition 1.2 The (α,β)-*mixed deformation* of Γ is the conic:

$$\Gamma^m = \Gamma^m_{\alpha,\beta} : f^m(x,y) := g_{\alpha,\beta}(In(x,y), \frac{1}{2}\nabla f(x,y)) = \alpha y\left[\frac{1}{2}f_x\right] + \beta x\left[\frac{1}{2}f_y\right] = 0. \tag{1.3}$$

Examples 1.3: i) Fix other non-vanishing scalars a, b. The ellipse $E(a,b) : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ and the hyperbola $H(a,b) : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ have the following (α,β)-mixed deformation:

$$E^m_{\alpha,\beta} : \left(\frac{\alpha}{a^2} + \frac{\beta}{b^2}\right)xy = 0, \quad H^m_{\alpha,\beta} : \left(\frac{\alpha}{a^2} - \frac{\beta}{b^2}\right)xy = 0. \tag{1.4}$$

Hence if $\left(\frac{\alpha}{a^2} + \frac{\beta}{b^2}\right)\left(\frac{\alpha}{a^2} - \frac{\beta}{b^2}\right) \neq 0$ we get the coordinate axes Ox, Oy. The equilateral hyperbola $\Gamma : 2xy = constant$ has the (α,β)-mixed deformation:

$$\Gamma^m : \beta x^2 + \alpha y^2 = 0 \tag{1.5}$$

which is the origin $O(0,0)$ for $\alpha\beta > 0$ and two secant lines through O if $\alpha\beta < 0$. These lines are orthogonal if and only if $\beta = -\alpha$ which means that Γ^m is exactly the pair of canonical bisectrices $B_\pm : y = \pm x$.
ii) For $p > 0$ let the parabola $P(p) : y^2 - 2px = 0$. Its (α,β)-mixed deformation is:

$$P^m_{\alpha,\beta} : \beta xy - \alpha py = 0 \tag{1.6}$$

which consists in two orthogonal lines: $y = 0$ respectively $x = \frac{p\alpha}{\beta} = constant$.
iii) Consider again the ellipse $E(a,b)$ with $a > b > 0$. The family of all *confocal* conics with $E(a,b)$ is given by:

$$\Gamma_\lambda : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} - 1 = 0 \tag{1.7}$$

for $\lambda \in \mathbb{R} \setminus \{a,b\}$. The (α,β)-mixed deformation of Γ_λ is:

$$(\Gamma_\lambda)^m_{\alpha,\beta} : \left(\frac{\alpha}{a-\lambda} + \frac{\beta}{b-\lambda}\right)xy = 0 \tag{1.8}$$

and it follows a discussion similar to the first example. $\square$

In order to study the (α,β)-mixed deformations we recall the algebraic invariants associated to Γ:

$$\Delta = \begin{vmatrix} r_{11} & r_{12} & r_{10} \\ r_{12} & r_{22} & r_{20} \\ r_{10} & r_{20} & r_{00} \end{vmatrix}, \quad D = \delta + Ir_{00} - r_{10}^2 - r_{20}^2, \quad I = r_{11} + r_{22}, \quad \delta = r_{11}r_{22} - r_{12}^2. \tag{1.9}$$

More precisely, the main result of this Section is:

Theorem 1.4 *The conic $\Gamma^m_{\alpha,\beta}$ has the following invariants:*

$$I^m = (\alpha+\beta)r_{12}, \quad \delta^m = \alpha\beta r_{12}^2 - \frac{1}{4}(\alpha r_{11} + \beta r_{22})^2, \quad D^m = \alpha\beta r_{12}^2 - \frac{1}{4}(\alpha r_{11} + \beta r_{22})^2 - \alpha^2 r_{10}^2 - \beta^2 r_{20}^2$$

$$\Delta^m = \frac{\alpha\beta}{4}\left[r_{10}r_{20}(\alpha r_{11} + \beta r_{22}) - r_{12}(\alpha r_{10}^2 + \beta r_{20}^2)\right]. \tag{1.10}$$

Then, if the initial conic Γ *does not have a linear part, i.e.* $r_{10} = r_{20} = 0$*, then* Γ^m *is a degenerate conic*: $\Delta^m = 0$.

Proof A straightforward computation yields the coefficients of Γ^m:

$$r^m_{11} = \beta r_{12}, \quad r^m_{12} = \alpha r_{11} + \beta r_{22}, \quad r^m_{22} = \alpha r_{12}, \quad r^m_{10} = \beta r_{20}, \quad r^m_{20} = \alpha r_{10}, \quad r^m_{00} = 0. \tag{1.11}$$

One obtain immediately the claimed relations. □

A special attention deserves the case $\alpha = \beta$ for which we have:

$$I^m = 2\alpha r_{12}, \quad \delta^m = -\frac{\alpha^2}{4}(r_{11} - r_{22})^2 \leq 0, \quad D^m = -\frac{\alpha^2}{4}(r_{11} - r_{22})^2 - \alpha^2(r^2_{10} + r^2_{20}) \leq 0,$$

$$\Delta^m = \frac{\alpha^3}{4}\left[r_{10}r_{20}I - r_{12}(r^2_{10} + r^2_{20})\right]. \tag{1.12}$$

By performing a second mixed deformation for this case we obtain:

$$\left(\Gamma^m_{\alpha,\alpha}\right)^m_{\alpha,\alpha} : \frac{I}{2}\left(x^2 + y^2\right) + 2r_{12}xy + \frac{r_{10}}{2}x + \frac{r_{20}}{2}y = 0. \tag{1.13}$$

Also, an interesting fact is the comparison between the iterations of (α, β) and (α, β)-mixed deformation of Γ:

$$\widetilde{\Gamma^m} : [r^2_{12} + \frac{(\alpha r_{11} + \beta r_{22})^2}{\alpha\beta}](\beta x^2 + \alpha y^2) + 2r_{12}[2(\alpha r_{11} + \beta r_{22})xy + \beta r_{20}x + \alpha r_{10}y] + \alpha r^2_{10} + \beta r^2_{20} = 0, \tag{1.14}$$

$$(\tilde{\Gamma})^m : \alpha(\beta r_{12}x + \alpha r_{11}y)(r_{11}x + r_{12}y + r_{10}) + \beta(\beta r_{22}x + \alpha r_{12}y)(r_{12}x + r_{22}y + r_{20}) = 0. \tag{1.15}$$

Again the equality case $\alpha = \beta$ gives for the last two relations:

$$\widetilde{\Gamma^m} : (r^2_{12} + I^2)(x^2 + y^2) + 4r_{12}Ixy + 2(r_{12}r_{20})x + 2(r_{12}r_{20})y + (r^2_{10} + r^2_{20}) = 0, \tag{1.16}$$

$$(\tilde{\Gamma})^m : (r_{12}x + r_{11}y)(r_{11}x + r_{12}y + r_{10}) + (r_{22}x + r_{12}y)(r_{12}x + r_{22}y + r_{20}) = 0. \tag{1.17}$$

Returning to the general case of α and β we treat the mixed deformation with complex numbers following the model of [7]. More precisely, with the usual notation $z = x + iy \in \mathbb{C}$ we derive the complex expression of Γ:

$$\Gamma : F(z, \bar{z}) := Az^2 + Bz\bar{z} + \bar{A}\bar{z}^2 + Cz + \bar{C}\bar{z} + r_{00} = 0 \tag{1.18}$$

with:

$$A = \frac{r_{11} - r_{22}}{4} - \frac{r_{12}}{2}i \in \mathbb{C}, \quad 2B = r_{11} + r_{22} = I \in \mathbb{R}, \quad C = r_{10} - r_{20}i \in \mathbb{C}. \tag{1.19}$$

It follows that the usual rotation performed with the angle φ to eliminate the mixed term xy has the meaning to reduce/rotate A in the real line while the translation which eliminates the term y has a similar meaning with respect to C. The inverse relationship between f and F is:

$$r_{11} = B + 2\Re A, \quad r_{22} = B - 2\Re A, \quad r_{12} = -2\Im A, \quad r_{10} = \Re C, \quad r_{20} = -\Im C \tag{1.20}$$

with $\Re$ and $\Im$ respectively the real and imaginary part. Hence the angle φ is provided by the formula:

$$\tan 2\varphi := \frac{2r_{12}}{r_{11} - r_{22}} = -\frac{\Im A}{\Re A} = -\tan \arg A \to 2\varphi = -\arg A. \tag{1.21}$$

The expression of the invariants of Γ in terms of A, B, C is:

$$I = 2B, \quad \delta = B^2 - 4|A|^2, \quad D = \delta + 2r_{00}I - |C|^2 \tag{1.22$_1$}$$

$$\Delta = r_{00}(B^2 - 4|A|^2) - B|C|^2 + 2\Re C(\Re A\Re C + \Im A\Im C) + 2\Im C(\Re C\Im A - \Re A\Im C). \tag{1.22$_2$}$$

By using (1.11) we derive the transformation of the complex coefficients:

$$\tilde{A} = (\beta - \alpha)(-\frac{\Im A}{2} + \Re Ai) - \frac{\alpha + \beta}{2}Bi, \quad \tilde{B} = -(\alpha + \beta)\Im A, \quad \tilde{C} = -\beta\Im C - \alpha\Re Ci. \tag{1.23}$$

For the considered particular case $\alpha = \beta$ we obtain:

$$\tilde{A} = -\alpha Bi, \quad \tilde{B} = -2\alpha\Im A, \quad \tilde{C} = -\alpha(\Im C + \Re Ci) = -\alpha\bar{C}i. \tag{1.24}$$

Returning to the general complex formalism above, in the case of a non-degenerate Γ, which means $\Delta \neq 0$, we can also express *the eccentricity* e by:

$$e^2 := 2 - \frac{I}{\lambda} = 1 - \frac{\delta}{\lambda^2}, \quad \lambda^2 - I\lambda + \delta = 0. \tag{1.25}$$

It follows that μ and e are provided by:

$$\lambda_{\pm} := B \pm 2|A| \to e^2 = \frac{\pm 4|A|}{B \pm 2|A|}. \tag{1.26}$$

2. The mixed deformation of two-dimensional Finsler structures

Let M be an open subset of $\mathbb{R}^m$ considered as a smooth m-dimensional manifold with $m \geq 2$ and $\pi : TM \to M$ its tangent bundle. Let $x = (x^i) = (x^1, ..., x^m)$ be the coordinates on M and $(x, y) = (x^i, y^i) = (x^1, ..., x^m, y^1, ..., y^m)$ the induced coordinates on TM. Denote by O the null-section of π.

Recall after [10] that a *Finsler fundamental function* on M is a map $F : TM \to \mathbb{R}_+$ with the following properties:
F1) F is smooth on the slit tangent bundle $T_0M := TM \setminus O$ and continuous on O,
F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$,
F3) the matrix $(g_{ij}) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is invertible and its associated quadratic form is of constant rank.
The tensor field $g = \{g_{ij}(x, y); 1 \leq i, j \leq m\}$ is called *the Finsler metric* and the homogeneity of F implies:

$$F^2(x, y) = g_{ij}y^iy^j = y_iy^i \tag{2.1}$$

where $y_i = g_{ij}y^j$. The pair (M, F) is called *Finsler manifold.* We point out the possibility of singular Finsler metrics as in [9].

Fix now the dimension $m = 2$ for which we change the notation: $(x^1, x^2) \to (x, y)$, $(y^1, y^2) \to (\dot{x}, \dot{y})$. Fix also the vector $\bar{\alpha} = (\alpha, \beta) \in \mathbb{R}^2$ with all strictly positive components although there are cases when some of them can be null or even negative. Inspired by the previous Section we introduce:

Definition 2.1 The $\bar{\alpha}$-*mixed deformation* of F is $F^m = F^m_{\bar{\alpha}} : TM \to \mathbb{R}$ given by:

$$F^m = \sqrt{\alpha\dot{y}\left[\frac{1}{2}(F^2)_{\dot{x}}\right] + \beta\dot{x}\left[\frac{1}{2}(F^2)_{\dot{y}}\right]}. \tag{2.2}$$

From (2.1) due to homogeneity it results a basic equation of Finsler geometry:

$$\frac{1}{2}(F^2)_{y^i} = g_{ij}y^j \tag{2.3}$$

and then the $\bar{\alpha}$-mixed deformation of F is:

$$F^m = \sqrt{\alpha\dot{y}\,(g_{1j}y^j) + \beta\dot{x}\,(g_{2j}y^j)}. \tag{2.4}$$

This new Finslerian fundamental function yields a new Finslerian metric $g^m = g^{\bar{\alpha}}$ which we call *the* $\bar{\alpha}$-*mixed deformation* of g. A straightforward computation yields"

$$g^m_{11} = \beta g_{12} + \frac{1}{2}\left(\alpha\dot{y}\frac{\partial g_{11}}{\partial \dot{x}} + \beta\dot{x}\frac{\partial g_{12}}{\partial \dot{x}}\right), \quad g^m_{22} = \alpha g_{12} + \frac{1}{2}\left(\alpha\dot{y}\frac{\partial g_{12}}{\partial \dot{y}} + \beta\dot{x}\frac{\partial g_{22}}{\partial \dot{y}}\right)$$

$$g^m_{12} = \frac{1}{2}(\alpha g_{11} + \beta g_{22}) + \frac{1}{2}\left(\alpha\dot{y}\frac{\partial g_{11}}{\partial \dot{y}} + \beta\dot{x}\frac{\partial g_{12}}{\partial \dot{y}}\right). \tag{2.5}$$

Example 2.2 (Riemannian geometry) Let $a = (a_{ij}(x))$ be a Riemannian metric on M. It is well-known that $F = \sqrt{a_{ij}y^iy^j}$ is a Finslerian structure on M with $g = a$. Then (2.5) yields a semi-Riemannian metric a^m:

$$a^m := \begin{pmatrix} \beta a_{12} & \frac{1}{2}(\alpha a_{11} + \beta a_{22}) \\ \frac{1}{2}(\alpha a_{11} + \beta a_{22}) & \alpha a_{12} \end{pmatrix}. \tag{2.6}$$

For example, the Euclidean metric g_{can} is transformed into the hyperbolic metric: $g^m_{can} = \frac{\alpha+\beta}{2}dxdy$. By performing a second deformation in the case $\alpha = \beta$ we get $(g^m_{can})^m = \alpha^2 g_{can}$ which is a homothetical transformation. Hence, if $\alpha = \beta = 1$ we get an involution on the positive cone of conformal Euclidean metrics $ConfEuclidean = \{\lambda g_{can}; \lambda \in (0, +\infty)\}$. □

Example 2.3 (Randers geometry) Let F be a Randers fundamental function of Minkowski type:

$$F_b(x, y, \dot{x}, \dot{y}) = F_b(\dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} + b\dot{x} \tag{2.7}$$

with $0 < b < 1$. The corresponding Finsler metric is:

$$g^b_{11} = 1 + b^2 + b\frac{2\dot{x}^3 + 3\dot{x}\dot{y}^2}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}, \quad g^b_{12} = \frac{b\dot{y}^3}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}, \quad g^b_{22} = 1 + \frac{b\dot{x}^3}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}. \tag{2.8}$$

The new Finslerian metric with $\alpha = \beta$ is:

$$g^{b,m} = \alpha \begin{pmatrix} \frac{b\dot{y}^3}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} & 1 + \frac{b^2}{2} + \frac{3b\dot{x}}{(2\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}} \\ 1 + \frac{b^2}{2} + \frac{3b\dot{x}}{2(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}} & \frac{b\dot{y}^3}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} \end{pmatrix}. \tag{2.9}$$

This proves that the new Finslerian structure F^m defines a completely new Finsler geometry on M. □

Example 2.4 (Spherically symmetric Finsler functions) Let $I \subseteq \mathbb{R}_+$ be an interval and $A, B : I \to \mathbb{R}$ two smooth functions. We define the orthogonal invariant Finsler function:

$$F(x, y, \dot{x}, \dot{y}) = \sqrt{A(x^2 + y^2)(\dot{x}^2 + \dot{y}^2) + B(x^2 + y^2) < (x, y), (\dot{x}, \dot{y}) >^2_{can}}. \tag{2.10}$$

Its Finsler metric is a non-diagonal Riemannian one:

$$g_{11} = A + Bx^2, \quad g_{12} = Bxy, \quad g_{22} = A + By^2. \tag{2.11}$$

The new Finslerian fundamental function is:

$$F^m(x, y, \dot{x}, \dot{y}) = \sqrt{A(\alpha + \beta)\dot{x}\dot{y} + \frac{B}{2}(y\dot{x} + x\dot{y})} \tag{2.12}$$

which generates the semi-Riemannian metric:

$$g^m = (\alpha + \beta) \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}. \tag{2.13}$$

□

3. Finslerian flows

For the given manifold M let $Finsler(M \times \mathbb{R})$ be the infinite space of all possible time-dependent Finslerian metrics on M as well as $T^s_2(TM \times \mathbb{R})$ the space of all time-dependent symmetric tensor fields of $(0, 2)$-type on TM. Having the theory of geometric (more precisely Riemannian) flows as example we introduce:

Definition 3.1 A *Finslerian flow* on M is a dynamical system modeled by the partial differential equations:

$$\partial_t g_t = \mathcal{F}(g_t) \tag{3.1}$$

where $\mathcal{F} : Finsler(M \times \mathbb{R}) \to T^s_2(TM \times \mathbb{R})$ is a given map and g_t is a family of Finslerian metrics depending on the parameter t belonging to the interval $I \subseteq \mathbb{R}$.

Examples 3.2 i) (Special Riemannian flows) If we restrict the functional $\mathcal{F}$ to $Riemann(M \times \mathbb{R})$ to be the (-2)Ricci curvature then we obtain the famous Ricci flow provided the proof of two outstanding conjectures: Poincaré Conjecture and Thurston Geometrization Conjecture. For a relationship between Randers metrics and Ricci solitons via the Zermelo navigation problem, see [5].
ii) Other famous Riemannian flows are: the Calabi flow and the Yamabe flow.
iii) Time-dependent Randers metrics are recently used in the study of causal relationships on space-time manifolds in [12]. □

Returning to the general Finslerian framework and vector $\bar{\alpha}$ of previous Section we consider:

Definition 3.3 The *Finslerian $\bar{\alpha}$-mixed flow* is that given by:

$$\mathcal{F}(g) = g^m = g^{\bar{\alpha}}. \tag{3.2}$$

Inspired by [6, p. 96] we introduce the corresponding *area variation* as the smooth function $A : TM \times \mathbb{R} \to TM \times \mathbb{R}$ given by:

$$\partial_t A(x, y, \dot{x}, \dot{y}, t) = \sum_{i,j=1}^{m} \bar{g}_{ij} g^{ij} \tag{3.3}$$

where, as usual, g^{ij} are the components of inverse g^{-1}.

Example 3.4 (Riemannian $\bar{\alpha}$-flow) With the notations of Example 2.2 we have:

$$\partial_t a_{11} = \beta a_{12}, \quad \partial_t a_{12} = \frac{1}{2}(\alpha a_{11} + \beta a_{22}), \quad \partial a_{22} = \alpha a_{12}. \tag{3.4}$$

with the direct consequence:

$$\partial_t^2 a_{12} = \alpha\beta a_{12}. \tag{3.5}$$

Case 1) $0 < \alpha\beta = \rho^2$. Then $a_{12}(t) = A(x, y)\cosh(\rho t) + B(x, y)\sinh(\rho t)$ on $I = \mathbb{R}$. We get: $a_{11}(t) = \frac{\beta}{\rho}\left[A(x, y)\sin h(\rho t) + B(x, y)\cosh(\rho t)\right]$ and $a_{22}(t) = \frac{\alpha}{\rho}\left[A(x, y)\sin h(\rho t) + B(x, y)\cosh(\rho t)\right]$.
Case 2) $0 > \alpha\beta = -\rho^2$. Then $a_{12}(t) = A(x, y)\cos(\rho t) + B(x, y)\sin(\rho t)$ also on $I = \mathbb{R}$. We obtain: $a_{11}(t) = \frac{\beta}{\rho}\left[A(x, y)\sin(\rho t) - B(x, y)\cos(\rho t)\right]$ and $a_{22}(t) = \frac{\alpha}{\rho}\left[A(x, y)\sin(\rho t) - B(x, y)\cos(\rho t)\right]$.

We have immediately that $\partial_t A(x, y, \dot{x}, \dot{y}, t) = 0$ which means that the Riemannian $\bar{\alpha}$-mixed flow preserves the area, and hence can be considered as a symplectic tool of study. □

We finish with the following remark: in the reference [13], from two Finsler functions F_+, F_-, it is obtained a *bi-metric*:

$$F = \sqrt{F_+ \cdot F_-}. \tag{3.6}$$

The negative result of [13] concerning the physical implications of this metric as well as the considerations of our Section 1 suggests other two deformations:

$$F_{2,\alpha,\beta} = \sqrt{\alpha F_+^2 + \beta F_-^2}, \quad F_{m,\alpha,\beta} = \sqrt[m]{\alpha F_+^m + \beta F_-^m}, \quad m \in \mathbb{N}^* \tag{3.7}$$

which will be studied in a future work.

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Faculty of Mathematics
University "Al. I.Cuza"
Iasi, 700506
România
E-mail: mcrasm@uaic.ro
http://www.math.uaic.ro/~mcrasm