Noether theorem for time-dependent higher order Lagrangians

Mircea Crâşmăreanu

Proceedings of Romanian Academy, Series A, 1(2000), no. 2, 77-79. MR 2027026, Zbl 1034.53019

Abstract

In this paper we obtain the time-dependent version of Noether theorem, given for time-independent higher order Lagrangians in [3], [4] and [5]. Altough our characterization (16) for Noether symmetries is exacly equation (17) from [6, p. 224], our Noetherian first integral (19) appears in another form that the similar one (21) in [6, p. 225] because we used the so-called "higher-order energies".

2000 Subject Classification: 70H33, 37J15.
Key words and phrases: Lagrangian of order k, k-jet bundle, symmetry transformation, Noether theorem.

In order to give a geometrical approach of analytical mechanics one must specify an appropriate space. If the mechanical system under consideration has n degree of freedom then the first requirement is a fibred manifold $\pi : E \longrightarrow I\!R$ with standard fibre a manifold M which has dimension n. If we discuss the mechanics of Lagrangians involving accelerations of order k then the main arena for our account is the bundle $\pi_k : J^k\pi \longrightarrow E$ of k-jets of sections of π.

Given a trivialization $E \cong I\!R \times M$ one may identify $J^k\pi$ with $I\!R \times T^kM$ where T^kM is the k-tangent bundle of M([2]). Fibered coordinates in E

will be denoted by (t, x^i) where $(x^i)_{1 \leq i \leq n}$ are the local coordinates in the fibre M. The fibred manifold $\pi : E \longrightarrow I\!R$ is *the configuration manifold* and the k-jet prologation $J^k\pi$ is *the generalized phase space*. We introduce local coordinates $(t, x^i, x^i_1, \ldots, x^i_k)$ in $J^k\pi$ with:

$$x^i_\alpha \left(j^k c\right) = \frac{d^\alpha \left(x^i \circ c\right)}{dt^\alpha},\ 1 \leq \alpha \leq k \tag{1}$$

for c a section of π and $j^k c \in J^k\pi$ the associated k-jet.

Definition 1 A Lagrangian of k-order is a smooth mapping $L : J^k\pi \longrightarrow I\!R$.

If $c : [0, 1] \longrightarrow M$ is a regular curve on M, i. e. a section of π, then the *integral of action* of L on c is given by:

$$I\left(c\right) = \int_0^1 L\left(t, x\left(t\right), \frac{dx}{dt}\left(t\right), \frac{d^2x}{dt^2}\left(t\right) \ldots, \frac{d^kx}{dt^k}\left(t\right)\right) dt. \tag{2}$$

Applying the usual variational principle to $I(c)$ we get:

Proposition 2([2]) *If c is an extremal curve of the functional $I(c)$ then the following Euler-Lagrange equations hold:*

$$E_i\left(L\right) = 0 \tag{3}$$

where:

$$E_i = \frac{\partial}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial}{\partial x^i_1}\right) + \ldots + (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial}{\partial x^i_k}\right) \tag{4}$$

is the Euler-Lagrange operator of k-order.

Let us consider on $J^k\pi$ *the Liouville vector fields* $\overset{1}{\Gamma}, \ldots, \overset{k}{\Gamma}$ given by([5]):

$$\overset{\alpha}{\Gamma} = x^i_1 \frac{\partial}{\partial x^i_{k-\alpha+1}} + \ldots + x^i_\alpha \frac{\partial}{\partial x^i_k} \tag{5}$$

and *the higher order energies* of L([5]):

$$\begin{cases} \mathcal{E}^k(L) = \overset{k}{\Gamma}(L) - \frac{1}{2!}\frac{d}{dt}\overset{k-1}{\Gamma}(L) + \ldots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\overset{1}{\Gamma}(L) - L, \\ \mathcal{E}^{k-1}(L) = -\frac{1}{2!}\overset{k-1}{\Gamma}(L) + \frac{d}{dt}\overset{k-2}{\Gamma}(L) + \ldots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-2}}{dt^{k-2}}\overset{1}{\Gamma}(L), \\ \mathcal{E}^{k-2}(L) = \frac{1}{3!}\overset{k-2}{\Gamma}(L) - \overset{k-3}{\Gamma}(L) + \ldots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-3}}{dt^{k-3}}\overset{1}{\Gamma}(L), \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \mathcal{E}^1(L) = (-1)^{k-1}\frac{1}{k!}\overset{1}{\Gamma}(L) \end{cases} \tag{6}$$

A straightforward computation yields:

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{dx^i}{dt}E_i(L) + \frac{d}{dt}\overset{k}{\Gamma}(L) - \frac{d^2}{dt^2}\overset{k-1}{\Gamma}(L) + \ldots + (-1)^{k-1}\frac{d^k}{dt^k}\overset{1}{\Gamma}(L) \tag{7}$$

$$\frac{d\mathcal{E}^k(L)}{dt} = -E_i(L)\frac{dx^i}{dt} - \frac{\partial L}{\partial t} \tag{8}$$

and for $\tau \in C^\infty(I\!R \times M)$:

$$\frac{d\tau}{dt}L - \left[\frac{d\tau}{dt}\overset{k}{\Gamma}(L) + \frac{d^2\tau}{dt^2}\overset{k-1}{\Gamma}(L) + \ldots + \frac{d^k\tau}{dt^k}\overset{1}{\Gamma}(L)\right] = \tau\frac{d\mathcal{E}^k(L)}{dt} +$$

$$+\frac{d}{dt}\left[-\tau\mathcal{E}^k(L) + \frac{d\tau}{dt}\mathcal{E}^{k-1}(L) - \frac{d^2\tau}{dt^2}\mathcal{E}^{k-2}(L) + \ldots + (-1)^k\frac{d^{k-1}\tau}{dt^{k-1}}\mathcal{E}^1(L)\right]. \tag{9}$$

From relation (8) it results *conservation of energy for TID(time-independent) Lagrangians*([5]):

Theorem 3(Miron) *If L is TID then $\mathcal{E}^k(L)$ is a conservation law, that is $\mathcal{E}^k(L)$ is conserved along the extremals of L.*

Now, after these preparatory results, we state the Noether theorem concerning the invariance of higher-order Lagrangians. For the original Noether theorem see [7].

For V a vector field on E with local expression:

$$V = \tau\frac{\partial}{\partial t} + V^i\frac{\partial}{\partial x^i}$$

let us consider the induced point-transformation:

$$\tilde{t} = t + \varepsilon\tau \tag{10a}$$

$$\tilde{x}^i = x^i + \varepsilon V^i, \quad 1 \le i \le n \tag{10b}$$

with ε being a real number with sufficiently small absolute value.

Definition 4 The transformation (10) is called *symmetry transformation* for L if there exists $\Phi \in C^\infty\left(J^k\pi\right)$, called *gauge*, such that:

$$\tilde{L}d\tilde{t} = \left(L + \varepsilon\frac{d\Phi}{dt}\right)dt + o\left(\varepsilon\right) \tag{11}$$

where $\tilde{L}$ is $L\left(\tilde{t}, \tilde{x}^i, \tilde{x}^i_1, \ldots, \tilde{x}^i_k\right)$.

Retained the terms of first order in ε we get:

$$\frac{d\tilde{t}}{dt} = 1 + \varepsilon\frac{d\tau}{dt} \tag{12a}$$

$$\frac{dt}{d\tilde{t}} = 1 - \varepsilon\frac{d\tau}{dt} \tag{12b}$$

and:

$$\begin{cases} \frac{d\tilde{x}^i}{d\tilde{t}} = \frac{dx^i}{dt} + \varepsilon\Phi^i_1, \\ \frac{d^2\tilde{x}^i}{d\tilde{t}^2} = \frac{d^2x^i}{dt^2} + \varepsilon\Phi^i_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d^k\tilde{x}^i}{d\tilde{t}^k} = \frac{d^kx^i}{dt^k} + \varepsilon\Phi^i_k \end{cases} \tag{13}$$

where:

$$\Phi^i_{\alpha+1} = \frac{d\Phi^i_\alpha}{dt} - x^i_{\alpha+1}\frac{d\tau}{dt}, 0 \le \alpha \le k-1 \tag{14}$$

with $\Phi^i_0 = V^i$.

Returning to L it results:

$$\tilde{L} = \left(L + \varepsilon\frac{d\Phi}{dt}\right)\frac{dt}{d\tilde{t}}$$

and therefore:

$$\tilde{L} - L = \varepsilon\left(\frac{d\Phi}{dt} - L\frac{d\tau}{dt}\right). \tag{15}$$

Using the Taylor expansion of first order for $\tilde{L}$ and identifying the coefficients of ε we have:

Theorem 5 *The transformation* (10) *is symmetry transformations for L with the gauge* Φ *if and only if:*

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x^i}V^i + \frac{\partial L}{\partial x_1^i}\Phi_1^i + \ldots + \frac{\partial L}{\partial x_k^i}\Phi_k^i = \frac{d\Phi}{dt} - L\frac{d\tau}{dt}. \tag{16}$$

The last relation is equivalent with:

$$E_i(L) + \frac{d}{dt}I_V^k(L) - \frac{d^2}{dt^2}I_V^{k-1}(L) + \ldots + (-1)^{k-1}\frac{d^k}{dt^k}I_V^1(L) +$$
$$+\tau\left(\frac{d}{dt}\mathcal{E}^k(L) + \frac{\partial L}{\partial t}\right) +$$
$$+\frac{d}{dt}\left[-\tau\mathcal{E}^k(L) + \frac{d\tau}{dt}\mathcal{E}^{k-1}(L) - \ldots + (-1)^k\frac{d^{k-1}\tau}{dt^{k-1}}\mathcal{E}^1(L)\right] = \frac{d\Phi}{dt} \tag{17}$$

where the vector fields $I_V^1, \ldots, I_V^k$ are:

$$\begin{cases} I_V^1 = k!V^i\frac{\partial}{\partial x_k^i}, \\ I_V^2 = (k-1)!V^i\frac{\partial}{\partial x_{k-1}^i} + k!\frac{dV^i}{dt}\frac{\partial}{\partial x_k^i}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ I_V^k = V^i\frac{\partial}{\partial x_1^i} + 2\frac{dV^i}{dt}\frac{\partial}{\partial x_2^i} + \ldots + k\frac{d^{k-1}V^i}{dt^{k-1}}\frac{\partial}{\partial x_k^i}. \end{cases} \tag{18}$$

Following the terminology of classical mechanics, that is $k = 1$([1, p. 184]), we call equation (16) or equivalenty (17) *the generalized Killing equations of L.*

In the final we obtain the main result of this paper:

Theorem 6(Noether) *If* (10) *is symmetry transformation for L with the gauge* Φ *then the following function* $\mathcal{F} = \mathcal{F}(L, V, \Phi)$ *is conserved along the extremals of L:*

$$\mathcal{F} = I_V^k(L) - \frac{1}{2!}\frac{d}{dt}I_V^{k-1}(L) + \ldots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}I_V^1(L) -$$
$$-\tau\mathcal{E}^k(L) + \frac{d\tau}{dt}\mathcal{E}^{k-1}(L) - \ldots + (-1)^k\frac{d^{k-1}\tau}{dt^{k-1}}\mathcal{E}^1(L) - \Phi. \tag{19}$$

For $k = 1$ we reobtain a well-known result([1, p. 184]):

$$\mathcal{F} = V^i \frac{\partial L}{\partial x_1^i} - \tau \left(\frac{\partial L}{\partial x_1^i} x_1^i - L \right) - \Phi. \tag{20}$$

Another version of Noether theorem for higher-order time-dependent Lagrangians, based on exterior differential calculus applied to the Cartan forms associated to the Lagrangian, appears in [2].

He hope to report on the applications elsewhere.

References

[1] Bocaletti, D., Pucacco, G., *Killing equations in classical mechanics*, Il Nuovo Cimento, Vol. 112(1997), N. 2-3, 181-212.

[2] de León, M., De Diego, D. M., *Symmetries and constants of motion for higher-order Lagrangian systems*, J. Math. Phys., 36(1995), 4138-4161.

[3] Miron, R., *Noether theorems in higher-order Lagrangian mechanics*, Int. J. Theor. Phys., 34(1995), no. 7, 1123-1146.

[4] Miron, R., *Noether type theorems in higher order analytical mechanics*, in Tamassy, L.(ed.)-"*New developments in differential geometry*", Kluwer, vol. 350, 1996, 313-317.

[5] Miron, R., *The Geometry of Higher-Order Lagrange Spaces(Applications to Mechanics and Physics)*, Fundamental Theories of Physics, no. 82, Kluwer Academic Publishers, 1997.

[6] Moyo, S., Leach, P. G. L., *Noether's theorem in classical mechanics*, Sci. Bull. U. P. Bucureşti, Series A, 60(1998), no. 3-4, 221-234.

[7] Noether, E., *Invariante Variationsprobleme*, in "Gesammelte Abhandlungen", Springer Verlag, 1983, 249-270.

Faculty of Mathematics
University "Al. I. Cuza"
Iaşi, 6600, Romania
E-mail: mcrasm@uaic.ro

and
Institute of Mathematics "Octav Mayer"
Iaşi Branch of Romanian Academy
Iaşi, 6600, Romania