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# INEQUALITIES FOR GRADIENT EINSTEIN AND RICCI SOLITONS

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**Abstract.** This short note concerns with two inequalities in the geometry of gradient Einstein solitons $(g, f, \lambda)$ on a smooth manifold $M$. These inequalities provide some relationships between the curvature of the Riemannian metric $g$ and the behavior of the scalar field $f$ through two quadratic equations satisfied by the scalar $\lambda$. The similarity with gradient Ricci solitons and a slight generalization involving a $g$-symmetric endomorphism $A$ are provided.
**Keywords**: gradient Einstein solitons; smooth manifold; Riemannian metric; $g$-symmetric endomorphism.

## 1. Introduction

Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold endowed with a smooth function $f \in C^\infty(M)$; an excellent textbook in Riemannian geometry is [6]. The scalar field $f$ yields *the Hessian endomorphism*: $h_f : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $h_f(X) = \nabla_X \nabla f$, where $\nabla$ is the Levi-Civita connection of $g$. Then we know the symmetry of *the Hessian tensor field* of $f$: $H_f(X, Y) := g(h_f(X), Y)$, namely $H_f(X, Y) = H_f(Y, X)$. What follows is the existence of a $g$-orthonormal frame field $E = \{E_i\}_{i=1,\dots,n} \subset \mathfrak{X}(M)$ and the existence of the eigenvalues $\lambda = \{\lambda_i\}_{i=1,\dots,n} \subset C^\infty(M)$:

$$h_f(E_i) = \lambda_i E_i. \tag{1.1}$$

Hence we express all the geometric objects related to $f$ in terms of the pair $(E, \lambda)$ which we call *the spectral data* of $f$:

$$\nabla f = \sum_{i=1}^{n} E_i(f) E_i, \quad \|\nabla f\|_g^2 = \sum_{i=1}^{n} [E_i(f)]^2, \quad h_f(X) = \sum_{i=1}^{n} (\lambda_i X^i) E_i, \tag{1.2}$$

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for $X = \sum_{i=1}^{n} X^i E_i$. Also *the Hessian* and *the Laplacian* of $f$ are:

$$H_f(X,Y) = \sum_{i=1}^{n} \lambda_i (X^i Y^i), \quad \Delta f := Tr_g H_f = \sum_{i=1}^{n} \lambda_i. \tag{1.3}$$

Let us remark that if $\nabla f$ does not have zeros and $E_1$ is exactly its unit vector field i.e. $E_1 = \frac{\nabla f}{\|\nabla f\|_g}$, then $\nabla f$ is a *geodesic vector field*: $\nabla_{\nabla f} \nabla f = \lambda_1 \nabla f$ which means that the flow of $\nabla f$ consists in geodesics of $g$.

## 2. Results

Assume now that the triple $(g, f, \lambda \in \mathbb{R})$ is a *gradient Einstein soliton* on $M$, [2, p. 67]:

$$H_f + Ric + \left(\lambda - \frac{R}{2}\right) g = 0, \tag{2.1}$$

where $Ric$ is the Ricci tensor field of $g$ and $R$ is the scalar curvature. Einstein solitons generate self-similar solutions of the Einstein flow (1.1) of [2] and are more rigid than the well-known Ricci solitons. By considering the Ricci endomorphism $Q \in \mathcal{T}_1^1(M)$ provided by:

$$Ric(X,Y) = g(QX,Y), \tag{2.2}$$

we can express (2.1) as:

$$h_f + Q + \left(\lambda - \frac{R}{2}\right) I = 0 \tag{2.3}$$

with $I$ the Kronecker endomorphism. From (2.3) we get that $Q$ is also of diagonal form with respect to the frame $E$:

$$Q(X) = -\sum_{i=1}^{n} \left(\lambda_i + \lambda - \frac{R}{2}\right) X^i E_i, \quad \|Q\|_g^2 = \sum_{i=1}^{n} \left(\lambda_i + \lambda - \frac{R}{2}\right)^2. \tag{2.4}$$

By developing the second formula above we derive:

$$\|Ric\|_g^2 = \sum_{i=1}^{n} \lambda_i^2 + (2\lambda - R) \sum_{i=1}^{n} \lambda_i + n\left(\lambda^2 - \lambda R + \frac{R^2}{4}\right) =$$

$$= \|H_f\|_g^2 + (2\lambda - R)\Delta f + n\left(\lambda^2 - \lambda R + \frac{R^2}{4}\right). \tag{2.5}$$

Hence the scalar $\lambda$ is a solution of the quadratic equation:

$$n\lambda^2 + 2\left(\Delta f - \frac{nR}{2}\right)\lambda + \left(\|H_f\|_g^2 - \|Ric\|_g^2 + \frac{nR^2}{4} - R\Delta f\right) = 0 \tag{2.6}$$

which means the non-negativity:

$$0 \leq \Delta' := \left(\Delta f - \frac{nR}{2}\right)^2 - n\left(\|H_f\|_g^2 - \|Ric\|_g^2 + \frac{nR^2}{4} - R\Delta f\right). \tag{2.7}$$

It follows a lower boundary of the geometry of $g$ in terms of $f$:

$$\|Ric\|_g^2 \geq \|H_f\|_g^2 - \frac{1}{n}(\Delta f)^2. \tag{2.8}$$

An "exotic" consequence is provided by the case of strict inequality in (2.7), more precisely, it follows that the data $(g, f, \lambda)$ is doubled by $\left(g, f, \frac{2\Delta f}{n} - R - \lambda = -\frac{2}{n}R - \lambda\right)$.

**Example 1** i) (*Gaussian soliton*) We have $(M = \mathbb{R}^n, g_{can})$ and $f(x) = -\frac{\lambda}{2}\|x\|^2$. It results $h_f = -\lambda I_n$ and $\Delta f = -n\lambda$. Since $\|H_f\|^2 = n\lambda^2$, the left hand side of (2.6) is:

$$n\lambda^2+2\left(\Delta f - \frac{nR}{2}\right)\lambda+\left(\|H_f\|_g^2 - \|Ric\|_g^2 + \frac{nR^2}{4} - R\Delta f\right) = n\lambda^2+2(-n\lambda)\lambda+n\lambda^2$$

which is exactly zero. Also: $\Delta' = (n\lambda)^2 - n(n\lambda^2 - 0) = 0$ which means the uniqueness of $\lambda$ and the equality case in (2.8): $0 = n\lambda^2 - \frac{(n\lambda)^2}{n}$.
ii) A generalization of the previous example is provided on a Ricci-flat manifold by a smooth function $f$ satisfying a generalization of Hessian structures:

$$H_f = -\lambda g. \tag{2.9}$$

Then $\Delta f = -n\lambda$ and $\|H_f\|^2 = n\lambda^2$ exactly as for the Gaussian soliton. Using Lemma 4.1. of [3, p. 1540] it results form (2.9) that $\nabla f$ is a particular *concircular vector field*: $h_f = -\lambda I$; hence $\lambda_1 = ... = \lambda_n = -\lambda$ is the spectral part of the spectral data of $f$. If $\nabla f$ is without zeros it follows from Theorem 3.1. of [3, p. 1539] that $(M, g)$ is locally a warped product manifold with a 1-dimensional basis: $(M, g) = (I \subseteq \mathbb{R}, g_{can}) \times_\varphi (F^{n-1}, g_F)$. In fact, $\nabla f = \varphi(s)\frac{\partial}{\partial s}$ with $\varphi'(s) = -\lambda$ which means an affine warping function, $\varphi(s) = -\lambda s + C$. □

A new quadratic equation, similar to (2.6), follows from:

$$\Delta f + \left(1 - \frac{n}{2}\right)R + n\lambda = 0 \tag{2.10}$$

obtained by tracing (2.1). Hence the companion equation of (2.6) is:

$$n\lambda^2 + 2\left(1 - \frac{n}{2}\right)R\lambda + \left(\|Ric\|_g^2 - \|H_f\|_g^2 + \frac{n-4}{4}R^2\right) = 0. \tag{2.11}$$

The new inequality is then:

$$0 \leq \Delta' := \left(1 - \frac{n}{2}\right)^2 R^2 - n\left(\|Ric\|_g^2 - \|H_f\|_g^2 + \frac{n-4}{4}R^2\right) \tag{2.12}$$

and it results a lower boundary of the behavior of $f$ in terms of the geometry of $g$:

$$\|H_f\|_g^2 \geq \|Ric\|_g^2 - \frac{R^2}{n} = \frac{1}{n}\sum_{i \neq j}(\lambda_i - \lambda_j)^2. \tag{2.13}$$

We remark that (2.8) and (2.13) can be unified in the double inequality:

$$\|H_f\|_g^2 - \frac{1}{n}(\Delta f)^2 \leq \|Ric\|_g^2 \leq \|H_f\|_g^2 + \frac{R^2}{n} \tag{2.14}$$

and the simultaneous equalities for $n \geq 3$ hold if and only if $R = \Delta f = 0 = \lambda$ and $H_f = -Ric$; hence $f$ is a harmonic map on a steady gradient Einstein soliton. The vanishing of the right-hand side of (2.13) means that $g$ is an Einstein metric; other interesting aspects concerning the functional $F_g := \frac{R^2}{\|Ric\|_g^2}$ on the space of non-flat metrics appear in [5]. This raises the first future problem to study the similar functional $F_f^g := \frac{(\Delta f)^2}{\|H_f\|_g^2}$ on the space of smooth functions which are not *linear on* $M$ after the name from [6, p. 283]. Remark that for the Hessian structures (2.9) we have a constant and maximal $F_f^g = n$.

**Example 1 revisited** i) (*Gaussian soliton*) The inequality (2.13) becomes $n\lambda^2 \geq 0$.
ii) Again, (2.13) means $n\lambda^2 \geq 0$.
iii) (*relationship with gradient Ricci solitons*) If $R = 0$, then the gradient Einstein soliton becomes a gradient Ricci soliton and we remark that (2.14) is exactly the double inequality (20) of [4, p. 3339]. The explication of this fact is provided by the following remark. □

**Remark** An unified proof of the double inequality (2.14) is provided by the following relation satisfied by an Einstein soliton, which is a direct consequence of the equations (2.5) and (2.10):

$$n\left(\|H_f\|_g^2 - \|Ric\|_g^2\right) = (\Delta f)^2 - R^2 \tag{2.15}$$

and it is important to point out that this equation does not involves the scalar $\lambda$. In other words, (2.15) is a universal formula of the gradient Einstein solitons. With $\lambda \to \lambda + \frac{R}{2}$ we get that (2.15) holds also for gradient Ricci solitons and hence we obtain the similarity between gradient Ricci and Einstein solitons with respect to (2.14). □

Returning to (2.3) we remark that the Ricci endomorphism $Q$ commutes with $h_f$ for an Einstein or Ricci gradient soliton. It results the commuting property also for the Einstein endomorphism:

$$Einst_g := Q - \frac{R}{n}I \tag{2.16}$$

which is the trace-free part of $Q$. We will assume now that the data $(g, f, \lambda, \mu \in \mathbb{R})$ satisfies:

$$h_f + Q + \lambda I + \mu Einst_g = 0. \tag{2.17}$$

The corresponding relation in terms of Ricci endomorphism is:

$$h_f + (1+\mu)Q + \left(\lambda - \frac{\mu R}{n}\right)I = 0 \tag{2.18}$$

or, for $\mu \neq -1$:

$$h_{\frac{f}{1+\mu}} + Q + \left( \frac{\lambda}{1+\mu} - \frac{\mu R}{n(1+\mu)} \right) I = 0. \tag{2.19}$$

This last equation is an example of $\rho$-Einstein soliton as is introduced in Definition 1.1 of [2, p. 67] with $\rho = \frac{\mu}{n(1+\mu)}$ and $(f, \lambda)$ of [2] replaced by $\frac{1}{1+\mu}(f, \lambda)$.

Hence we naturally arrive to the following slight generalization of all the above considerations. Fix a $g$-symmetric endomorphism $A \in \mathcal{T}^1_1(M)$ which is also diagonal with respect to the frame $E$:

$$A(E_i) = \rho_i E_i, \quad \rho_i \in C^\infty(M). \tag{2.20}$$

Hence $A$ and $h_f$ commutes: $A \circ h_f = h_f \circ A$. We introduce:

**Definition** The data $(g, f, \lambda, \mu \in \mathbb{R})$ is an *A-Ricci gradient soliton* if:

$$h_f + Q + \lambda I + \mu A = 0. \tag{2.21}$$

We get that $A$ commutes also with $Q$ and the corresponding generalization of (2.15) is:

$$n \left[ \|H_f\|^2_g - \|Ric\|^2_g + \mu^2 \|A\|^2_g + 2\mu Tr_g(h_f \circ A) \right] = (\Delta f + \mu Tr_g A)^2 - R^2 \tag{2.22}$$

yielding the double inequality:

$$\|H_f\|^2_g - \frac{1}{n}(\Delta f + \mu Tr_g A)^2 + \mu^2 \|A\|^2_g + 2\mu Tr_g(h_f \circ A) \leq \|Ric\|^2_g \leq$$

$$\leq \|H_f\|^2_g + \frac{R^2}{n} + \mu^2 \|A\|^2_g + 2\mu Tr_g(h_f \circ A). \tag{2.23}$$

There is another problem: to find remarkable endomorphisms commuting with a given $h_f$. We will finish this note with an example.

**Example 2** Suppose that $(M, g)$ is a hypersurface in $(N^{n+1}, g)$ and let $A = S$ be the shape endomorphism of $M$ commuting with $h_f$ for the fixed scalar field $f \in C^\infty(M)$. If $(g, f, \lambda, \mu \in \mathbb{R})$ is a *shape-Ricci gradient soliton* on $M$ i.e. (2.21) holds for $S$, then denoting by $H$ the mean curvature of $M$, we get:

$$\|H_f\|^2_g - \frac{1}{n}(\Delta f + \mu H)^2 + \mu^2 \|S\|^2_g + 2\mu Tr_g(h_f \circ S) \leq \|Ric\|^2_g \leq$$

$$\leq \|H_f\|^2_g + \frac{R^2}{n} + \mu^2 \|S\|^2_g + 2\mu Tr_g(h_f \circ S). \tag{2.24}$$

We point out that immersions of (almost) Ricci solitons into another Riemannian manifold are studied in [1]. □

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