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RESEARCH ARTICLE

Semi-invariant Submanifolds in Metric Geometry of Endomorphisms

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Abstract Attempts have been made to introduce generalization of structured manifolds as the most general Riemannian metric g associated with an endomorphism F and initiate a study of their semi-invariant submanifolds. These submanifolds are generalizations of CR submanifolds of almost complex geometry and semi-invariant submanifolds of several interesting geometries (almost product, almost contact and others). Characterization of the integrability of both invariant and anti-invariant distribution are done; the special case when F is covariant constant with respect to g.

Keywords
(g, F, μ)-manifold: semi-invariant submanifold · (Integrable) distribution

Mathematics Subject Classification 53C40 · 53C15 · 53C12 · 53C25

1 Introduction

The geometry of manifolds endowed with geometrical structures has been intensively studied, and several important results have been published; see Yano-Kon [1]. The more important classes of such manifolds are formed by almost complex, almost product, almost contact, almost paracontact manifolds for which the cited book offers a good introduction. The geometry of submanifolds in these manifolds is very rich and interesting; as well, see for example the classical [2] or the more recent survey [3]. CR submanifolds introduced by Bejancu in [4] (for almost complex geometry) and [5] (for almost contact geometry), respectively, have had a great impact on the developing of the theory of submanifolds in these ambient manifolds; a proof of this fact is given by the books [6, 7].

In the present paper, we first introduce the concept of (g, F, μ)-manifold which contains as particular cases all the above types of structures. Then, we study semi-invariant submanifolds of a (g, F, μ)-manifold, which are extensions of CR submanifolds to this general class of manifolds. We find necessary and sufficient conditions for the integrability of both distributions on a semi-invariant submanifold; see Theorems 3.1 and 3.3. In particular, we prove that some semi-invariant submanifolds carry a natural foliation, Theorem 4.4 and we obtain characterizations of totally geodesic foliations on semi-invariant submanifolds in Theorems 4.8 and 4.10. For a particular value of the real parameter μ we can connect our study with the almost symplectic geometry and this fact opens some possible further applications in physical sciences having as example the relationship between CR-structures and Relativity pointed out in the last Chapter of [6].

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2 Metric Geometry of Endomorphisms and Submanifolds

Let M be an m-dimensional manifold for which we denote by $C^\infty(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $C^\infty(M)$-module of smooth sections of the

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tangent bundle TM of M; let $X, Y, Z, \ldots$ denote such vector fields. We use the same notation $\Gamma(V)$ for any other vector bundle V over M. Let also $\mathcal{T}^1_1 M$ be the $C^\infty(M)$-module of $\Gamma(TM \otimes T^*M)$ i.e. the real space of tensor fields of (1, 1)-type on M. Consider a fixed $F \in \mathcal{T}^1_1 M$ usually called *endomorphism* or *affinor* or *vector 1-form*; the remarkable endomorphism of every manifold is the Kronecker tensor field $I = (\delta^i_j)$.

Fix $\mu \in \{-1, +1\}$. Let now g be a Riemannian metric on M.

Definition 1.1 M is called a (g, F, μ)*-manifold* if for any X, Y:

$$g(FX, Y) + \mu g(X, FY) = 0. \tag{1.1}$$

The geometry of the data (M, g, F, μ) is called *endomorphism-metric geometry*. If in particular, F_x is nondegenerate at any point $x \in M$ then we say that M is a *nondegenerate* (g, F, μ)*-manifold*; otherwise, M is called *degenerate* (g, F, μ)*-manifold*.

Relation (1.1) says that the g-adjoint of F is $F^* = -\mu F$. In the literature, there is an abundance of examples of (g, F, μ)-manifolds. Some of the main examples are presented here:

Example 1.2

1. An *almost Hermitian manifold* ([6, p. 11]) (M, g, J) is a nondegenerate $(g, F, \mu = +1)$-manifold; the nondegeneration is a consequence of $J^2 = -I$.
2. An *almost para-Hermitian manifold* ([8]) (M, g, P) is a nondegenerate $(g, F, \mu = +1)$-manifold, while an *almost Riemannian product manifold* is a nondegenerate $(g, F, \mu = -1)$-manifold; the nondegeneration is a consequence of $P^2 = I$.
3. An *almost contact metric manifold* ([6, p. 15]) $(M, g, \varphi, \xi, \eta)$ is a $(g, F, \mu = +1)$-manifold; as $\varphi(\xi) = 0$, M is degenerate.
4. An *almost paracontact manifold* ([9]) $(M, g, \varphi, \xi, \eta)$ is a $(g, F, \mu = +1)$-manifold. As in the previous example, we have $\varphi(\xi) = 0$ and therefore M is degenerate.
5. The general case of a nondegenerate $(g, F, \mu = +1)$-manifold is called *structured manifold* in [10].

Recall that a real $2k$-dimensional manifold M is called an *almost symplectic manifold* if it is endowed with a nondegenerate 2-form $\Omega \in \Omega^2(M)$. We derive the following characterization:

Proposition 1.3 *Let M be a $(g, F, \mu = +1)$-manifold. Then, M is nondegenerate if and only if Ω defined by:*

$$\Omega(X, Y) = g(FX, Y) \tag{1.2}$$

is an almost symplectic structure. In this case, m is even.

Proof Ω is skew-symmetric from $\mu = +1$. A straightforward computation yields that Ω is nondegenerate if and only if M is a nondegenerate $(g, F, \mu = +1)$-manifold. □

Example 1.4 For Example 1.2.1 Ω is exactly the *fundamental* or *Kähler* 2-form and then inspired by this fact we introduce:

Definition 1.5 For a nondegenerate $(g, F, \mu = +1)$-manifold Ω is call *the fundamental* 2*-form*.

In the last part of this section, let us recall briefly the geometry of Riemannian submanifolds. Consider an n-dimensional submanifold N of M. Then, the main objects induced by the Levi-Civita connection $\widetilde{\nabla}$ of (M, g) on N are involved in the well-known Gauss–Weingarten equations:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{1.3}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$. Here ∇ is the Levi-Civita connection on N, h is the second fundamental form of N, A_V is the Weingarten operator with respect to the normal section V and $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp N$ of N. Let us pointed out that h and A_V are related by:

$$g(h(X, Y), V) = g(A_V X, Y). \tag{1.4}$$

If h vanishes identically on N then N is called *totally geodesic*.

3 Submanifolds in Endomorphism-Metric Geometry

Next, we consider a submanifold N of a (g, F, μ)-manifold M. Then, g induces a Riemannian metric on N which we denote by the same symbol g. Following the definition given by Bejancu [4] for CR submanifolds, we introduce a special class of submanifolds of M as follows:

Definition 2.1 N is a *semi-invariant submanifold* of M if there exists a distribution D on N satisfying the conditions:

(i) D is F-invariant:

$$F(D_x) \subset D_x, \quad \forall\, x \in N. \tag{2.1}$$

(ii) The complementary orthogonal distribution $D^\perp$ to D in TN is F-anti-invariant, that is:

$$F(D^\perp_x) \subset T^\perp_x N, \quad \forall\, x \in N. \tag{2.2}$$

(iii) $F^2(D^\perp)$ is a distribution on N.

Some particular classes of semi-invariant submanifolds are defined as follows. Let p and q be the ranks of the distributions D and $D^{\perp}$, respectively. If $q=0$, that is $D^{\perp}=\{0\}$, we say that N is an *F-invariant submanifold* of M. If $p=0$, that is $D=\{0\}$, we call N an *F-anti-invariant submanifold* of M.

If $pq \neq 0$, then N is called a *proper* semi-invariant submanifold. Now, we denote by $\widetilde{D}$ the complementary orthogonal vector bundle to $F(D^{\perp})$ in $T^{\perp}N$. If $\widetilde{D}=\{0\}$ then we say that N is a *normal* F-semi-invariant submanifold.

Thus, N is an F-invariant, respectively F-anti-invariant, if and only if:

$$F(TN) \subset TN \quad (\mathrm{resp.F(TN)} \subset \mathrm{T^{\perp}N}). \tag{2.3}$$

N is normal F-semi-invariant if and only if:

$$F(D^{\perp}) = T^{\perp}N. \tag{2.4}$$

Examples 2.2

1. For Example 1.2.1, we obtain the classical concept of CR submanifold of Bejancu [6, p. 20]; the condition (iii) is satisfied from $J^2=-I$.
2. For Example 1.2.2, we obtain the notion of semi-invariant submanifold; for the almost para-Hermitian case; see [8], while for the second case; see [11]. The condition (iii) is satisfied again from $P^2=-I$.
3. For Example 1.2.3, we obtain the notion of semi-submanifold [6, p. 100] with $\xi \in T^{\perp}N$. This last condition implies $TN \subset \ker \eta$ and since $\varphi|_{\ker \eta}$ is an almost complex structure we get (iii). Semi-invariant $\xi^{\perp}$-submanifolds of generalized quasi-Sasakian manifolds are studied in [12].
4. For Example 1.2.4, we obtain the concept of semi-submanifold from [13] with $\xi \in T^{\perp}N$. Again this condition means $TN \subset \ker \eta$ and since $\varphi|_{\ker \eta}$ is an almost product structure, we have (iii).
5. The condition (iii) does not appear in [10].

Returning to the Definition 2.1, we deduce that the tangent bundle TN and the normal bundle $T^{\perp}N$ of a semi-invariant submanifold N have the orthogonal decompositions:

$$TN = D \oplus D^{\perp}, \quad T^{\perp}N = F(D^{\perp}) \oplus \widetilde{D}. \tag{2.5}$$

Then, we denote by P and Q the projection morphisms of TN on D and $D^{\perp}$, respectively, and obtain for $X = PX + QX \in \Gamma(TN)$:

$$FX = \varphi X + \omega X \tag{2.6}$$

where we put:

$$\varphi = F \circ P, \quad \omega = F \circ Q. \tag{2.7}$$

Thus, φ is a tensor field of (1, 1)-type on N, while ω is a $F(D^{\perp})-$valued vector 1-form on N. Hence we derive:

Proposition 2.3 *Let N be a semi-invariant submanifold of a (g, F, μ)-manifold M. Then*:

(iv) *N is a (g, φ, μ)-manifold.*
(v) *$F^2(D^{\perp})$ is a vector subbundle of $D^{\perp}$.*
(vi) *The vector bundle $\widetilde{D}$ is F-invariant i.e. for all $x \in N$ we have: $F(\widetilde{D}_x) \subset \widetilde{D}_x$.*

Proof

(iv) By definition, g is a Riemannian metric on N and φ is a tensor field of (1, 1)-type on N; we need only to show (2.1). By using (1.1) for F we obtain for $X, Y \in \Gamma(TN)$:

$$\begin{aligned} g(\varphi X, Y) &= g(FPX, Y) = g(FPX, PY) \\ &= -\mu g(PX, FPY) = -\mu g(X, FPY) = -\mu g(X, \varphi Y). \end{aligned}$$

(v) Take $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$ in (2.1): $g(X, F^2 Y) = -\mu g(FX, FY) = 0$ since $FX \in \Gamma(D)$ and $FY \in \Gamma(T^{\perp}N)$. Hence, $F^2(D^{\perp})$ is orthogonal to D and by condition (iii), we deduce that $F^2(D^{\perp})$ is a vector subbundle of $D^{\perp}$.
(vi) Take $X \in \Gamma(TN)$, $Y \in \Gamma(D^{\perp})$ and $V \in \Gamma(\widetilde{D})$. Then we obtain:

$$g(FV, X) = -\mu g(V, FX) = -\mu g(V, \varphi X + \omega X) = 0$$

and:

$$g(FV, FY) = -\mu g(V, F^2 Y) = 0$$

since $\varphi X \in \Gamma(D)$, $\omega X \in \Gamma(FD^{\perp})$ and $F^2 Y \in \Gamma(D^{\perp})$. Thus $F\widetilde{D}$ is orthogonal to $TN \oplus FD^{\perp}$, that is $F\widetilde{D}$ is a vector subbundle of $\widetilde{D}$. This completes the proof of the proposition.

□

In the non-degenerated case, we have equalities for the above inclusions:

Corollary 2.4 *Let N be a semi-invariant submanifold of a nondegenerate (g, F, μ)-manifold M. Then*:

(1) *the above distributions satisfy*:

$$F(D) = D, \quad F^2(D^{\perp}) = D^{\perp}, \quad F(\widetilde{D}) = \widetilde{D}. \tag{2.8}$$

(2) *if $\mu = +1$ then $D^{\perp}$ and $F(D^{\perp})$ are Lagrangian distribution on (TM, Ω). In particular if N is a normal semi-invariant submanifold, then $T^{\perp}N$ is a Lagrangian submanifold in (TM, Ω).*

Proof We need to prove only (2).

(2.1) Let $X, Y \in \Gamma(D^{\perp})$; then $\Omega(X, Y) = g(FX, Y) = 0$ since $FX \in \Gamma(T^{\perp}N)$ while $Y \in \Gamma(TN)$.
(2.2) Let $X, Y \in \Gamma(F(D^{\perp}))$; then $\Omega(X, Y) = g(FX, Y) = 0$ since $FX \in \Gamma(TN)$ while $Y \in \Gamma(T^{\perp}N)$.

□

The second part of the above Corollary is extremely important since it relates the geometry of semi-invariant submanifolds with the almost symplectic geometry, a topic very studied from the point of view of applications in Analytical Mechanics.

4 Integrability of Distributions on a Semi-invariant Submanifold

Let N be a semi-invariant submanifold of a (g, F, μ)-manifold M. Then, we recall that the Nijenhuis tensor field of F is defined as follows ([6, p.11]):

$$N_F(X, Y) = [FX, FY] + F^2[X, Y] - F[FX, Y] - F[X, FY], \tag{3.1}$$

for any $X, Y \in \Gamma(TM)$. In a similar way, the Nijenhuis tensor field of φ on N is given by:

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \tag{3.2}$$

for any $X, Y \in \Gamma(TN)$. We recall that a tensor field of (1, 1)-type defines an *integrable structure* on a manifold if and only if its Nijenhuis tensor field vanishes identically on the manifold. Now we obtain necessary and sufficient conditions for the integrability of D and $D^\perp$ in terms of Nijenhuis tensor fields of F and φ.

Theorem 3.1 *Let N be a semi-invariant submanifold of a (g, F, μ)-manifold M. Then, the following assertions are equivalent*:

(1) *D is an integrable distribution.*
(2) *The Nijenhuis tensor field of φ satifies*:

$$Q \circ N_\varphi = 0, \quad \forall X, Y \in \Gamma(D). \tag{3.3}$$

(3) *The Nijenhuis tensor fields of F and φ satisfy the equality*: *$N_F = N_\varphi$ on D.*

Proof Firstly, we note that D is integrable if and only if:

$$Q([X, Y]) = 0, \quad \forall X, Y \in \Gamma(D). \tag{3.4}$$

Since the last three terms in the right side of (3.2) lie in $\Gamma(D)$ we deduce that:

$$Q \circ N_\varphi(X, Y) = Q([FX, FY]), \quad \forall X, Y \in \Gamma(D). \tag{3.5}$$

As M is nondegenerate we deduce that φ is an automorphism on $\Gamma(D)$. Thus, the equivalence of (1) and (2) it follows directly. Next, we obtain for any $X, Y \in \Gamma(D)$:

$$N_F(X, Y) = N_\varphi(X, Y) + F\omega([X, Y]) - \omega([\varphi X, Y]) - \omega([X, \varphi Y]). \tag{3.6}$$

If D is integrable, then the last three terms of (3.6) vanishes and this yields 3). Conversely, suppose that $N_F = N_\varphi$ on D; then:

$$F\omega([X, Y]) = \omega([\varphi X, Y] + [X, \varphi Y]). \tag{3.7}$$

Obviously, the right-hand-side of the previous equation is in $\Gamma(F(D^\perp)) \subset \Gamma(T^\perp N)$. On the other hand, the left-hand-side is in $\Gamma(F^2 D^\perp) \subset \Gamma(TN)$; we conclude that both sides in (3.7) must vanish. Finally, from: $F^2 Q([X, Y]) = 0$ and F^2 automorphism of $\Gamma(TM)$, we deduce 1). □

Remark 3.2 For Example 1.2.1, the equivalence of (1) and (2) is exactly the Theorem 2.2. of [6, p. 25], while the equivalence of (1) and (3) is the Theorem 2.1. of [6, p. 25].

Now, we consider $X, Y \in \Gamma(D^\perp)$. Then, taking into account that $\varphi X = \varphi Y = 0$, we get:

$$N_\varphi(X, Y) = F^2 P[X, Y] \tag{3.8}$$

and this enables us to state the following:

Theorem 3.3 *Let N be a semi-invariant submanifold of a nondegenerate $(g, F, \mu)-$manifold. Then $D^\perp$ is integrable if and only if the Nijenhuis tensor field of φ vanishes identically on $D^\perp$.*

Remark 3.4 For Example 1.2.1, the above result is the Theorem 2.3. of [6, p. 26].

5 A Natural Foliation on a Semi-invariant Submanifold

Let $\tilde{\nabla}$ be the Levi-Civita connection on M with respect to the Riemannian metric g. Then, F is a *parallel tensor field* on M if:

$$\tilde{\nabla} F = 0. \tag{4.1}$$

Examples 4.1

(1) For Example 1.2.1, we have the notion of *Kähler manifold*.
(2) For Example 1.2.2, the first part we have the concept of *para-Kähler manifold* while for the second part the notion of *locally Riemannian product manifold*.
(3) For Example 1.2.3, we get the notion of *cosymplectic manifold*.

In the present section, we study the geometry of semi-invariant submanifolds of (g, F, μ)-manifolds with parallel tensor field F. First, we prove the following:

Proposition 4.2 *Let N be a semi-invariant submanifold of a nondegenerate (g, F, μ)-manifold with parallel tensor field F. Then, for all $X, Y \in \Gamma(D^{\perp})$:*

$$A_{FX}Y - A_{FY}X = \varphi([X, Y]). \tag{4.2}$$

Proof By using the Weingarten equation and the parallelism condition, we get:

$$A_{FX}Y = \nabla^{\perp}_{Y}FX - \widetilde{\nabla}_{Y}FX = \nabla^{\perp}_{Y}FX - F(\widetilde{\nabla}_{X}Y). \tag{4.3}$$

Writing a similar equation by interchanging X and Y and then subtracting, we obtain:

$$A_{FX}Y - A_{FY}X = \nabla^{\perp}_{Y}FX - \nabla^{\perp}_{X}FY + F([X, Y]), \tag{4.4}$$

since ∇ is a torsion-free linear connection. Thus, (4.2) is obtained by equalizing the tangent parts to N in the above equation. □

Example 4.3 Relation (4.2) becomes, for Example 1.2.1, equation (2.2) of [6, p. 43].

Now, we can state the following main result:

Theorem 4.4 *Let N be a semi-invariant submanifold of a nondegenerate $(g, F, \mu = +1)$-manifold with parallel tensor field F. Then, the F-anti-invariant distribution $D^{\perp}$ is integrable.*

Proof For any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D)$ we have:

$$\begin{aligned} g(A_{FX}Y, Z) &= -g(F\widetilde{\nabla}_{Y}X, Z) = +\mu g(\widetilde{\nabla}_{Y}X, FZ) = \\ &\quad - \mu g(X, \widetilde{\nabla}_{Y}FZ) \\ &= \mu^2 g(FX, \widetilde{\nabla}_{Y}Z) = \mu^2 g(FX, [Y, Z] + \widetilde{\nabla}_{Z}Y) \\ &= \mu^2 g(FX, \widetilde{\nabla}_{Z}Y). \end{aligned} \tag{4.5}$$

Also, we have:

$$\begin{aligned} g(A_{FY}X, Z) &= \mu^2 g(FY, \widetilde{\nabla}_{Z}X) \\ &= -\mu^2 g(F\widetilde{\nabla}_{Z}Y, X) = \mu^3 g(\widetilde{\nabla}_{Z}Y, FX). \end{aligned} \tag{4.6}$$

Comparing (4.5) and (4.6), we deduce that for $\mu = +1$:

$$g(A_{FX}Y - A_{FY}X, Z) = 0$$

which means that $A_{FX}Y - A_{FY}X \in \Gamma(D^{\perp})$. On the other hand, from (4.2) we conclude that:

$$A_{FX}Y - A_{FY}X \in \Gamma(D).$$

and thus we have that:

$$A_{FX}Y - A_{FY}X = 0. \tag{4.7}$$

Finally, returning to (4.2) and taking into account that F is nondegenerate we deduce that:

$$P[X, Y] = 0,$$

that is, $D^{\perp}$ is integrable. □

Remark 4.5 For Example 1.2.1, the above result is part (i) of Theorem 1.1. of [6, p. 39].

Regarding the integrability of D, we prove the following:

Theorem 4.6 *Let N be a semi-invariant submanifold of a nondegenerate (g, F, μ)-manifold Mwith parallel tensor field F. Then, the F-invariant distribution D is integrable if and only if the second fundamental form h of N satisfies for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$:*

$$g(h(X, \varphi Y) - h(Y, \varphi X), FZ) = 0. \tag{4.8}$$

Proof By using the Gauss equation we deduce that:

$$\nabla_{X}\varphi Y + h(X, \varphi Y) = \varphi(\nabla_{X}Y) + \omega(\nabla_{X}Y) + Fh(X, Y). \tag{4.9}$$

Write a similar equation by interchanging X and Y, and then subtracting we obtain:

$$\nabla_{X}\varphi Y - \nabla_{Y}\varphi X + h(X, \varphi Y) - h(Y, \varphi X) = \varphi([X, Y]) + \omega([X, Y]) \tag{4.10}$$

since h is symmetric and ∇ is a torsion-free linear connection. Equalize the normal parts in the above equation and obtain:

$$h(X, \varphi Y) - h(Y, \varphi X) = \omega([X, Y]). \tag{4.11}$$

Now, suppose that D is integrable; then (4.8) is immediately. Conversely, if (4.8) is satisfied, then with (4.11) we deduce that:

$$0 = -\mu g(Q[X, Y], F^2 Z).$$

Since F is nondegenerate we infer that F^2 is an automorphism of $\Gamma(D^{\perp})$ and hence D is integrable. □

Remark 4.7 In particular, if F is an almost complex structure on M, then we obtain the results of Bejancu [6] and Blair–Chen [14], respectively, for CR submanifolds.

Now, for $\mu = +1$ we denote by $\mathcal{F}^{\perp}$ the natural foliation defined by the F-anti-invariant distribution $D^{\perp}$ and call it the *F-anti-invariant foliation* on N. We recall that $\mathcal{F}^{\perp}$ is called a *totally geodesic foliation* if each leaf of $\mathcal{F}^{\perp}$ is totally geodesic immersed in N. Thus, $\mathcal{F}^{\perp}$ is totally geodesic if and only if the Levi-Civita connection ∇ of N satisfies for all $Y, Z \in \Gamma(D^{\perp})$:

$$\nabla_{Y}Z \in \Gamma(D^{\perp}). \tag{4.12}$$

Theorem 4.8 *Let N be a semi-invariant submanifold of a nondegenerate (g, F, μ)-manifold M with parallel tensor field F. Then the following assertions are equivalent:*

(i) *The F-anti-invariant foliation is totally geodesic.*

(ii) *The second fundamental form h of N satisfies for all* $X \in \Gamma(D)$ *and* $Y \in \Gamma(D^{\perp})$:

$$h(X,Y) \in \Gamma(\widetilde{D}). \tag{4.13}$$

(iii) $D^{\perp}$ *is* A_V*-invariant for any* $V \in \Gamma(FD^{\perp})$ *that is we have for all* $Y \in \Gamma(D^{\perp})$:

$$A_V Y \in \Gamma(D^{\perp}). \tag{4.14}$$

Proof We have for any $X \in \Gamma(D)$ and $Y, Z \in \Gamma(D^{\perp})$:

$$\begin{aligned} g(\nabla_Y Z, FX) &= g(\widetilde{\nabla}_Y Z, FX) \\ &= -\mu g(\widetilde{\nabla}_Y FZ, X) = \mu g(A_{FZ}Y, X) = \mu g(h(X,Y), FZ). \end{aligned} \tag{4.15}$$

Now, suppose that $\mathcal{F}^{\perp}$ is totally geodesic; then the first term of (4.15) vanishes. Hence, the last term in (4.15) vanishes which implies ii). Conversely, suppose (4.14) is satisfied. Then from (4.15), we deduce (4.12) since F is an automorphism of $\Gamma(D)$. This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) is straightforward. □

Remark 4.9 For Example 1.2.1, the equivalence of (i) and (ii) is the Theorem 1.3. of [6, p. 41].

Finally, we can prove the following:

Theorem 4.10 *Let N be a semi-invariant submanifold of a nondegenerate* (g, F, μ)*-manifold with parallel tensor field F. Then, the F-invariant distribution D is integrable and the foliation* $\mathcal{F}$ *defined by D is totally geodesic if and only if the second fundamental form h of N satisfies for all* $X, Y \in \Gamma(D)$:

$$h(X,Y) \in \Gamma(\widetilde{D}). \tag{4.16}$$

Proof D is integrable and $\mathcal{F}$ is totally geodesic if and only if for all $X, U \in \Gamma(D)$:

$$\nabla_X U \in \Gamma(D). \tag{4.17}$$

This is equivalent to:

$$g(\widetilde{\nabla}_X U, Z) = 0, \tag{4.18}$$

for all $Z \in \Gamma(D^{\perp})$. As F is an automorphism of $\Gamma(D)$, we can write the above equality as follows:

$$g(\widetilde{\nabla}_X FY, Z) = 0, \tag{4.19}$$

for all $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, which is equivalent to:

$$g(\widetilde{\nabla}_X Y, FZ) = 0. \tag{4.20}$$

By using the Gauss equation, the last relation is equivalent with:

$$g(h(X,Y), FZ) = 0, \tag{4.21}$$

which completes the proof of the theorem. □

Remark 4.11 For Example 1.2.1, the above result is Theorem 1.2. of [6, p. 40].

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