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Clifford product of cycles in EPH geometries and EPH-square of elliptic curves

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Abstract The aim of this paper is to study the cycles of EPH geometries through a Clifford product generalizing a quaternion product of circles recently introduced by the author. A special topic is the square of a given cycle C with respect to both existence problem and equality case with C. Introducing a dual cycle C^d a main problem is the commutativity of the product between C and its dual as well as their orthogonality with respect to another $\{0, \pm 1\}$ parameter. We finish by introducing an EPH square for reduced third order polynomials and for a special class of elliptic curves.

Keywords EPH geometries · cycle · EPH square

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1 Introduction

It is well-known that up to isomorphisms there are three 2-dimensional real algebras: $\mathbb{C} = \mathbb{R}[X]/(x^2+1)$, $\mathbb{D} = \mathbb{R}[X]/(x^2)$ and $\mathbb{A} = \mathbb{R}[X]/(x^2-1)$. The theory of the first algebra is richer than the following two, a fact corresponding to the field property of $\mathbb{C}$. Inspired by the terminology of [4, p. 1458] or [5, p. 2] we call *EPH geometries* these spaces and a common image consists in $A(\sigma) := \mathbb{R}[X]/(x^2-\sigma)$ with $\sigma := i^2 \in \{-1, 0, 1\}$ respectively and i the corresponding imaginary unit.

In the recent paper [2] we study some types of deformations (of gradient-type, Blaschke, similarity, rotation) for cycles of these geometries. In the present work, we continue their study by introduction of a product inspired by the Clifford algebra $Cl(\sigma)$. Previously, we have considered a quaternionic product of circles C in the paper [3] as well as a product of cycles (C, ε). Here $\varepsilon = \pm 1$ corresponds to a fixed trigonometrical orientation for the given circle C. This quaternionic product correspond to the particular case $\sigma = -1$ of elliptic (i.e. Euclidean) geometry.

A special attention of the first section below is devoted to squares and degeneration cases for the considered cycles C. For example, in elliptic geometry we have $C^2 \neq C$ while the equilateral hyperbolas are idempotent elements in the hyperbolic geometry of $\sigma = +1$. An important role in our analysis is given by the projective character of the coefficients (m, l, n, k) of a cycle C as well as the interpretation of the zero element $0 \in Cl(\sigma)$. Hence we point out that our study will be a mix of elements from $A(\sigma)$ and projective geometry and the parabolic geometry appears sometimes as a special case.

The Clifford product is treated in the second section using the own algebraic structure of $A(\sigma)$. Due to technical difficulties we restrict our discussion here to the non-vanishing case $\sigma \neq 0$ and it follows relative complicated expressions in terms of real and imaginary parts of the number $B = -l - \sigma ni \in A(\sigma)$. Also in this section we express the Clifford product in a matrix way and we introduce a dual cycle C^d. All cases of equality $C = C^d$ and commutativity for the product between C and its dual are determined. We discuss also the orthogonality between C and C^d with respect to a new parameter $\breve{\sigma} \in \{-1, 0, 1\}$ independently of σ. Also, this orthogonality is study with respect to C^d and two transformations of C: an inversion and a rotation.

In the last section we introduce a σ-Clifford product for reduced third order polynomials f^r. In fact, due to technical restrictions, we restrict only to compute its σ-Clifford product. With one step further we introduce this square in a special class of elliptic curves.

2 A Clifford product of cycles in EPH geometries

A main object in EPH geometries is given by:

Definition 2.1 *([4, p. 1459], [5, p. 4]) The common name cycle will be used to denote circles, parabolas and hyperbolas (as well as straight lines as their limits) in the respective EPH geometry.*

An analytical study of a cycle can be done via the general equation given in [4, p. 1460] or [5, p. 6]:

$$C : f(u, v) := k(u^2 - \sigma v^2) - 2lu - 2nv + m = 0 \tag{2.1}$$

and hence C is a conic section completely defined by the data $(k, l, n, m) \in \mathbb{P}^3$. As usual, if $k = 0$ then C can be called *a degenerate cycle*. In fact, in the cited works C is identified with the matrix:

$$C^s_{\breve{\sigma}} := \begin{pmatrix} l + \breve{\imath} sn & -m \\ k & -l + \breve{\imath} sn \end{pmatrix} \tag{2.2}$$

where s is a new parameter, usually equal to ± 1, and a new imaginary unit $\breve{\imath}$. Its square $\breve{\sigma} := \breve{\imath}^2$ belongs again to $\{-1, 0, 1\}$ but independently of σ.

The Clifford algebra $Cl(\sigma)$ associated to $A(\sigma)$ is presented in [5, p. 160] as the unit algebra with quaternionic-type generators e_0, e_1 and e_0e_1 satisfying:

$$e_0^2 = -1, \quad e_1^2 = \sigma, \quad e_1e_0 = -e_0e_1. \tag{2.3}$$

The starting point of this paper is the identification of the given cycle with the element $q(C) \in Cl(\sigma)$:

$$q(C) = k(e_0e_1) - 2le_0 - 2ne_1 + m = (m, -2l, -2n, k) \in \mathbb{R}^4. \qquad (2.4)$$

The element $q(C)$ is pure imaginary i.e. $m = 0$ if and only if the origin $O(0,0)$ belongs to C. Let us point out that our study will be a mix of elements from $A(\sigma)$ and projective geometry.

From the real algebra structure of $Cl(\sigma)$ it follows a product of cycles:

$$C_1 \odot_c C_2 := q^{-1}(q(C_1) \cdot q(C_2)). \qquad (2.5)$$

For $C_i, i = 1, 2$ given by $(m_i, -2l_i, -2n_i, k_i)$ we derive immediately:

$$q(C_1 \odot_c C_2) = (m_1k_2 + m_2k_1 + 4l_1n_2 - 4l_2n_1)(e_0e_1)-$$

$$-2(l_2m_1 + l_1m_2 - n_1k_2\sigma + n_2k_1\sigma)e_0-$$

$$-2(n_1m_2 + n_2m_1 + k_1l_2 - k_2l_1)e_1 + (m_1m_2 - 4l_1l_2 + 4n_1n_2\sigma + k_1k_2\sigma) \quad (2.6)$$

which gives a non-commutative expression for the coefficients of e_0, e_1 and e_0e_1. We remark that the degenerate case:

$$m_1k_2 + m_2k_1 + 4l_1n_2 = 4l_2n_1 \qquad (2.7)$$

and the coefficient of e_1 do not depend on σ and we point out the possibility to obtain a zero q in (2.6) which do not have a corresponding cycle; see the equilateral hyperbolas of example 2.1. We remark that only two coefficients, namely that of e_0 and the free coefficient, depend on σ.

The square of the cycle (2.1) is:

$$C^2 : 2m[k(u^2 - \sigma v^2) - 2lu - 2nv] + (m^2 - 4l^2 + 4\sigma n^2 + \sigma k^2) = 0 \qquad (2.8)$$

and then the case $m = 0$ implies the constraint:

$$4l^2 = \sigma(k^2 + 4n^2). \qquad (2.9)$$

Therefore, in the parabolic geometry of $\sigma = 0$ the condition $m = 0$ implies $l = 0$ and, indeed an element $q = k(e_0e_1) - 2ne_1 \in Cl(\sigma)$ has a zero square. We treat this lack of significance immediately.

If $m \neq 0$ then the square cycle C^2 is:

$$C^2 : k(u^2 - \sigma v^2) - 2lu - 2nv + \left[\frac{m}{2} + \frac{\sigma(k^2 + 4n^2) - 4l^2}{2m}\right] = 0 \qquad (2.10)$$

and a direct computation gives that $C^2 = C$ if and only if:

$$\sigma(k^2 + 4n^2) = m^2 + 4l^2. \qquad (2.11)$$

We recast the condition $m = 0 = l$ of the parabolic geometry and hence, we definite the square of a cycle $C : ku^2 - 2nv = 0$ from this geometry as being exactly C. The remaining geometries are covered by:

Proposition 2.2 *In elliptic geometry any cycle C is different from its square C^2. In hyperbolic geometry a cycle is equal to its square if and only if: $k^2 + 4n^2 = m^2 + 4l^2$.*

Example 2.1 A remarkable particular solution of the equation provided in previous proposition is: $m = \pm k \neq 0$, $n = \pm l$ or, due to the projective setting:

$$C : u^2 - v^2 - 2l(u \pm v) \pm 1 = 0. \tag{2.12}$$

The following two hyperbolas of the hyperbolic geometry correspond to $l = 0$:

$$H_\pm : u^2 - v^2 \pm 1 = 0 \tag{2.13}$$

and coincide with their square. Their matrices are independent from s and ĭ:

$$C^s_{\breve{\sigma}}(H_\pm) := \begin{pmatrix} 0 & \mp 1 \\ 1 & 0 \end{pmatrix}, \quad det C^s_{\breve{\sigma}}(H_\pm) = \pm 1. \tag{2.14}$$

The square and the product of these matrices are:

$$C^s_{\breve{\sigma}}(H_\pm)^2 = \mp I_2, \quad C^s_{\breve{\sigma}}(H_+) \cdot C^s_{\breve{\sigma}}(H_-) = -C^s_{\breve{\sigma}}(H_-) \cdot C^s_{\breve{\sigma}}(H_+) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.15}$$

The $Cl(1)$-product of their representations is zero, $q(H_+) \cdot q(H_-) = 0 = q(H_-) \cdot q(H_-)$, and hence we do not have the products $H_+ \odot_c H_-$, $H_- \odot_c H_+$. But the matrix product (2.15) yields the possibility to define another product:

$$H_+ \odot_m H_- = H_- \odot_m H_+ : u = 0 \tag{2.16}$$

which is the degenerate case of the vertical line of $A(1)$. The opposite case of this product is provided by the matrix I_2 of (2.15) which is not of (2.2)-form and hence the products $H_+ \odot_m H_+$, $H_- \odot H_-$ do not exist. □

Example 2.2 Due to the projective character of coefficients we fix the non-degenerate case of cycle C as being $k = 1$. In [2] we call C as being *decomposable* if:

$$m = l^2 - \sigma n^2. \tag{2.17}$$

Its square (2.10) is:

$$C^2 : u^2 - \sigma v^2 - 2lu - 2nv + \left(\frac{m}{2} + \frac{\sigma}{2m} - 2\right) = 0. \tag{2.18}$$

C^2 is also decomposable if and only if $C^2 = C$ which means:

$$\frac{m}{2} + \frac{\sigma}{2m} - 2 = m \to m^2 + 4m - \sigma = 0. \tag{2.19}$$

The quadratic equation has the solutions:

$$m_\pm = -2 \pm \sqrt{4 + \sigma}. \tag{2.20}$$

The elliptic case of (2.16) gives the impossible equation: $l^2+n^2=-2\pm\sqrt{3}<0$ while the parabolic case gives a second and third impossible situation, $m_1=0$, and $m_2=-4=l^2$ respectively. It remains the hyperbolic geometry with:

$$m_{\pm}=-2\pm\sqrt{5}=l^2-n^2 \tag{2.21}$$

and then we have the cases:
1) $n_1\in(-\infty,-\sqrt{\sqrt{5}-2}=-0.4858...]\cup[\sqrt{\sqrt{5}-2}=0.4858...,+\infty)$,
2) $n_2\in(-\infty,-\sqrt{\sqrt{5}+2}=-2.0581...]\cup[\sqrt{\sqrt{5}+2}=2.0581...,+\infty)$,
with:

$$l_{1,\pm}=\pm\sqrt{n_1^2-2+\sqrt{5}},\quad l_{2,\pm}=\pm\sqrt{n_2^2-2-\sqrt{5}}. \tag{2.22}$$

The determinant of the associated matrix is:

$$detC^s_{\breve{\sigma}}=\breve{\sigma}n^2-l^2+km=n^2[\breve{\sigma}-\sigma]. \tag{2.23}$$

For $\breve{\sigma}=0$ and n provided by (2.22) of hyperbolic geometry it results a strictly negative $detC^s_0$. $\square$

In the following we present several particular cases provided by the vanishing of some coefficients:
1) $k_2=0$:

$$\begin{aligned}C_1\odot C_2:(m_2k_1+4l_1n_2-4l_2n_1)(u^2-\sigma v^2)-2(l_2m_1+l_1m_2-n_2k_1)u-\\-2(n_1m_2+n_2m_1+k_1l_2)v+\\+(m_1m_2-4l_1l_2+4\sigma n_1n_2)=0.\end{aligned} \tag{2.24}$$

2) $k_1=0$:

$$\begin{aligned}C_1\odot_c C_2:(m_1k_2+4l_1n_2-4l_2n_1)(u^2-\sigma v^2)-2(l_2m_1+l_1m_2+n_1k_2)u-\\-2(n_1m_2+n_2m_1-k_2l_1)v+\\+(m_1m_2-4l_1l_2+4\sigma n_1n_2)=0.\end{aligned} \tag{2.25}$$

3) $k_1=k_2=0$:

$$\begin{aligned}C_1\odot_c C_2:4(l_1n_2-l_2n_1)(u^2-\sigma v^2)-2(l_2m_1+l_1m_2)u-2(n_1m_2+n_2m_1)v+\\+(m_1m_2-4l_1l_2+4\sigma n_1n_2)=0.\end{aligned} \tag{2.26}$$

Is a degenerate cycle if and only if:

$$\frac{l_1}{l_2}=\frac{n_1}{n_2} \tag{2.27}$$

which means that the linear parts of C_1 and C_2 are proportional. Hence the Clifford product of two parallel lines commutes and is also a line parallel with them. More precisely, considering $l_1\cdot l_2\neq 0$ and denoting:

$$\rho=\frac{n_1}{l_1}=\frac{n_2}{l_2} \tag{2.28}$$

it results the three lines:

$$\begin{cases} C_1 : u + \rho v + \frac{m_1}{l_1} = 0, \quad C_2 : u + \rho v + \frac{m_2}{l_2} = 0, \\ C_1 \odot_c C_2 = C_2 \odot_c C_1 : u + \rho v + \frac{m_1 m_2 - 4(1+\sigma\rho^2) l_1 l_2}{-2(l_2 m_1 + l_1 m_2)} = 0. \end{cases} \tag{2.29}$$

supposing the non-vanishing of the last fraction. In fact with $\mu_a = \frac{m_a}{l_a}$ for $a = 1, 2$ we have the lines:

$$C_a : u+\rho v+\mu_a = 0, \quad C_1 \odot_c C_2 = C_2 \odot_c C_1 : u+\rho v+\frac{4(1+\sigma\rho^2) - \mu_1\mu_2}{2(\mu_1+\mu_2)} = 0. \tag{2.30}$$

Hence, the square of the line $C : u + \rho v + \mu = 0$ not containing the origin is the parallel line:

$$C^2 : u + \rho v + \frac{4(1+\sigma\rho^2) - \mu^2}{4\mu} \tag{2.31}$$

which coincides with C if and only if:

$$5\mu^2 = 4(1+\sigma\rho^2). \tag{2.32}$$

For example, in the parabolic geometries the lines:

$$L_\pm : u + \rho v \pm \frac{2}{\sqrt{5}} = 0 \tag{2.33}$$

coincide with their square. The same fact holds in hyperbolic geometries for the lines:

$$L_{\pm,\rho} : u + \rho v \pm \frac{2\sqrt{1+\rho^2}}{\sqrt{5}} = 0. \tag{2.34}$$

4) $l_1 = n_1 = 0$:

$$C_1 \odot_c C_2 : (m_1 k_2 + m_2 k_1)(u^2 - \sigma v^2) - 2(l_2 m_1 - n_2 k_1)u - 2(n_2 m_1 + k_1 l_2)v+$$
$$+(m_1 m_2 + \sigma k_1 k_2) = 0. \tag{2.35}$$

Is a degenerate cycle if and only if $\frac{m_1}{m_2} = -\frac{k_1}{k_2}$ which for $k_1 = k_2 = 1$ means $m_1 + m_2 = 0$ and then:

$$C_1 \odot_c C_2 : (l_2 m_1 - n_2)u + (n_2 m_1 + l_2)v + \frac{1}{2}(m_1^2 - \sigma) = 0. \tag{2.36}$$

5) $l_1 = n_1 = l_2 = n_2 = 0$:

$$C_1 \odot_c C_2 : (m_1 k_2 + m_2 k_1)(u^2 - \sigma v^2) + (m_1 m_2 + \sigma k_1 k_2) = 0. \tag{2.37}$$

The elliptic case is developed in [3].
6) $l_2 = n_2 = 0$:

$$C_1 \odot_c C_2 : (m_1 k_2 + m_2 k_1)(u^2 - \sigma v^2) - 2(l_1 m_2 + n_1 k_2)u - 2(n_1 m_2 - k_2 l_1)v+$$

$$+(m_1m_2+\sigma k_1k_2)=0 \tag{2.38}$$

and the degeneration condition is the same as in the case 4). For $k_1=k_2=1$ this means $m_1+m_2=0$ and then:

$$C_1\odot_c C_2:(n_1-l_1m_1)u-(n_1m_1+l_1)v+\frac{1}{2}(m_1^2-\sigma)=0. \tag{2.39}$$

7) $m_1=m_2=0$:

$$C_1\odot_c C_2:4(l_1n_2-l_2n_1)(u^2-\sigma v^2)-2(n_1k_2-n_2k_1)u-2(k_1l_2-k_2l_1)v+$$
$$+(k_1k_2\sigma+4n_1n_2\sigma-4l_1l_2)=0. \tag{2.40}$$

The degeneration condition is again (2.28) and then we have the cycles:

$$\begin{cases} C_a:\frac{k_a}{l_a}(u^2-\sigma v^2)-2u-2\rho v=0, \quad a=1,2, \\ C_1\odot_c C_2:\left(\frac{k_2}{l_2}-\frac{k_1}{l_1}\right)(\rho u-v)-\left(\frac{\sigma}{2}\frac{k_1}{l_1}\frac{k_2}{l_2}+2\sigma\rho^2-2\right)=0. \end{cases} \tag{2.41}$$

□

Remark 2.1 The relation (2.30) inspires a symmetric function parametrised by σ and μ and defined on the set $\Delta(\mathbb{R})=\{(x,y)\in\mathbb{R}^2; x+y\neq 0\}$ which is the usual plane without the second bisectrix $x+y=0$:

$$f_{\sigma,\mu}:\Delta(\mathbb{R})\to\mathbb{R}, \quad f_{\sigma,\mu}(x,y):=\frac{4(1+\sigma\mu^2)-xy}{2(x+y)}. \tag{2.42}$$

Its general study remains an open problem but the transformation $(x,y)\to 2(x,y)$ reduces it to the function:

$$F_{\sigma,\mu}:\Delta(\mathbb{R})\to\mathbb{R}, \quad F_{\sigma,\mu}(x,y):=\frac{1+\sigma\mu^2-xy}{x+y}. \tag{2.43}$$

In the parabolic geometry we obtain:

$$F_{0,\mu}(\tan\varphi_1,\tan\varphi_2)=\cot(\varphi_1+\varphi_2). \tag{2.44}$$

3 The Clifford product within $A(\sigma)$ and dual cycles

In this section we express the Clifford product by working into the setting of $A(\sigma)$. More precisely, with the usual notation $z=u+iv\in A(\sigma)$ we derive the expression of the generic cycle C:

$$C:C(z,\bar{z}):=kz\bar{z}+Bz+\bar{B}\bar{z}+m=0, \ B=B(C):=-l-\frac{n}{\sigma}i\in A(\sigma)\ (\sigma\neq 0). \tag{3.1}$$

For $\sigma=0$ we have: $B=-l-\frac{n}{i}$. The inverse relationship between C and $C(z,\bar{z})$ is:

$$l=-\Re B, \quad n=-\sigma\Im B \tag{3.2}$$

with $\Re$ and $\Im$ respectively the real and imaginary part. In the following we restrict our study to the case $\sigma \neq 0$ and we point out that decomposability means: $m = |B|^2$.

In this framework the expression (2.1) becomes:

$$C_1 \odot_c C_2 : (m_1k_2 + m_2k_1 + 4\sigma\Re B_1\Im B_2 - 4\sigma\Re B_2\Im B_1)(u^2 - \sigma v^2)- \\ -2(k_2\Im B_1 - k_1\Im B_2 - m_1\Re B_2 - m_2\Re B_1)u- \\ -2(k_2\Re B_1 - k_1\Re B_2 - \sigma m_2\Im B_1 - \sigma m_1\Im B_2)v+ \\ +(m_1m_2 + \sigma k_1k_2 + 4\Re B_1\Re B_2 + 4\sigma\Im B_1\Im B_2) = 0. \tag{3.3}$$

At the level of coefficients B's we have:

$$B(C_1 \odot_c C_2) = (k_2\Im B_1 - k_1\Im B_2 - m_1\Re B_2 - m_2\Re B_1)+ \\ +i(m_2\Im B_1 + m_1\Im B_2 + \sigma k_1\Re B_2 - \sigma k_2\Re B_1). \tag{3.4}$$

In particular, the $A(\sigma)$-number $B(C^2)$ does not depends on k of C since:

$$B(C^2) = -2m\overline{B(C)} \tag{3.5}$$

and for $m \neq 0$ we have from (2.10) the equality $B(C^2) = B(C)$. The degeneration condition for $C_1 \odot_c C_2$ is:

$$m_1k_2 + m_2k_1 = 4\sigma(\Re B_2\Im B_1 - \Re B_1\Im B_2) \tag{3.6}$$

or else:

$$\begin{vmatrix} \Re B_1 & \Re B_2 \\ \Im B_1 & \Im B_2 \end{vmatrix} = -\frac{\sigma}{4}(m_1k_2 + m_2k_1). \tag{3.7}$$

The condition (2.11) of equality $C^2 = C$ available only for the hyperbolic geometry is expressed as:

$$k^2 = m^2 + 4|B|^2 \tag{3.8}$$

and the nondegenerate case $k = 1$ implies the situation of B into the (hyperbolic) $\frac{1}{2}$-disc:

$$|B| = l^2 - n^2 \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{3.9}$$

Concretely, the hyperbolas of example 2.3 have $B(H_{\pm}) = 0$ while for the decomposable cycles of example 2.4 we arrive at $|B|^4 + 4|B|^2 = 1$ with solutions $|B| = \pm\sqrt{\sqrt{5} \pm 2}$ from (2.22).

One remark now that the product (2.6) can be expressed in a matrix way:

$$q(C_1 \odot_c C_2) = (m_2, -2l_2, -2n_2, k_2) \cdot \begin{pmatrix} m_1 & -2l_1 & -2n_1 & k_1 \\ 2l_1 & m_1 & k_1 & 2n_1 \\ -2\sigma n_1 & \sigma k_1 & m_1 & -2l_1 \\ \sigma k_1 & 2\sigma n_1 & 2l_1 & m_1 \end{pmatrix}. \tag{3.10}$$

Denoting as $A(C)$ the real 4×4 matrix above without the indices it results:

$$det A(C) = [(m^2 + 4l^2) - \sigma(k^2 + 4n^2)]^2 = [m^2 - \sigma k^2 + 4|B|^2]^2 \tag{3.11}$$

and comparing with Proposition 2.2 we have:

Proposition 3.1 *A cycle C of the hyperbolic geometry is idempotent if and only if $detA(C) = 0$. A non-degenerate cycle C of the elliptic geometry has $detA(C) > 0$.*

We remark beginning with relation (2.11) a duality between the coefficients: $(k, 2n) \leftrightarrow (m, 2l)$ and hence we introduce:

Definition 3.2 *The dual of the cycle C is:*

$$C^d : m(u^2 - \sigma v^2) - 2nu - 2lv + k = 0. \tag{3.12}$$

Due to the projective character we have $C^d = C$ if and only if $n = \pm l, m = \pm k$ and hence the only self-dual cycles are:

$$C_{\pm}(k, l) : k[(u^2 - \sigma v^2) \pm 1] - 2l(u \pm v) = 0, \quad B(C_{\pm}(k, l)) = -l(1 \pm \sigma i). \tag{3.13}$$

For example, the hyperbolas $H_{\pm}$ of (2.13) are self-dual; in elliptic geometry $C_+(1, 0)$ is the void set and $C_-(1, 0)$ is the unit circle S^1. The associated matrix has the determinant:

$$detA(C_{\pm}(k, l)) = (1 - \sigma)[k^2 + 4l^2]^2 \tag{3.14}$$

and then in elliptic and parabolic geometry this determinant is strictly positive while in hyperbolic geometry is zero.

Returning to the general case of C the Clifford product is:

$$C \odot_c C^d : (k^2 + m^2 + 4l^2 - 4n^2)(u^2 - \sigma v^2) - 2[(1 + \sigma)kl + (1 - \sigma)mn]u - 4knv +$$
$$+[(1 + \sigma)km + 4(\sigma - 1)ln] = 0, \tag{3.15}$$
$$C^d \odot C : (k^2 + m^2 + 4n^2 - 4l^2)(u^2 - \sigma v^2) - 2[(1 - \sigma)kl + (1 + \sigma)mn]u - 4lmv +$$
$$+[(1 + \sigma)km + 4(\sigma - 1)ln] = 0. \tag{3.16}$$

Since we have the equality of the free coefficients it follows that the commutativity $C \odot_c C^d = C^d \odot_c C$ implies the equality of first coefficients and then:

$$n = \pm l, \quad |B(C)|^2 = l^2(1 - \sigma). \tag{3.17}$$

In fact, the required commutativity holds only in the following cases:
1) $l = n = 0$ and then:

$$\begin{cases} C : k(u^2 - \sigma v^2) + m = 0, \quad C^d : m(u^2 - \sigma v^2) + k = 0, \\ C \odot_c C^d = C^d \odot_c C : (k^2 + m^2)(u^2 - \sigma v^2) + (1 + \sigma)km = 0. \end{cases} \tag{3.18}$$

Remark that for a non-degenerate cycle C in elliptic geometry the product $C \odot_c C^d = C^d \odot_c C$ reduces to the origin $\{O\}$.
2) $k = 0 = m$, $l = \pm n$ and we have the particular cases of self-duality (3.13) provided by the first and second bisectrix:

$$B_{\pm} : u \pm v = 0 \tag{3.19}$$

but the only possible geometry is elliptic with the degeneration $q(C \odot_c C^d) = q(C^d \odot_c C) = 0$.
3) the hyperbolic geometry with $m = \pm k \neq 0$ and we have the particular case $C^d = C$ with $\sigma = 1$ in (3.13) and the commuting product is C^2 discussed in Proposition 2.1 and Example 2.3 □

In [4, p. 1462] or [5, p. 2] a Möbius-invariant (indefinite) inner product (depending on $\breve{\sigma}$) is defined on the set of cycles through:

$$< C^s_{\breve{\sigma}}, \hat{C}^s_{\breve{\sigma}} >:= Tr(C^s_{\breve{\sigma}} \cdot \overline{\hat{C}^s_{\breve{\sigma}}}) \tag{3.20}$$

which yields an associated $\breve{\sigma}$-orthogonality. Here, the bar means the conjugation with respect to $\breve{\imath}$. Let us remark that:

$$det\ C^s_{\breve{\sigma}} = km + \breve{\sigma}n^2 - l^2 \rightarrow \|C^s_{\breve{\sigma}}\|^2 = -2det\ C^s_{\breve{\sigma}}. \tag{3.21}$$

For C and C^d we have:

$$C^s_{\breve{\sigma}} \cdot \overline{C^{ds}_{\breve{\sigma}}} = \begin{pmatrix} ln(1-\breve{\sigma}) + \breve{\imath}s(n^2 - l^2) - m^2 & mn - kl + \breve{\imath}s(lm - kn) \\ kn - ml + \breve{\imath}s(mn - kl) & ln(1-\breve{\sigma}) - k^2 + \breve{\imath}s(l^2 - n^2) \end{pmatrix} \tag{3.22}$$

and then C is $\breve{\sigma}$-orthogonal to C^d if and only if:

$$0 =< C, C^d >= 2ln(1-\breve{\sigma}) - k^2 - m^2. \tag{3.23}$$

In particular, C is 1-orthogonal to C^d if and only if C reduces to a line through origin and hence its matrix is a diagonal one:

$$C : lu + nv = 0, \quad C^d : nu + lv = 0, \quad C^s_{\breve{\sigma}} = \begin{pmatrix} l + \breve{\imath}sn & 0 \\ 0 & -l + \breve{\imath}sn \end{pmatrix}. \tag{3.24}$$

Other two transformations of a cycle C are considered in [2]:
i) the inversion $J(C)$: $(k, l, n, m) \rightarrow (m, l, -n, k)$,
ii) the rotation $R(C)$: $(k, l, n, m) \rightarrow (k, -\sigma n, -l, -\sigma m)$.
It follows that:
i) $J(C) = C^d$ if and only if and $n = l = 0$ and we have the relations (3.18),
ii) $R(C) = C^d$ if and only if $k = m = -\sigma m$, $l = 0$ and $(1 + \sigma)n = 0$. These three conditions exclude the parabolic and hyperbolic geometries since then all coefficients of C become zero.

Proposition 3.3 *A cycle C of elliptic geometry satisfies $R(C) = C^d$ if and only if it has the form:*

$$C(k, n) : k[u^2 + v^2 + 1] - 2nv = 0. \tag{3.25}$$

For $k = 0$ we have the Ou axis and for $k \neq 0$ we have a circle centered in $(0, \frac{n}{k})$ and with radius $R = \left(\frac{n^2}{k^2} - 1\right)^{\frac{1}{2}}$ which exists only for $k^2 \leq n^2$.

Concerning the inner product and associated orthogonality we have:

$$< C^d, J(C) >=$$
$$= Tr\begin{pmatrix} ln(1+\breve{\sigma})+\breve{\imath}s(l^2+n^2)-km & k(l-n)-\breve{\imath}sk(l+n) \\ m(l-n)+\breve{\imath}sm(l+n) & -km+ln(1+\breve{\sigma})-\breve{\imath}s(l^2+n^2)\end{pmatrix} =$$
$$= 2[(1+\breve{\sigma})ln-km] = 0 \tag{3.26}$$
$$< C^d, R(C) >=$$
$$= Tr\begin{pmatrix} \breve{\sigma}l^2-\sigma n^2-k^2+\breve{\imath}sln(1-\sigma) & -n(m+\sigma k)-\breve{\imath}sl(k+m) \\ -n(k+\sigma m)+\breve{\imath}sl(k+m) & \breve{\sigma}l^2-\sigma n^2-m^2-\breve{\imath}sln(1-\sigma)\end{pmatrix} =$$
$$= 2(\breve{\sigma}l^2-\sigma n^2)-k^2-m^2 = 0. \tag{3.27}$$

Hence the possible orthogonality between C^d and $J(C)$ is common to all EPH geometries.

4 The σ-Clifford product of third degree polynomials and elliptic curves

Let $\mathbb{R}_3[X]$ be the 4-dimensional real vector space of polynomials:

$$f(X) = aX^3 + bX^2 + cX + d, \quad a \neq 0. \tag{4.1}$$

Recall that the Cardano substitution $X = Y - \frac{b}{3a}$ gives *the reduced form* of f:

$$f^r(Y) = Y^3 + pY + q, \quad p = \frac{c}{a} - \frac{b^2}{3a^2}, \quad q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \tag{4.2}$$

with *the discriminant*:

$$\Delta(f^r) := \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2. \tag{4.3}$$

The roots of f^r are:

$$y_1 := P + Q, \quad y_2 := \overline{y_3} := -\frac{y_1}{2} + i\frac{P-Q}{2}\sqrt{3},$$
$$P^3 := -\frac{q}{2} + \sqrt{\Delta(f^r)}, \quad Q = -\frac{q}{2} - \sqrt{\Delta(f^r)}. \tag{4.4}$$

With the method of the first section we define *the σ-Clifford product* on (r=reduced) $\mathbb{R}^r_3[Y]$:

$$f^r_1 \odot_\sigma f^r_2(Y) := (q_1+q_2)Y^3 + (p_2-p_1)Y^2 + (p_1q_2+p_2q_1)Y + (\sigma+q_1q_2-p_1p_2),$$
$$q_1 + q_2 \neq 0, p_1 = p_2 \tag{4.5}$$

In particular, *the σ-Clifford square* of f^r from (4.2) with $q \neq 0$ (hence 0 is not a root of f^r) is:

$$f^{r2}_\sigma(Y) := Y^3 + pY + \frac{\sigma + q^2 - p^2}{2q}, \quad \Delta(f^{r2}_\sigma) = \left(\frac{p}{3}\right)^3 + \left(\frac{\sigma + q^2 - p^2}{4q}\right)^2. \tag{4.6}$$

From the expression (4.4) of roots it results that these are all real and distinct if and only if $\Delta(f^r) < 0$. In order to obtain the same results for f^{r2}_σ we search for:

$$\begin{aligned} 0 \leq \Delta(f^r) - \Delta(f^{r2}_\sigma) &= \left(\frac{q}{2}\right)^2 - \left(\frac{\sigma + q^2 - p^2}{4q}\right)^2 = \\ &= \frac{(\sigma + 3q^2 - p^2)(p^2 + q^2 - \sigma)}{(4q)^2}. \end{aligned} \tag{4.7}$$

Example 4.1 If $p = 0$ and $q \neq 0$ then:

$$\begin{aligned} f^r(Y) = Y^3 + q, \quad f^{r2}_\sigma(Y) = Y^3 + \frac{q^2 + \sigma}{2q}, \\ \Delta(f^r) = \frac{q^2}{4} > 0, \quad \Delta(f^{r2}_\sigma) = \frac{(q^2 + \sigma)^2}{(4q)^2} \geq 0. \end{aligned} \tag{4.8}$$

For the example of third order roots of unity we choose $q = -1$ and hence:

$$f^{r2}_{-1}(Y) = Y^3, \quad f^{r2}_0(Y) = Y^3 - \frac{1}{2}, \quad f^{r2}_1(Y) = f^r(Y) = Y^3 - 1. \tag{4.9}$$

□

Example 4.2 Let $-p = q = 6$. Then:

$$f^r(Y) = Y^3 - 6Y + 6, \Delta(f^r) = 1, f^{r2}_\sigma(Y) = Y^3 - 6Y + \frac{\sigma}{12}, \Delta(f^{r2}_\sigma) = \left(\frac{\sigma}{24}\right)^2.$$

□

Fix now a field F with characteristic $\neq 2, 3$ and an elliptic curve in short Weierstrass form:

$$E : y^2 = x^3 + ax + b, \quad \Delta(E) := -2^4(4a^3 + 27b^2) \neq 0 \tag{4.10}$$

where $a, b \in F$ with $2b$ in the group of units $U(F)$ of F. With the computations above we define its *the σ-Clifford square*:

$$\begin{aligned} E^2_\sigma : y^2 = x^3 + ax + [(2b)^{-1}(\sigma + b^2 - a^2)], \\ \Delta(E^2_\sigma) = -2^4\{4a^3 + 27[(2b)^{-1}(\sigma + b^2 - a^2)]^2\} \end{aligned} \tag{4.11}$$

and then we have the invariance $\Delta(E) = \Delta(E^2_\sigma)$ if and only if $\sigma \in \{a^2 - 3b^2, a^2 + b^2\}$.

Example 4.3 In [1] the following elliptic curve is associated to a specific hyperbola called *Barning*:

$$E_b : y^2 = x^3 - \frac{13}{3}x + \frac{92}{27}, \quad \Delta(E_b) = 16 \cdot 12 = 2^6 \cdot 3 = 192. \tag{4.12}$$

The intersection points of E_b with the Ox-axis are: $P(\frac{4}{3}, 0)$,$(-\frac{2}{3} \pm \sqrt{3}, 0)$. Its σ-Clifford square is then:

$$(E_b)^2_\sigma : y^2 = x^3 - \frac{13}{3}x + \frac{27}{184}\left[\sigma + \frac{92^2}{27^2} - \frac{13^2}{9}\right] = x^3 - \frac{13}{3}x + \frac{27}{184}\left[\sigma - \frac{5225}{27^2}\right]. \tag{4.13}$$

We can consider also the elliptic curve:

$$C_b : y^2 + y = x^3 - \frac{13}{3}x + \frac{92}{27} \tag{4.14}$$

which intersections with the Oy-axis are: $(0, -\frac{1}{2} \pm \frac{1}{6}\sqrt{\frac{395}{3}})$.

The minimal integral model for E_b is:

$$E_b : y^2 = x^3 + x^2 - 4x + 2 \tag{4.15}$$

and The L-functions and Modular Forms Database http://www.lmfdb.org/ provides that its torsion points are the point at infinity $\mathcal{O}[0 : 1 : 0]$ and $P[4 : 0 : 3]$, the last having order 2. Hence the group $E_b(\mathbb{Q})$ is isomorphic with $\mathbb{Z}/2\mathbb{Z}$. □

Remark 4.1 Let $n \in \mathbb{N}^*$ be a positive integer and the elliptic curve $E_n : y^2 = x^3 - n^2x$. This E_n appears in the *problem of congruent numbers*, [6, p. 4], but our EPH-square is not available since $b = 0$. □

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