Adjoint variables for higher-order equations in kinematical form

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Abstract

We obtain a type of first integrals for ODE in kinematical form by using a generalization of the method of adjoint variables to higher-order systems. A generalization of harmonic oscillator and classical spinning particle is completely discussed in order to solve the associated inverse problem.

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Introduction

Until now, the method of *adjoint*(or *added*) *variables* was used only for systems of differential equations which involve the derivatives of first and second order. So, for second order ODE see [5], [6] and for second order PDE see [1], [2]. For other informations about this approach see [10, chapter 3].

In the first paragraph we present the generalization of this method to higher-order ODE obtained in [3]. Next is considered systems in kinematical form. The results of this section are generalizations of similar results from [5]. As example is given a system which generalizes both the harmonic oscillator and spinning particle. By using the obtained first integrals the coresponding inverse problem is resolved.

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1 The adjoint variables approach

Let us consider a general system of k-order ODE, with k and n two natural numbers:

$$F^j\left(t,x,x_1,\ldots,x_k\right)=0, 1\leq j\leq n \tag{1.1}$$

with the unknow function $x=\left(x^i\right)_{1\leq i\leq n}$ and:

$$x^i_\alpha=\frac{d^\alpha x^i}{dt^\alpha}, 1\leq\alpha\leq k. \tag{1.2}$$

A *first integral* (or *conservation law* or *conserved quantity*) for the system (1.1) is a function $\mathcal{F}=\mathcal{F}\left(t,x,x_1,\ldots,x_{k-1}\right)$ satisfying:

$$\frac{d\mathcal{F}}{dt}=0, mod\,(1.1) \tag{1.3}$$

where $mod\,(1.1)$ means "on the solutions of (1.1)" and $\frac{d}{dt}$ denotes the total differentiation with respect to t: $\frac{d}{dt}=\frac{\partial}{\partial t}+x^i_1\frac{\partial}{\partial x^i}+\ldots+x^i_k\frac{\partial}{\partial x^i_{k-1}}$.

In [3] it is proved:

Theorem *If the functions* $\xi=\left(\xi^i\right)$, $\mu=\left(\mu_j\right)$ *of* (t,x) *satisfy the system:*

$$\frac{d}{dt}T_{(k)i}+(-1)^k\mu_j\frac{\partial F^j}{\partial x^i}=0 \tag{1.4a}$$

$$\mu_j\sum_{\alpha=1}^{k+1}\frac{\partial F^j}{\partial x^i_{k-\alpha+1}}\frac{d^{k-\alpha+1}\xi^i}{dt^{k-\alpha+1}}=\frac{dK}{dt} \tag{1.4b}$$

on the solutions of (1.1) *where* $K=K\left(t,x,\ldots,x_{k-1}\right)$ *is a given function and:*

$$T_{(\alpha)i}=\frac{d}{dt}T_{(\alpha-1)i}+(-1)^{\alpha+1}\mu_j\frac{\partial F^j}{\partial x^i_{k-\alpha+1}}, 1\leq\alpha\leq k,\ T_{(0)i}=0 \tag{1.4a'}$$

then $\mathcal{F}$ *given by:*

$$\mathcal{F}=\sum_{\alpha=1}^{k}(-1)^{\alpha+1}T_{(\alpha)i}\frac{d^{k-\alpha}\xi^i}{dt^{k-\alpha}}-K \tag{1.5}$$

is first integral for (1.1).

The relation (1.4b) is exactly:

$$\mu_j dF^j(x)(\xi) = 0 \tag{1.6}$$

where $dF^j(x)$ means the Frechet derivative of F^j and then the equations (1.4a) + (1.4a') are:

$$(dF(x))_i^*(\mu) = 0 \tag{1.7}$$

where $(dF(x))_i^*(\mu)$ is the adjoint of the linear operator $dF(x)$:

$$(-1)^k (dF(x))_i^*(\mu) = \frac{d^k}{dt^k}\left(\frac{\partial F^j}{\partial x_k^i}\mu_j\right) - \frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial F^j}{\partial x_{k-1}^i}\mu_j\right) + \ldots +$$

$$+ (-1)^{k-1}\frac{d}{dt}\left(\frac{\partial F^j}{\partial x_1^i}\mu_j\right) + (-1)^k \frac{\partial F^j}{\partial x^i}\mu_j. \tag{1.8}$$

Recall that the vectorial operator $F = (F^j)$ is called *self-adjoint* if the Frechet derivative of F is self-adjoint and this property is the necessary and sufficient condition for the system $F^j = 0$ to be yield by a variational principle i.e. there exists a Lagrangian L such that the given system is exactly the Euler-Lagrange system of L.

2 Systems in kinematical form

The system (1.1) is called *in kinematical form* if:

$$F^j = x_k^j - f^j(t, x, x_1, \ldots, x_{k-1}). \tag{2.1}$$

Then, on solutions of (1.1) the total differentiation operator $\frac{d}{dt}$ is reduced to the vector field $\frac{D}{Dt} = \frac{\partial}{\partial t} + x_1^i \frac{\partial}{\partial x^i} + \ldots + x_{k-1}^i \frac{\partial}{\partial x_{k-2}^i} + f^i \frac{\partial}{\partial x_{k-1}^i}$ and the theorem becomes:

Proposition 1 *If* ξ *and* μ *satisfy:*

$$\frac{D}{Dt}T_{(k)i} = (-1)^k \mu_j \frac{\partial f^j}{\partial x^i} \tag{2.2a}$$

$$\mu_j\left(\frac{D^k \xi^j}{Dt^k} - \sum_{\alpha=2}^{k+1} \frac{\partial f^j}{\partial x_{k-\alpha+1}^i}\frac{D^{k-\alpha+1}\xi^j}{Dt^{k-\alpha+1}}\right) = \frac{DK}{Dt} \tag{2.2b}$$

where:
(i) $K = K(t, x, \ldots, x_{k-1})$ *is a given function*
(ii) $T_{(1)i} = \mu_i$ *and:*

$$T_{(\alpha)i} = \frac{D}{Dt} T_{(\alpha-1)i} + (-1)^{\alpha} \mu_j \frac{\partial f^j}{\partial x^i_{k-\alpha+1}}, \ 2 \le \alpha \le k \qquad (2.2a')$$

then $\mathcal{F}$ *given by* (1.5) *is first integral for* (2.1).

Using the relations:

$$\left[\frac{\partial}{\partial x^i_{\alpha}}, \frac{D}{Dt} \right] = \frac{\partial f^j}{\partial x^i_{\alpha}} \frac{\partial}{\partial x^i_{k-1}} + \frac{\partial}{\partial x^i_{\alpha-1}}, \ 0 \le \alpha \le k-1 \qquad (2.6)$$

it results:

Proposition 2 *If* $\mathcal{F}_1, \mathcal{F}_2$ *are first integrals of* (2.1) *then:*

$$\mu_i = \mathcal{F}_1 \frac{\partial \mathcal{F}_2}{\partial x^i_{k-1}} \qquad (2.7)$$

satisfy $(2.2a) + (2.2a')$.

Proof By induction we get:

$$T_{(\alpha)i} = (-1)^{\alpha+1} \mathcal{F}_1 \frac{\partial \mathcal{F}_2}{\partial x^i_{k-\alpha}}, \ 0 \le \alpha \le k. \qquad (2.8)$$

Indeed, for $\alpha = 1$ we have (2.7) from $T_{(1)i} = \mu_i$(cf. proposition 1) and (2.7). Let us suppose that we have (2.8) for a fixed α. Then, using $(2.2a')$:

$$T_{(\alpha+1)i} = (-1)^{\alpha+1} \mathcal{F}_1 \frac{D}{Dt} \frac{\partial \mathcal{F}_2}{\partial x^i_{k-\alpha}} + (-1)^{\alpha+1} \mu_j \frac{\partial f^j}{\partial x^i_{k-\alpha}} \overset{(2.6)}{=} (-1)^{\alpha+2} \mathcal{F}_1 \frac{\partial \mathcal{F}_2}{\partial x^i_{k-(\alpha+1)}}$$

what we require. Then $T_{(k)i} = (-1)^{k+1} \mathcal{F}_1 \frac{\partial \mathcal{F}_2}{\partial x^i}$ which gives:

$$\frac{D}{Dt} T_{(k)i} = (-1)^{k+1} \mathcal{F}_1 \frac{D}{Dt} \frac{\partial \mathcal{F}_2}{\partial x^i} \overset{(2.6)}{=} (-1)^{k+2} \mathcal{F}_1 \frac{\partial f^j}{\partial x^i} \frac{\partial \mathcal{F}_2}{\partial x^j_{k-1}} = (-1)^k \mu_j \frac{\partial f^j}{\partial x^i}$$

i.e. the conclusion. □

Particularly by putting $\mathcal{F}_1 = 1$ we obtain:

Proposition 3 *A first integral* Ω *of* (2.1) *gives rise to a new one:*

$$\mathcal{F}_{\Omega}=\sum_{\alpha=1}^{k}\frac{\partial\Omega}{\partial x_{k-\alpha}^{i}}\frac{D^{k-\alpha}\xi^{i}}{Dt^{k-\alpha}}-K \tag{2.9}$$

where ξ *satisfy*:

$$\frac{\partial\Omega}{\partial x_{k-1}^{j}}\left(\frac{D^{k}\xi^{j}}{Dt^{k}}-\sum_{\alpha=2}^{k+1}\frac{\partial f^{j}}{\partial x_{k-\alpha+1}^{i}}\frac{D^{k-\alpha+1}\xi^{i}}{Dt^{k-\alpha+1}}\right)=\frac{DK}{Dt}. \tag{2.10}$$

Let us consider the particular case when the functions (f^j) does not depend of time, that is the considered system is *autonomous*(time-independent). Then a straightforward calculus gives that $\xi^i = x_1^i$ satisfy $(2.2b)$(although we require only $\xi = \xi(t,x)$!) with $K = 0$ and therefore:

Proposition 4 *If* $f^j = f^j(x,\ldots,x_{k-1})$ *and* μ *satisfy* $(2.2a) + (2.2a')$ *then:*

$$\mathcal{F}_{\mu}=\mu_i f^i+\sum_{\alpha=2}^{k}(-1)^{\alpha+1}T_{(\alpha)i}x_{k-\alpha+1}^{i} \tag{2.11}$$

is a first integral of (2.1).

3 A generalization of harmonic oscillator and spinning particle

Let us consider the system of order $k = 2m$:

$$F^i = x_{2m}^i + x_{2m-2}^i = 0, 1 \leq i \leq 3 \tag{3.1}$$

which is a system in kinematical form with $f^i = -x_{2m-2}^i$. For $m = 1$ this system describes the harmonic oscillator and for $m = 2$ the spinning particle([4]). The associated system (2.2) is:

$$\frac{d^{2m}\mu_i}{dt^{2m}}+\frac{d^{2m-2}\mu_i}{dt^{2m-2}}=0 \tag{3.2}$$

and we have two solutions:

$$\mu_i = \cos t \tag{3.3a}$$

$$\mu_i = \sin t \tag{3.3b}$$

with coresponding first integrals given by proposition 4:

$$\mathcal{F}^i_{\cos t} = x^i_{2m-1} \sin t - x^i_{2m-2} \cos t \tag{3.4a}$$

$$\mathcal{F}^i_{\sin t} = x^i_{2m-1} \cos t + x^i_{2m-2} \sin t. \tag{3.4b}$$

Let us remark that from proposition 3 we have $\mathcal{F}_{\mathcal{F}_{\cos t}} = \mathcal{F}_{\cos t}$ and $\mathcal{F}_{\mathcal{F}_{\sin t}} = \mathcal{F}_{\sin t}$.

The system (3.1) is self-adjoint and then we are interested in finding the associated Lagrangian. Also, from (3.1) it results the first integral:

$$C^i = x^i_{2m-1} + x^i_{2m-3} \tag{3.5}$$

which, in order to eliminate the variable t, yields the first integral:

$$\Psi^i = \left(C^i\right)^2 - \left(\mathcal{F}^i_1\right)^2 - \left(\mathcal{F}^i_2\right)^2. \tag{3.6}$$

A straightforward computation gives:

$$\Psi^i = \left(x^i_{2m-3}\right)^2 - \left(x^i_{2m-2}\right)^2 + 2x^i_{2m-3}x^i_{2m-1} \tag{3.7}$$

and then we have the first integral:

$$H = \frac{1}{2}\left(\Psi^1 + \Psi^2 + \Psi^3\right) =$$

$$= \frac{1}{2}\sum_{i=1}^{3} \left(x_{2m-3}\right)^2 - \frac{1}{2}\sum_{i=1}^{3} \left(x_{2m-2}\right)^2 + \sum_{i=1}^{3} x^i_{2m-3}x^i_{2m-1} \tag{3.8}$$

which is exactly the Hamiltonian for (3.1). The associated Lagrangian is:

$$L = \frac{1}{2}\sum_{i=1}^{3} \left(x^i_{m-1}\right)^2 - \frac{1}{2}\sum_{i=1}^{3} \left(x^i_{m}\right)^2 \tag{3.9}$$

(that is (3.1) are the Euler-Lagrange equations for L) a result very important from the point of view of Inverse Problem of Analytical Mechanics([9]). Thus we solve on this way the inverse problem for harmonic oscillator and spinning particle.

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