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Classes of harmonic functions in 2D generalized Poincaré geometry

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Abstract. By using the additive and multiplicative separation of variables we find some classes of solutions of the Laplace equation for a generalization of the Poincaré upper half plane metric. Non-constant totally geodesic functions implies the flat metric and several examples are studied including the Hamilton's cigar Ricci soliton. The Bochner formula is discussed for our generalized Poincaré metric and for its important particular cases.

1. Introduction

This paper is devoted to the Laplace equation on a two-dimensional (semi-) Riemannian geometry generalizing the well-known Poincaré metric of the upper half plane model of hyperbolic geometry:

$$g(x, y) = \frac{1}{y^2}\left(dx^2 + dy^2\right) \tag{1.1}$$

or, in complex variable z and its conjugate $\bar{z}$:

$$g(z, \bar{z}) = \frac{-4}{(z - \bar{z})^2}|dz|^2 = -\left(\frac{2|dz|}{z - \bar{z}}\right)^2. \tag{1.2}$$

For a geometrical method to obtain this metric see [3]. More precisely, we make two variations on this metric: 1) allow the sign of dy^2 to be also negative and then we place in a semi-Riemannian framework, 2) consider arbitrary functions $a = a(y)$, $b = b(y)$ as coefficients. We have this choice since this class of metrics was used recently in [1] in order to find classes of gradient Ricci solitons; for the very interesting interplay between Ricci flow techniques and hyperbolic geometry see [13] and [16]. Also, the example 8 of [10, p. 5661] proves that the inverse metric of (1.1) is a Liouville metric satisfying the Wünschmann-type condition.

The motivation for this subject is triple. Firstly, from a geometrical point of view it is well-known the importance of harmonic functions in the (semi-) Riemannian geometry; see for example [11]. Let us remark that harmonic functions for the metric (1.1) are called *hyperbolic harmonic functions* (see the survey [12]) and are studied recently in [14] as being contractions with respect to the induced distance. Secondly, from the point of view of applications, by using Proposition 3.1 of [6, p. 463], the square root of positive harmonic

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functions are autonomous solutions of the Liouville equation of transport corresponding to their gradient vector field. Thirdly, it is well-known that the dimension two is a special (critical) one for the theory of harmonic functions on (semi-)Riemnanian manifolds.

In fact, we search for harmonic functions with a prescribed separation of variables: i) additive i.e. $f(x,y) = u(x) + v(y)$, and ii) multiplicative i.e. $f(x,y) = u(x)v(y)$. In the first case we obtain one type of harmonic function while for the second case we have five types of harmonic functions, two of them splitting again in two subclasses according to the sign of ratio $h = a/b$. We remark also that the functions a, b appear in the expression of these harmonic functions only through h and not separately. A whole section is devoted to a particular class of harmonic function namely that with vanishing Hessian, called here *totally geodesic*, and for which we compute also the energy density. An important fact is that the existence of non-constant totally geodesic functions implies the flatness of the metric g.

We treat also a subject related to the theory of harmonic functions namely Bochner formula which is given for the generalized metric and then derived for some remarkable particular cases. The final section is devoted to the weighted harmonic functions.

2. Two dimensional generalized Poincaré geometry

Let (M^n, g) be a smooth, n-dimensional semi-Riemannian manifold. We call it *generalized Poincaré manifold* if there exists an atlas of M such that in its local coordinates systems $x = (x^1, ..., x^n)$ the metric has the expression:

$$g(x) = \mathrm{diag}(f^1(x^n), ..., f^n(x^n)). \tag{2.1}$$

In the following we restrict to the case $n = 2$, for which we denote:

$$g(x,y) = \mathrm{diag}(a(y), b(y)) \tag{2.2}$$

where the smooth functions a, b satisfy $a \cdot b \neq 0$ for g being a metric.

Example 2.1. *Let $S \subset \mathbb{R}^3$ be a surface of revolution parametrized as $S : \bar{r}(x,y) = (\varphi(y)\cos x, \varphi(y)\sin x, \psi(y))$. It is well-known that its first fundamental form is: $g = [(\varphi')^2 + (\psi')^2]dx^2 + \varphi^2 dy^2$ and then a comparison with* (2.2) *gives the Riemannian metric ($a > 0$, $b > 0$) with:*

$$\varphi(y) = \sqrt{b(y)}, \quad \psi(y) = \int_{y_0}^{y} \sqrt{a(t) - \frac{(b'(t))^2}{4b(t)}}dt$$

if the condition: $4a(y)b(y) > (b'(y))^2$ holds for any y. □

Returning to the general case (2.2) a straightforward computation yields that the only non-vanishing Christoffel coefficients are:

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{a'}{2a}, \quad \Gamma^2_{11} = -\frac{a'}{2b}, \quad \Gamma^2_{22} = \frac{b'}{2b} \tag{2.3}$$

where the prime denotes the derivative with respect to the variable y. Its Gaussian curvature is:

$$K(y) = -\frac{1}{2\sqrt{|ab|}}\left(\frac{a'}{\sqrt{|ab|}}\right)'. \tag{2.4}$$

Let $f : (M^n, g) \to \mathbb{R}$ be a smooth function on an arbitrary semi-Riemannian manifold. Recall that the gradient vector field ∇f and the Hessian H_f are locally:

$$\nabla f = (g^{ij}f_j)\frac{\partial}{\partial x^i}, \quad H_f = f_{ij}dx^i \otimes dx^j \tag{2.5}$$

with components:

$$f_i = \frac{\partial f}{\partial x^i}, \quad f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}f_k. \tag{2.6}$$

Finally, the Laplacian of f is ([11, p. 142]):

$$-\Delta f = Tr_g(H_f) = g^{ij} f_{ij} \tag{2.7}$$

with (g^{ij}) the contravariant components of the metric $g = (g_{ij})$. For our framework (2.2) we get:

$$-\Delta f = \frac{1}{a}\left[\frac{\partial^2 f}{\partial x^2} + \frac{a'}{2b}\frac{\partial f}{\partial y}\right] + \frac{1}{b}\left[\frac{\partial^2 f}{\partial y^2} - \frac{b'}{2b}\frac{\partial f}{\partial y}\right]. \tag{2.8}$$

In order to simplify the computations we consider the functions:

$$h(y) = \frac{a(y)}{b(y)}, \quad I(y) = \int_{y_0}^{y} \frac{dt}{\sqrt{|h(t)|}} \tag{2.9}$$

with y_0 a fixed point in the common domain of definition for the functions a and b. Let us remark that h is well-defined and non-vanishing, since a and b are the components of the semi-Riemannian metric g. We get:

$$-a \cdot \Delta f = \frac{\partial^2 f}{\partial x^2} + \operatorname{sgn}(h)\sqrt{|h|}\frac{\partial}{\partial y}\left(\sqrt{|h|}\frac{\partial f}{\partial y}\right). \tag{2.10}$$

For the completeness of the subject we study very briefly the geodesics of (M, g):

$$\begin{cases} \ddot{x} + \frac{a'}{a}\dot{x}\dot{y} = 0 \\ \ddot{y} - \frac{a'}{2b}\dot{x}^2 + \frac{b'}{2b}\dot{y}^2 = 0. \end{cases} \tag{2.11}$$

From the first equation a solution is $\dot{x} = 0$ which means $x(t)$ =constant. If $\dot{x} \neq 0$ then the first equation gives a first integral:

$$a(y(t))\dot{x}(t) = constant. \tag{2.12}$$

3. Separable harmonic functions and examples

We are interested in the harmonic functions of the metric (2.2). According to (2.8) such a harmonic function f satisfies the equation:

$$\frac{\partial^2 f}{\partial x^2} = -\operatorname{sgn}(h)\sqrt{|h|}\frac{\partial}{\partial y}\left(\sqrt{|h|}\frac{\partial f}{\partial y}\right). \tag{3.1}$$

Firstly we search for a separation of variables in the expression of f of type: $f(x, y) = u(x) + v(y)$. We obtain:

Proposition 3.1. *With c, d and e arbitrary real constants, the function:*

$$f_1(x, y) = \frac{cx^2}{2} + dx + e - \frac{c}{\operatorname{sgn}(h)}\int_{y_0}^{y} \frac{I(t)}{\sqrt{|h(t)|}}dt \tag{3.2}$$

is harmonic for the metric g.

Proof. The equation (3.1) becomes:

$$u''(x) = -\operatorname{sgn}(h)\sqrt{|h|}\frac{\partial}{\partial y}\left(\sqrt{|h|}v'\right). \tag{3.3}$$

and therefore there exists a real number c such that:

$$u''(x) = c, \quad \operatorname{sgn}(h)\sqrt{|h|}\frac{\partial}{\partial y}\left(\sqrt{|h|}v'\right) = -c. \tag{3.4}$$

It results that $u = \frac{cx^2}{2} + dx + e$ and:

$$v'(y) = \frac{-c}{\mathrm{sgn}(h)\sqrt{|h|}} I(y)$$

which yields the conclusion.

Let us remark that if we search for a "horizontally" periodic f_1 i.e. $f_1(x+T, y) = f_1(x, y)$ for all x, y and fixed $T > 0$ then we obtain $c = d = 0$ and hence $f_1 = f_1(y)$ is "horizontally" constant. Also, the "horizontally" convexity i.e. $\frac{\partial^2 f_1}{\partial x^2} > 0$ requires $c > 0$. □

Secondly, we search for a separation of variables of multiplicative type: $f = u(x)v(y)$. We obtain:

Proposition 3.2. *With α, β, c, d, e and $k > 0$, arbitrary real constants the functions:*
1. $c > 0$:

$$\begin{cases} f_{2,\pm}(x,y) = \frac{1}{2}\left(\alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}\right)\left(e^{\pm\sqrt{c}I(y)} - \frac{k}{c}e^{\mp\sqrt{c}I(y)}\right) \\ f_{3,\pm}(x,y) = \pm\sqrt{\frac{k}{c}}\left(\alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}\right)\sin(\sqrt{c}I(y)) \end{cases} \tag{3.5}$$

2. $c = 0$: $f_4(x,y) = k(dx+e)I(y)$
3. $c < 0$:

$$\begin{cases} f_{5,\pm}(x,y) = \pm\sqrt{\frac{k}{-c}}[\alpha\cos(\sqrt{-c}x) + \beta\sin(\sqrt{-c}x)]\sin(\sqrt{-c}I(y)) \\ f_{6,\pm}(x,y) = \frac{1}{2}[\alpha\cos(\sqrt{-c}x) + \beta\sin(\sqrt{-c}x)]\left(e^{\pm\sqrt{-c}I(y)} + \frac{k}{c}e^{\mp\sqrt{-c}I(y)}\right) \end{cases} \tag{3.6}$$

are harmonic with respect to the metric g. The functions $f_{2,\pm}$ and $f_{5,\pm}$ correspond to the pseudo-Riemannian case i.e. sgn$(h) = -1$ while $f_{3,\pm}$ and $f_{6,\pm}$ correspond to the Riemannian case i.e. sgn$(h) = +1$.

Proof. As in the proof of previous Proposition we get:

$$\frac{u''}{u}(x) = -\frac{\mathrm{sgn}(h)\sqrt{|h|}}{v(y)}\frac{\partial}{\partial y}\left(\sqrt{|h|}v'(y)\right), \tag{3.7}$$

which means the existence of a real number c such that:

$$u''(x) = cu(x), \quad \sqrt{|h|}\frac{\partial}{\partial y}\left(\sqrt{|h|}v'(y)\right) = \frac{-cv(y)}{\mathrm{sgn}(h)}. \tag{3.8}$$

If $c = 0$, then we have the function f_4. Suppose now that $c > 0$ and then: $u(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$. We multiply the second equation of (3.7) with $v'(y)$ and thus:

$$\frac{\partial}{\partial y}\left(\sqrt{|h|}v'\right)^2 = \frac{-c}{\mathrm{sgn}(h)}\frac{\partial v^2}{\partial y}. \tag{3.9}$$

An integration yields:

$$\left(\sqrt{|h|}v'\right)^2 = k - \frac{cv^2}{\mathrm{sgn}(h)} \tag{3.10}$$

and then we have a splitting after the sign of h. If sgn$(h) = -1$, then from:

$$\frac{v'}{\sqrt{\frac{k}{c} + v^2}} = \pm\sqrt{\frac{c}{|h|}} \tag{3.11}$$

we have:

$$\ln|v + \sqrt{\frac{k}{c} + v^2}| = \pm\sqrt{c}I(y) \tag{3.12}$$

yielding:

$$v_{\pm}(y)=\frac{e^{\pm 2\sqrt{c}I(y)}-\frac{k}{c}}{2e^{\pm\sqrt{c}I(y)}}, \tag{3.13}$$

which corresponds to $f_{2,\pm}$. If $sgn(h)=+1$ then (3.10) implies:

$$\frac{v'}{\sqrt{\frac{k}{c}-v^2}}=\pm\sqrt{\frac{c}{|h|}} \tag{3.14}$$

and therefore:

$$\arcsin\left(\frac{v}{\sqrt{\frac{k}{c}}}\right)=\pm\sqrt{c}I(y), \tag{3.15}$$

which means $f_{3,\pm}$. A similar analysis yields the functions of case 3. □

Example 3.3. *i) Since the Laplacian is a linear operator we can delete the constant $\frac{1}{2}$ or k from f_4.*

An important remark concerning the expressions of these two Propositions is that for the additive and multiplicative case the harmonic functions depends not on a and b properly but on their ratio h. Hence the warped metric $h(y)dx^2+dy^2$ and its conformal transformations $b(y)[h(y)dx^2+dy^2]$ share the same additive and multiplicative separable harmonic functions.

ii) Suppose that $a=b$; thus we work with the conformal transformation $g=a(y)\,\mathrm{diag}(1,1)$ of the Euclidean metric which is a particular case of the Liouville metrics $[u(x)+v(y)]\,\mathrm{diag}(1,1)$. The equation (3.1) is the Euclidean Laplace equation and then the Euclidean metric and its conformal transformation above have the same harmonic functions. Since $I(y)=y-y_0$ we obtain the harmonic functions:

$$\begin{cases} f_1^+(x,y)=\frac{c}{2}(x^2-y^2)+dx+\gamma y+e, \quad f_4^+(x,y)=(dx+e)(ky+\gamma),\\ f_3^+(x,y)=\left(\alpha e^{\sqrt{c}x}+\beta e^{-\sqrt{c}x}\right)\sin(\sqrt{c}y+d),\\ f_{6,\pm}^+(x,y)=[\alpha\cos(\sqrt{-c}x)+\beta\sin(\sqrt{-c}x)]\left(e^{\pm\sqrt{-c}y}+\frac{k}{c}e^{\mp\sqrt{-c}y}\right)\end{cases} \tag{3.16}$$

with γ real. For $\alpha=\beta$ we get that $f_3^+(x,y)=2\alpha\cosh(\sqrt{c}x)\sin(\sqrt{c}y+d)$. The Gaussian curvature of this metric is:

$$K(y)=-\frac{1}{2a}\left(\frac{a'}{a}\right)'=\frac{(a')^2-aa''}{2a^3} \tag{3.17}$$

while for the geodesics we have the equation:

$$x=x(y)=\pm C\int\frac{dy}{\sqrt{a(y)-C^2}}$$

with arbitrary $C>0$. In [8, p. 95] a Poincaré flow $a(y)=b(y)=y^t$ with a real t is discussed as an example of Riemannian flow.

iii) Suppose now that $a=-b$; then the pseudo-metric has the form $g=a(y)\,\mathrm{diag}(1,-1)$. We obtain the harmonic functions:

$$\begin{cases} f_1^-(x,y)=\frac{c}{2}(x^2+y^2)+dx+\gamma y+e, \quad f_4^-=f_4^+,\\ f_{2,\pm}^-(x,y)=\left(\alpha e^{\sqrt{c}x}+\beta e^{-\sqrt{c}x}\right)\left(e^{\pm\sqrt{c}y}-\frac{k}{c}e^{\mp\sqrt{c}y}\right),\\ f_5^-(x,y)=[\alpha\cos(\sqrt{-c}x)+\beta\sin(\sqrt{-c}x)]\sin(\sqrt{-c}y+d).\end{cases} \tag{3.18}$$

For $\alpha=\beta$ we get $f_{2,\pm}^-(x,y)=2\alpha\cosh\sqrt{c}x\left(e^{\pm\sqrt{c}y}-\frac{k}{c}e^{\mp\sqrt{c}y}\right)$. The equation (3.1) is exactly the homogeneous one-dimensional wave equation:

$$\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial^2 y}=0 \tag{3.19}$$

studied in great detail in [2]. It is well-known that the general solution of this wave equation is: $f(x,y) = u(x-y) + u(x+y)$ *with* u *an arbitrary smooth function.*

iv) The particular case $a = k^2$*=constant was studied in [5] from the point of view of vanishing Hessian; see also the Section 5. So, the functions obtained in the third Section of [5] are particular cases of* (3.16)*. From* (2.4) *it results that the metric* g *is flat, i.e.* $K = 0$*.*

More precisely, the metric is $g = (kdx)^2 + b(y)dy^2$*, with* $k > 0$ *and* $b(y) > 0$ *everywhere. The change of coordinates:*

$$(x,y) \to (\tilde{x},\tilde{y}) = (kx, \int_{y_0}^{y} \sqrt{b(t)}dt) \tag{3.20}$$

gives the Euclidean expression: $g(\tilde{x},\tilde{y}) = d\tilde{x}^2 + d\tilde{y}^2$*.*

v) Let us consider now the particular case $b = k^2$ *and* $a(y) > 0$ *everywhere. With* $a(y) = A^2(y)$ *where* $A(y) > 0$ *again, we obtain:*

$$I(y) = k\int_{y_0}^{y} \frac{dt}{A(t)}, \quad K(y) = -\frac{A''(y)}{kA(y)}. \tag{3.21}$$

The metric reads:

$$g(x,y) = A^2(y)\left(dx^2 + \left(\frac{kdy}{A(y)}\right)^2\right), \tag{3.22}$$

which is of the type studied in ii) above via a change of coordinates: $(x,y) \to (\tilde{x},\tilde{y}) = (x, I(y))$*.*

vi) A remarkable example of the 2D generalized Poincaré metric is the Hamilton's cigar metric written in polar coordinates as ([4, p. 160]) or in complex variable:

$$g(\theta,r) = \frac{1}{1+r^2}\left(r^2d\theta^2 + dr^2\right) = \frac{1}{1+|z|^2}|dz|^2. \tag{3.23}$$

It follows our metric (2.2) *with* $x = \theta$*,* $y = r > 0$*,* $a(y) = \frac{y^2}{1+y^2} = \frac{(z-\bar{z})^2}{(z-\bar{z})^2-4}$ *and* $b(y) = \frac{1}{1+y^2} = \frac{4}{4-(z-\bar{z})^2}$*; hence the first integral* (2.12) *is* $y(t) - \arctan y(t) = C_1 t$ *and another first integral results from* (2.11_1) *as* $\left(1 + y^2(t)\right)\ln x'(t) = C_2$*. We have:*

$$h(y) = y^2, \quad I(y) = \int_{y_0}^{y} \frac{dt}{t}dt = \ln\frac{y}{y_0}. \tag{3.24}$$

Let us point out that I *is exactly the Poincaré length of vertical geodesics* $(x_0,y_0) \to (x_0,y)$ *for the metric* (1.1)*. A long but straightforward computation yields the following expression for* f_1*:*

$$f_1(\theta,r) = \frac{c\theta^2}{2} + d\theta + e - c\left(\frac{r^2}{2}\ln\frac{r}{r_0} + \frac{r_0^2 - r^2}{4}\right). \tag{3.25}$$

We recall that the Hamilton's cigar metric is a steady gradient (and complete) Ricci soliton, [9, p. 3338], and details about this class of metrics can be found in the book [4]. From the complex point of view the metric (3.23) *is a Kähler-Ricci soliton.*

vii) A natural generalization of the equation (3.1) *is:*

$$\frac{\partial^2 f}{\partial x^2} + A(y)\frac{\partial}{\partial y}\left(B(y)\frac{\partial f}{\partial y}\right) = 0 \tag{3.26}$$

with one solution generalizing (3.2)*:*

$$f_1(x,y) = \frac{c}{2}x^2 + dx + e - c\int_{y_0}^{y}\left(\frac{1}{B(t)}(\int_{y_0}^{t}\frac{ds}{A(s)})\right)dt. \tag{3.27}$$

viii) It is well-known that if f *is a harmonic function then its differential* df *is a harmonic* 1*-form. It follows that* df_1*,...,* df_6 *are examples of harmonic* 1*-forms in generalized Poincaré geometry.* □

4. The 2D generalized Poincaré-Bochner identity

Recall, after [15, p. 187], the Bochner formula:

$$-\frac{1}{2}\Delta\left(|\nabla f|_g^2\right) = |H_f|_g^2 + Ric(\nabla f, \nabla f) \tag{4.1}$$

for every harmonic function on the Riemannian manifold (M, g) where $|.|_g$ denotes the norm with respect to g and Ric is the Ricci tensor field (remark that the sign of our Laplacian is opposite to that of the cited book). For our framework since the dimension is 2 it follows that (M, g) is an Einstein manifold with: $Ric = Kg$ where K is provided by (2.4). In order to simplify the below expressions we use a subscript for the variable corresponding to a derivation: $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ and so on. The main result of this Section is the following relation which can be called 2D *generalized Poincaré-Bochner formula*:

Proposition 4.1. *If f is a harmonic function on the generalized Poincaré manifold (M^2, g) then:*

$$F_{yy} + \frac{h'F_y + 2F_{xx}}{2h} = \frac{2b}{a^2}(f_{xx} + \frac{a'}{2b}f_y)^2 + \frac{4}{a}\left(f_{xy} - \frac{a'}{2a}f_x\right)^2 + \frac{2}{b}\left(f_{yy} - \frac{b'}{2b}f_y\right)^2 - \frac{F}{\sqrt{h}}\left(\frac{a'}{\sqrt{ab}}\right)' \tag{4.2}$$

where $F = F(f) = |\nabla f|_g^2 = \frac{f_x^2}{a} + \frac{f_y^2}{b}$ is the double energy density of f.

Proof. Since $\nabla f = \frac{f_x}{a}\frac{\partial}{\partial x} + \frac{f_y}{b}\frac{\partial}{\partial y}$ for the above expression of F we have:

$$F = \left(\frac{f_x}{a}, \frac{f_y}{b}\right) \cdot \mathrm{diag}(a, b) \cdot \begin{pmatrix} \frac{f_x}{a} \\ \frac{f_y}{b} \end{pmatrix} = (f_x, f_y) \cdot \begin{pmatrix} \frac{f_x}{a} \\ \frac{f_y}{b} \end{pmatrix}$$

which yields the required expression. As in Section 2 the Laplacian of F is given by (2.8) with f replaced by F and we get the left hand side of (4.3) modulo the scalar 2 which appears in the right hand side. The matrix of the Hessian of f is:

$$H_f = \begin{pmatrix} f_{xx} + \frac{a'}{2b}f_y & f_{xy} - \frac{a'}{2a}f_x \\ f_{xy} - \frac{a'}{2a}f_x & f_{yy} - \frac{b'}{2b}f_y \end{pmatrix} \tag{4.3}$$

and then its norm is:

$$|H_f|_g^2 = f_{ij}f^{ij} = \frac{1}{a^2}\left(f_{xx} + \frac{a'}{2b}f_y\right)^2 + \frac{2}{ab}\left(f_{xy} - \frac{a'}{2a}f_x\right)^2 + \frac{1}{b^2}\left(f_{yy} - \frac{b'}{2b}f_y\right)^2 \tag{4.4}$$

which ends the claimed formula. □

Another expression can be done:

$$\frac{a}{2}\left[F_{xx} + hF_{yy} + \frac{h'F_y}{2} + F\sqrt{h}\left(\frac{a'}{\sqrt{ab}}\right)'\right] = \left(f_{xx} + \frac{a'}{2b}f_y\right)^2 + 2h\left(f_{xy} - \frac{a'}{2a}f_x\right)^2 + h^2\left(f_{yy} - \frac{b'}{2b}f_y\right)^2. \tag{4.5}$$

Example 4.2. *i) The 2D Euclidean-Bochner formula is:*

$$F_{xx} + F_{yy} = 2f_{xx}^2 + 4f_{xy}^2 + 2f_{yy}^2, \quad F = f_x^2 + f_y^2. \tag{4.6}$$

ii) The 2D Poincaré-Bochner formula is given by $a = b = y^{-2}$:

$$F_{xx} + F_{yy} + \frac{2F}{y^2} = 2y^2\left(f_{xx} - \frac{f_y}{y}\right)^2 + 4y^2\left(f_{xy} + \frac{f_x}{y}\right)^2 + 2y^2\left(f_{yy} + \frac{f_y}{y}\right)^2, \quad F = y^2(f_x^2 + f_y^2). \tag{4.7}$$

With some computations in the right hand side we get a more simple form:

$$F_{xx}+F_{yy}=2y^2\left(f_{xx}^2+2f_{xy}^2+f_{yy}^2\right)+4y\left(f_xf_{xy}+f_yf_{yy}-f_yf_{xx}\right)+2\left(f_x^2+f_y^2\right), F=y^2(f_x^2+f_y^2). \tag{4.8}$$

It is easy to verify the relations (4.6) *and* (4.8) *for the common harmonic function* $f=\frac{c}{2}\left(x^2-y^2\right)$*: the first relation gives in both members* $2c^2$ *while the second gives* $2c^2(x^2+7y^2)$.
iii) The Hamilton-Bochner formula is given for $a(r)=\frac{r^2}{1+r^2}$, $b(r)=\frac{1}{1+r^2}$ *and* $h(r)=r^2$*:*

$$F_{\theta\theta}+r^2F_{rr}+rF_r=\frac{2(1+r^2)}{r^2}\left(f_{\theta\theta}+\frac{rf_r}{1+r^2}\right)^2+4(1+r^2)\left(f_{\theta r}-\frac{f_\theta}{r(1+r^2)}\right)^2+$$
$$+2r^2(1+r^2)\left(f_{rr}+\frac{rf_r}{1+r^2}\right)^2+\frac{4(f_\theta^2+r^2f_r^2)}{1+r^2},\quad F=(1+r^2)\left(\frac{f_\theta^2}{r^2}+f_r^2\right). \tag{4.9}$$

□

Remark 4.3. *Recall the relationship between the partial derivatives in* (x,y) *respectively* $(z,\bar{z})$*:*

$$\partial_x=\partial_z+\partial_{\bar{z}},\quad \partial_y=i(\partial_z-\partial_{\bar{z}}) \tag{4.10}$$

which yields a new expression for the function $F(f)$*:*

$$F=\frac{1}{a}(f_z+f_{\bar{z}})^2-\frac{1}{b}(f_z-f_{\bar{z}})^2. \tag{4.11}$$

It follows the particular cases: 1) $a=b$, $F=\frac{4}{a}f_zf_{\bar{z}}$*; 2)* $a=-b$, $F=\frac{2}{a}(f_z^2+f_{\bar{z}}^2)$. □

5. Totally geodesic functions and their energy density

Having the expression (4.3) we determine now the totally geodesic functions namely that non-constant f with vanishing H_f; in [15, p. 283] these functions are called *linear*.

Proposition 5.1. *Let* f *be a totally geodesic function on the generalized Poincaré manifold* (M,g)*. Then we have two cases:*
I) $a'=\sqrt{ab}$ *and there exist constants* α *and* β *such that:*

$$f=f^1(x,y)=\left(\alpha\cos\frac{x}{2}+\beta\sin\frac{x}{2}\right)\sqrt{a(y)}. \tag{5.1}$$

II) a is a strictly positive constant and there exist constants α *and* β *such that:*

$$f=f^2(x,y)=\alpha x+\beta,\quad f=f^3(x,y)=\alpha x+\beta\int_{y_0}^{y}\sqrt{b(t)}dt. \tag{5.2}$$

In both cases the metric g is flat i.e. $K=0$.

Proof. We have the system:

$$\begin{cases} f_{xx}-\frac{a'}{2b}f_y=0\\ f_{xy}-\frac{a'}{2a}f_x=0\\ f_{yy}-\frac{b'}{2b}f_y=0\end{cases} \tag{5.3}$$

and search the compatibility relations: $f_{xxy}=f_{xyx}$ respectively $f_{xyy}=f_{yyx}$. So, we derive the first equation with respect to y and compare with the derivation of second relation with respect to x:

$$-\frac{1}{2}\left(\frac{a'}{b}\right)'f_y-\frac{a'}{2b}f_{yy}=\frac{a'}{2a}f_{xx}=\frac{a'}{2a}\left(-\frac{a'}{2b}\right)f_y. \tag{5.4}$$

It follows:

$$\left(\frac{a'}{b}\right)' + \frac{a'b'}{2b^2} = \frac{(a')^2}{2ab}$$

which means:

$$\left(\frac{a'}{b}\right)' = \frac{1}{2}\frac{a'}{b}\left(\frac{a'}{a} - \frac{b'}{b}\right). \tag{5.5}$$

Case I) $a' \neq 0$. Integrating the relation (5.5) gives:

$$\ln\frac{a'}{b} = \frac{1}{2}\ln\frac{a}{b} \tag{5.6}$$

which gives $a' = \sqrt{ab}$ and this last equation replaced in (2.4) yields $K = 0$.
In the last equation (5.3) we can not have $f_y = 0$ since this implies, via the second (5.3), that $f_x = 0$ and in conclusion f is a constant. From $f_y \neq 0$ we have:

$$\frac{f_{yy}}{f_y} = \frac{b'}{2b}$$

and then:

$$f_y = u(x)\sqrt{b(y)}. \tag{5.7}$$

Replacing this relation in the second (5.3) gives:

$$f_x = 2u'(x)\sqrt{a(y)}. \tag{5.8}$$

By deriving this relation and replacing in the first (5.3) we get:

$$u'' + \frac{1}{4}u = 0$$

with the solution:

$$u(x) = \alpha\cos\frac{x}{2} + \beta\sin\frac{x}{2}$$

and it results (5.1).
Case II) a is a strictly positive constant; again $K = 0$ from (2.4). From the first two equations (5.3) we get that f_x is a constant α and then $f = \alpha x + v(y)$. The last equation (5.3) means: $v'' = \frac{b'}{2b}v'$. If v is a constant we have f^2 while for a non-constant v we get f^3.

In fact we can express locally the metric as a product one $g = \pm dx^2 + dy^2$ since the vanishing a of the Hessian H_f means the parallelism of the gradient vector field ∇f and we apply the Exercise 22 of [15, p. 60]. □

Example 5.2. *i) Since a totally geodesic function is obviously a harmonic one is natural to ask if the functions f^* belong to the classes f_* of Section 3.*
Case I) From $a' = \sqrt{ab}$ it follows that $I(y) = \ln a(y)$ and with the choice: $k = 0$ and $c = -\frac{1}{4}$ we obtain that $f^1 = 2f_{6,+}$. The volume form of the metric g is $dVol := \sqrt{\det g}dx \wedge dy = dx \wedge da$.
Case II) a=constant. It is immediately that $f^2 = f_1$ for $c = 0$. The function f^3 does not belong to the class given by f_1 but it is covered by the Remark iv) of the Section 3. Namely, in the new coordinates $(\bar{x}, \bar{y})$ we have the linear expression:

$$f^3(\bar{x}, \bar{y}) = \alpha\bar{x} + \beta\bar{y}. \tag{5.9}$$

ii) An example of functions a, b satisfying the condition of Case I) is: $a(y) = b(y) = e^y$ and then:

$$f^1(x, y) = \left(\alpha\cos\frac{x}{2} + \beta\sin\frac{x}{2}\right)e^{\frac{y}{2}}. \tag{5.10}$$

The geodesics of the metric $g = e^y(dx^2 + dy^2)$ are given by:

$$\gamma(t) = \left(2\arctan\left(\frac{C_2 t}{2C_1}\right), \ln\left(\frac{C_1^2}{C_2} + \frac{C_2 t^2}{4}\right)\right) \tag{5.11}$$

where $C_1 > 0$ is the first integral (2.12) *and also $C_2 > 0$.* □

Recall that the energy density of the function $f : (M, g) \to \mathbb{R}$ is the function:

$$e(f) = \frac{1}{2}|df|_g^2 = \frac{1}{2}g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} = \frac{1}{2}F(f) \tag{5.12}$$

and the characterization of harmonic functions as critical points of the integrals of this density. Thus we are interested in the value of the energy density for the harmonic functions but for simplicity we compute it only for the particular case of totally geodesic functions above. With the calculus of Proposition 4.1 we have:

$$e(f) = \frac{1}{2}\left(\frac{f_x^2}{a} + \frac{f_y^2}{b}\right) \tag{5.13}$$

and then all three energy densities are constants:

$$e(f^1) = \frac{1}{8}\left(\alpha^2 + \beta^2\right), \quad e(f^2) = \frac{\alpha^2}{2a}, \quad e(f^3) = \frac{1}{2}\left(\frac{\alpha^2}{a} + \beta^2\right). \tag{5.14}$$

Remark that the $e(f^1)$ is independent of the flat metric g.

6. The weighted Laplacian in 2D generalized Poincaré geometry

We extend the framework of previous sections with the weighted case following [7]:

Definition 6.1. *Let $\mu \in C^\infty(M)$ with $\mu > 0$. The the weighted divergence of $X \in \mathcal{X}(M)$ is:*

$$div_\mu X = \frac{1}{\mu}div(\mu X) \tag{6.1}$$

and the weighted Laplacian is:

$$-\Delta_\mu = div_\mu \circ \nabla. \tag{6.2}$$

For our setting we have:

$$-\Delta_\mu f = \frac{1}{\mu\sqrt{|ab|}}\left[\frac{1}{\sqrt{|h|}}(\mu f_x)_x + (\sqrt{|h|}\mu f_y)_y\right] \tag{6.3}$$

and we restrict to the case $\mu = \mu(y)$. Inspired by the section 2 we introduce the function:

$$I_\mu(y) = \int_{y_0}^{y}\frac{dt}{\sqrt{|h(t)|}\mu(t)} \tag{6.4}$$

and we generalize the Propositions 3.1 and 3.2 as follows:

Proposition 6.2. *With c, d and e arbitrary real constants, the function:*

$$f_1^\mu(x, y) = \frac{cx^2}{2} + dx + e - \frac{c}{\mathrm{sgn}(h)}\int_{y_0}^{y}\frac{I_\mu(t)}{\sqrt{|h(t)|}\mu(t)}dt \tag{6.5}$$

is μ-harmonic for the metric g. For the multiplicative case we make the change $I \to I_\mu$ in (3.5)-(3.6).

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