

# Gradient-type deformations of cycles in EPH geometries

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**Abstract.** The aim of this paper is to study the cycles of EPH geometries through their homogeneous gradient-type deformations recently introduced by the author. A special topic is the orthogonality between a given cycle  $C$  and its deformations as well as between  $C$  and its rotated version  $R(C)$ .

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## 1. Introduction

It is well-known that up to isomorphisms there are three 2-dimensional real algebras:  $\mathbb{C} = \mathbb{R}[X]/(x^2 + 1)$ ,  $\mathbb{D} = \mathbb{R}[X]/(x^2)$  and  $\mathbb{A} = \mathbb{R}[X]/(x^2 - 1)$ . The theory of the first algebra is richer than the following two, a fact corresponding to the field property of  $\mathbb{C}$ . Inspired by the terminology of [6, p. 1458] or [7, p. 2] we call *EPH geometries* these spaces and a common image consists in  $A(\sigma) := \mathbb{R}[X]/(x^2 - \sigma)$  with  $\sigma := i^2 \in \{-1, 0, 1\}$  respectively and  $i$  the corresponding imaginary unit.

The recent papers [2] and [5], devoted to Finsler geometry, start with a deformation of a conic  $\Gamma$  obtained by deforming the gradient vector field for the quadratic form defining  $\Gamma$ . These deformations are inspired by the scaling (linear) transformation of Computer Graphics:  $(x, y) \in \mathbb{R}^2 \rightarrow (\lambda_x \cdot x, \lambda_y \cdot y) \in \mathbb{R}^2$ , following [8, p. 136]. Using the well-known invariants from the Euclidean geometry of conics we obtained the classifications of the new conics which depends on two scalars denoted  $\alpha$  and  $\beta$ , having the role of  $\lambda_x, \lambda_y$ . The new conic of [2], denoted  $\tilde{\Gamma}$ , is a degenerate one and we could interpret the map  $\Gamma \rightarrow \tilde{\Gamma}$  as a "curve shortening" transformation. The same fact holds for the new conic of [5], denoted  $\Gamma^m$ , if the initial conic  $\Gamma$  does not have linear terms.

In this note we use these classes of gradient-type deformation to a main object of EPH geometries, called *cycle*, which is a particular case of conic sections, invariant under the action of the group  $SL(2, \mathbb{R})$  through Möbius

transformations. A detailed analysis of the deformed cycles depends on the vanishing or not of  $\sigma$  as well as the vanishing or not of a parameter  $k$  separating the circles to lines. Also, we discuss the transformation of a square matrix associated to any cycle  $C$ .

Moreover, we treat these deformations in terms of  $A(\sigma)$ -numbers. In the second section we study the orthogonality of a given cycle  $C$  with its deformations restricting to the  $\sigma \neq 0$  case. In the last section we introduce a natural rotation  $R$  in  $A(\sigma)$  and we study the relationships between a given  $C$  and its rotated cycle  $R(C)$ .

## 2. The cycles of EPH geometries and their gradient-type deformations

In the two-dimensional Euclidean space  $\mathbb{R}^2$  let us consider the conic  $\Gamma$  implicitly defined by  $f \in C^\infty(\mathbb{R}^2)$  as:  $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$  where  $f$  is a quadratic function of the form  $f(x, y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$  with  $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$  for the non-degenerate conics.

It is well-known that the gradient vector field of  $f$ , namely  $\nabla f = (f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y})$ , gives important properties of  $\Gamma$ ; for example, the centers of  $\Gamma$  are exactly the critical points of  $\nabla f$ . Inspired by this fact we introduced recently:

**Definition 2.1.** Fix the scalars  $\alpha, \beta$  with  $\alpha\beta \neq 0$ .

i) ([2, p. 86-87], [3, p. 60]) The  $(\alpha, \beta)$ -deformation of  $\Gamma$  is the conic:

$$\tilde{\Gamma} = \Gamma_{\alpha, \beta} : \alpha \left[ \frac{1}{2} f_x \right]^2 + \beta \left[ \frac{1}{2} f_y \right]^2 = 0. \quad (2.1)$$

ii) ([5, p. 102]) The  $(\alpha, \beta)$ -mixed deformation of  $\Gamma$  is the conic:

$$\Gamma^m = \Gamma_{\alpha, \beta}^m : \alpha y \left[ \frac{1}{2} f_x \right] + \beta x \left[ \frac{1}{2} f_y \right] = 0. \quad (2.2)$$

A main object in EPH geometries is given in [6, p. 1459], [7, p. 4]:

**Definition 2.2.** The common name cycle will be used to denote circles, parabolas and hyperbolas (as well as straight lines as their limits) in the respective EPH geometry.

An analytical study of a cycle can be done via the general equation given in [6, p. 1460] or [7, p. 6]:

$$C : f(u, v) := k(u^2 - \sigma v^2) - 2lu - 2nv + m = 0 \quad (2.3)$$

and hence  $C$  is a conic section completely defined by the data  $(k, l, n, m) \in \mathbb{P}^3$ . As usual, if  $k = 0$  then  $C$  can be called a *degenerate cycle*. In fact, in the cited works  $C$  is identified with the matrix:

$$C_{\sigma}^s := \begin{pmatrix} l + \check{y}sn & -m \\ k & -l + \check{y}sn \end{pmatrix} \quad (2.4)$$

where  $s$  is a new parameter, usually equal to  $\pm 1$ , and a new imaginary unit  $\check{1}$ . Its square  $\check{\sigma} := \check{1}^2$  belongs again to  $\{-1, 0, 1\}$  but independently of  $\sigma$ .

Since  $C$  is a conic section we can apply the ideas of Definition 2.1 to introduce the gradient-type deformations of a cycle:

$$\begin{cases} \tilde{C} = C_{\alpha, \beta} : \alpha(ku - l)^2 + \beta(k\sigma v + n)^2 = 0, \\ C^m : \alpha v(ku - l) - \beta u(k\sigma v + n) = 0 \end{cases} \quad (2.5)$$

which yields immediately:

**Proposition 2.3.** *Since  $\alpha \neq 0$  we have:*

- i)  $\tilde{C}$  is a cycle if and only if  $\sigma(\alpha + \sigma\beta) = 0$ ,
- ii)  $C^m$  is a cycle if and only if  $k(\alpha - \beta\sigma) = 0$ . In this case  $C^m$  is the straight line:  $(\beta n)u + (\alpha l)v = 0$ .

**Example 2.4.** In the following we discuss the remarkable particular cases of the result above.

- i) Suppose  $\sigma = 0$ . Then  $\tilde{C}$  is the cycle:

$$\tilde{C} : (ku - l)^2 + \frac{\beta}{\alpha}n^2 = 0 \quad (2.6)$$

with the matrix:

$$\tilde{C}_{\check{\sigma}}^s = \begin{pmatrix} kl & -(l^2 + \frac{\beta}{\alpha}n^2) \\ k^2 & -kl \end{pmatrix}. \quad (2.7)$$

The degenerate case of an initial line i.e.  $k = 0$  is possible if and only if  $\alpha l^2 + \beta n^2 = 0$  which is relation (2.19) below. If  $k \neq 0$  then, due to the projective character of the coefficients of a cycle, we get the matrix:

$$\tilde{C}_{\check{\sigma}}^s = \begin{pmatrix} l & -\frac{1}{k}(l^2 + \frac{\beta}{\alpha}n^2) \\ k & -l \end{pmatrix}. \quad (2.8)$$

If  $\frac{\beta}{\alpha} > 0$  then  $\tilde{C}$  is a void set for  $n \neq 0$  while  $n = 0$  gives the deformation:

$$C : ku^2 - 2lu + m = 0 \rightarrow \tilde{C} : ku = l \text{ (line : } k \neq 0\text{)}. \quad (2.9)$$

If  $\frac{\beta}{\alpha} < 0$  then we have the lines:

$$\tilde{C} : ku - l = \pm \sqrt{-\frac{\beta}{\alpha}}n. \quad (2.10)$$

$C^m$  is a cycle if and only if  $k = 0$  which means that we have the mixed deformation:

$$C : 2lu + 2nv - m = 0 \text{ (line)} \rightarrow C^m : (\beta n)u + (\alpha l)v = 0 \text{ (line)}. \quad (2.11)$$

If  $\beta = -\alpha$  then these two lines are Euclidean orthogonal. From the matrix point of view the deformation (2.11) means:

$$C_{\check{\sigma}}^s = \begin{pmatrix} l + \check{1}sn & -m \\ 0 & -l + \check{1}sn \end{pmatrix} \rightarrow C_{\check{\sigma}}^{m,s} = \begin{pmatrix} -\beta n + \check{1}s(-\alpha l) & 0 \\ 0 & \beta n + \check{1}s(-\alpha l) \end{pmatrix}. \quad (2.12)$$

- ii) For  $\sigma \neq 0$  we have that  $\tilde{C}$  is a cycle only for  $\beta = -\frac{\alpha}{\sigma} = -\sigma\alpha$  and then:

$$\tilde{C} : [k(u + iv) - l + \frac{n}{i}][k(u - iv) - l - \frac{n}{i}] = 0. \quad (2.13)$$

Hence, if  $k \neq 0$  then  $\tilde{C}$  consists in a single point:  $M = (\frac{l}{k}, -\frac{n}{k\sigma})$ . Let us point out that for  $\sigma \neq 0$  we have  $\frac{1}{\sigma} = \sigma$  and hence  $M = (\frac{l}{k}, -\sigma\frac{n}{k})$  which is exactly the  $e/h$ -center of the initial cycle  $C$ , as it is introduced in formula (7) of [6, p. 1460] or [7, p. 7]. In conclusion, for  $\sigma \cdot k \neq 0$  we have the deformation:

$$C \rightarrow \tilde{C} = \text{its center.} \quad (2.14)$$

The matrix corresponding to  $\tilde{C}$  is:

$$\tilde{C}_{\sigma}^s = \begin{pmatrix} k(l + \text{isn}) & n^2\sigma - l^2 \\ k^2 & k(-l + \text{isn}) \end{pmatrix} \quad (2.15)$$

which for  $k = 0$  becomes:

$$\tilde{C}_{\sigma}^s = \begin{pmatrix} 0 & n^2\sigma - l^2 \\ 0 & 0 \end{pmatrix} \quad (2.16)$$

while for  $k \neq 0$ , due to the projective character of the parameters of a cycle:

$$\tilde{C}_{\sigma}^s = \begin{pmatrix} l + \text{isn} & \frac{1}{k}(n^2\sigma - l^2) \\ k & -l + \text{isn} \end{pmatrix}. \quad (2.17)$$

The same case  $\sigma \cdot k \neq 0$  for ii) of proposition above gives  $\beta = \frac{\alpha}{\sigma} = \sigma\alpha$  and  $C^m$  is the line:

$$C^m : nu + (\sigma l)v = 0. \quad (2.18)$$

For elliptic geometry the condition  $\beta = -\frac{\alpha}{\sigma} = -\sigma\alpha$  becomes the equality  $\alpha = \beta$  discussed in [2, p. 89] and [3, p. 62]; it can be called *the diagonal case*. Remark that the elliptic center  $\tilde{C}$  of (2.14) is obtained in [6, p. 1461] or [7, p. 8] from the vanishing condition  $\det C_{-1}^s = 0$ .  $\square$

**Remark 2.5.** The cycle  $C^m$  contains the origin  $(u, v) = (0, 0) = O$ . This fact holds for  $\tilde{C}$  if and only if:

$$\alpha l^2 + \beta n^2 = 0. \quad (2.19)$$

With the discussion of above particular cases it results:

i) for  $\sigma = 0$  the only available case is  $\frac{\beta}{\alpha} < 0$  yielding:

$$l_{\pm} = \pm \sqrt{-\frac{\beta}{\alpha}} n. \quad (2.20)$$

ii) for  $\sigma \neq 0$  since  $\beta = -\frac{\alpha}{\sigma} = -\sigma\alpha$  we get that for the elliptic geometry the only possible case is  $O = M$  the center of  $C$  while for the hyperbolic geometry:

$$l_{\pm} = \pm n. \quad (2.21)$$

The gradient-type deformation of a standard (i.e. Euclidean) ellipse is discussed in example 2.2i) of [2, p. 87]. Let us point out that (2.20) and (2.21) coincide for  $\beta = -\alpha$  which for the case ii) correspond to the hyperbolic geometry. Hence the above cases i) and ii) are completely different, both from  $\sigma$  and the sign of  $\frac{\beta}{\alpha}$  points of view.  $\square$

Returning to the general case of  $\alpha$  and  $\beta$  we treat the considered deformations within  $A(\sigma)$  following the model of [3] and [5]. More precisely, with the usual notation  $z = u + iv \in A(\sigma)$  we derive the expression of  $C$ :

$$C : F(z, \bar{z}) := kz\bar{z} + Bz + \bar{B}\bar{z} + m = 0, \quad B := -l - \frac{n}{\sigma}i \in A(\sigma) \quad (\sigma \neq 0). \quad (2.22)$$

For  $\sigma = 0$  we have:  $B = -l - \frac{n}{i}$ . The inverse relationship between  $f$  and  $F$  is:

$$l = -\Re B, \quad n = -\sigma \Im B \quad (2.23)$$

with  $\Re$  and  $\Im$  respectively the real and imaginary part. By replacing in (2.5) the usual relations:

$$u = \frac{1}{2}(z + \bar{z}), \quad v = \frac{1}{2i}(z - \bar{z}) \quad (2.24)$$

we get:

$$\begin{cases} \tilde{C} : \alpha[k(z + \bar{z}) - 2l]^2 + \beta[ki(z - \bar{z}) + 2n]^2 = 0, \\ C^m : \alpha(z - \bar{z})[k(z + \bar{z}) - 2l] - \beta(z + \bar{z})[k\sigma(z - \bar{z}) + 2ni] = 0. \end{cases} \quad (2.25)$$

For the case  $\sigma \neq 0$  we follow the discussion of Example 2.4ii and then:

$$\begin{cases} \tilde{C} : [k(z + \bar{z}) - 2l]^2 - \sigma[ki(z - \bar{z}) + 2n]^2 = 0, \\ C^m : (z - \bar{z})[k(z + \bar{z}) - 2l] - \sigma(z + \bar{z})[k\sigma(z - \bar{z}) + 2ni] = 0. \end{cases} \quad (2.26)$$

The second equation (2.26) reduces to:

$$C^m : Bz - \bar{B}\bar{z} = 0 \Leftrightarrow Bz \in \mathbb{R} \quad (2.27)$$

and hence, for  $B \neq 0$  we have the line:  $z = \bar{B} \cdot \mathbb{R}$ .

We finish this section by applying to the cycle  $C$  (not containing the origin, hence  $m \neq 0$ ) the *inversion*  $J : z \in A(\sigma)^* \rightarrow \frac{1}{z} = w$ . We get a new cycle, expressed in  $w$ :

$$J(C) : mw\bar{w} + \bar{B}w + B\bar{w} + k = 0 \quad (2.28)$$

which means  $J : (k, l, n, m) \rightarrow (m, l, -n, k)$ . With (2.26)-(2.27) its gradient deformations for  $\sigma \neq 0$  are:

$$\begin{cases} \widetilde{J(C)} : [m(w + \bar{w}) - 2l]^2 - \sigma[mi(w - \bar{w}) - 2n]^2 = 0, \\ J(C)^m : B\bar{w} - \bar{B}w = 0 \Leftrightarrow \bar{B}w \in \mathbb{R}. \end{cases} \quad (2.29)$$

Again, if  $B \neq 0$  then the second cycle from from above is the line:  $w = B \cdot \mathbb{R}$ .

### 3. Orthogonality in the geometry of cycles

In [6, p. 1462] or [7, p. 2] a Möbius-invariant (indefinite) inner product (depending on  $\check{\sigma}$ ) is defined on the set of cycles through:

$$\langle C_{\check{\sigma}}^s, \hat{C}_{\check{\sigma}}^s \rangle := Tr(C_{\check{\sigma}}^s \cdot \overline{\hat{C}_{\check{\sigma}}^s}) \quad (3.1)$$

which yields an associated  $\check{\sigma}$ -orthogonality. Here, the bar means the conjugation with respect to  $i$ .

For our setting we derive firstly the norms of a cycle and its gradient-type deformations for  $k\sigma \neq 0$ :

$$\begin{cases} \|C_{\check{\sigma}}^s\|^2 = 2(l^2 - km - \check{\sigma}n^2) = \|J(C)_{\check{\sigma}}^s\|^2, \\ \|\tilde{C}_{\check{\sigma}}^s\|^2 = 2(\sigma - \check{\sigma})n^2, \quad \|C_{\check{\sigma}}^{m,s}\|^2 = \frac{1}{2}(n^2 - \check{\sigma}l^2). \end{cases} \quad (3.2)$$

Let us remark that:

$$\det C_{\check{\sigma}}^s = km + \check{\sigma}n^2 - l^2 \rightarrow \|C_{\check{\sigma}}^s\|^2 = \|J(C)_{\check{\sigma}}^s\|^2 = -2\det C_{\check{\sigma}}^s. \quad (3.3)$$

Secondly, we study all the possible cases of orthogonality for our setting:

**Theorem 3.1.** *Let  $\sigma \neq 0$  and the cycle  $C$  with  $k \neq 0$ . Then:*

1)  $C$  is  $\check{\sigma}$ -orthogonal to its gradient deformation  $\tilde{C}$  if and only if:

$$l^2 - km + (\sigma - 2\check{\sigma})n^2 = 0. \quad (3.4)$$

2)  $C$  is  $\check{\sigma}$ -orthogonal to its mixed-gradient deformation  $C^m$  if and only if:

$$(1 - \sigma\check{\sigma})nl = 0. \quad (3.5)$$

3)  $\tilde{C}$  is  $\check{\sigma}$ -orthogonal to  $C^m$  if and only if (3.4) holds.

4) Suppose also  $m \neq 0$ . Then  $C$  is  $\check{\sigma}$ -orthogonal to  $J(C)$  if and only if:

$$2(l^2 + \check{\sigma}n^2) - k^2 - m^2 = 0. \quad (3.6)$$

*Proof.* 1) A straightforward computation gives:

$$\langle C_{\check{\sigma}}^s, \tilde{C}_{\check{\sigma}}^s \rangle = l^2 - km + (\sigma - 2\check{\sigma})n^2. \quad (3.7)$$

2) The matrix of  $C^m$  from (2.18) is:

$$C_{\check{\sigma}}^{m,s} = \frac{1}{2} \begin{pmatrix} n + \check{\sigma}sl & 0 \\ 0 & -n + \check{\sigma}sl \end{pmatrix} \quad (3.8)$$

and then:

$$\langle C_{\check{\sigma}}^s, C_{\check{\sigma}}^{m,s} \rangle = (1 - \sigma\check{\sigma})nl. \quad (3.9)$$

3) The same computation as above.

4) The matrix of  $J(C)$  is:

$$J(C)_{\check{\sigma}}^s := \begin{pmatrix} l - \check{\sigma}sn & -k \\ m & -l - \check{\sigma}sn \end{pmatrix} \quad (3.10)$$

and:

$$\langle C_{\check{\sigma}}^s, J(C)_{\check{\sigma}}^s \rangle = 2(l^2 + \check{\sigma}n^2) - m^2 - k^2. \quad (3.11)$$

which gives the conclusion.  $\square$

**Example 3.2.** Suppose  $\sigma = \check{\sigma}$ . Then  $1 - \sigma\check{\sigma} = 0$  since  $\sigma^2 = 1$  and then  $C^m$  is both orthogonally on  $C$  and  $\tilde{C}$ . In this case  $C$  is orthogonally to  $\tilde{C}$  if and only if  $l^2 - km - \check{\sigma}n^2 = 0$  but from the first equation (3.2) this means that  $\|C\| = 0$  i.e.  $C$  is also self-orthogonal.  $\square$

Returning to the Möbius-type study of cycles we continue this section considering some transformation of cycles. The first one is inspired by [1, p. 2706]. Let  $\alpha \in A(\sigma)$  with module  $|\alpha| \neq 1$  and consider the map  $T_\alpha : A(\sigma) \rightarrow A(\sigma)$ :

$$T_\alpha(z) = z + \alpha\bar{z} := w. \quad (3.12)$$

It follows directly that  $T_\alpha$  is a bijective map with the inverse:

$$z := T_\alpha^{-1}(w) = \frac{1}{1 - |\alpha|^2}(w - \alpha\bar{w}). \quad (3.13)$$

Replacing this expression of  $z$  in (2.22) we find the image of cycle  $C$  through  $T_\alpha$ :

$$T_\alpha(C) : k|w - \alpha\bar{w}|^2 + (1 - |\alpha|^2)[(B - \bar{\alpha}\bar{B})w + (\bar{B} - \alpha B)\bar{w} + (1 - |\alpha|^2)m] = 0 \quad (3.14)$$

but this curve is not a cycle for  $\alpha \cdot k \neq 0$ .

The second transformation is a Blaschke factor  $B_a$  defined by  $a \in A(\sigma)$  with module  $|a| < 1$ :

$$w := B_a(z) = \frac{z - a}{1 - \bar{a}z}, \quad (3.15)$$

having the inverse:

$$z = B_{-a}(w) = \frac{w + a}{1 + \bar{a}w}. \quad (3.16)$$

The Blaschke transformation of the cycle (2.22) is again a cycle:

$$B_a(C) : b_a(k)w\bar{w} + b_a(B)w + \overline{b_a(B)}\bar{w} + b_a(m) = 0 \quad (3.17)$$

with:

$$\begin{cases} b_a(k) = k + m|a|^2 + 2\Re(Ba), \\ b_a(B) = (k + m)\bar{a} + B + \bar{B}\bar{a}^2, \\ b_a(m) = m + k|a|^2 + 2\Re(Ba). \end{cases} \quad (3.18)$$

**Example 3.3.** Suppose that  $|B| < 1$  and let  $a = \bar{B}$ . Then the Blaschke transformation of the coefficients is:

$$\begin{cases} b_{\bar{B}}(k) = k + (m + 2)|B|^2, \\ b_{\bar{B}}(B) = (k + m + 1 + |B|^2)B, \\ b_{\bar{B}}(m) = m + (k + 2)|B|^2. \end{cases} \quad (3.19)$$

□

The last transformation is a similarity defined by  $a, b \in A(\sigma)$  with  $a \neq 0$ :

$$w := S_{a,b}(z) = az + b, \quad (3.20)$$

having the inverse:

$$z = \frac{1}{a}(w - b) = S_{\frac{1}{a}, \frac{-b}{a}}(w). \quad (3.21)$$

The similarity transformation of the cycle (2.22) is again a cycle:

$$S_{a,b}(C) : kw\bar{w} + (B\bar{a} - k\bar{b})w + (\bar{B}a - kb)\bar{w} + m|a|^2 + k|b|^2 - 2\Re(Bb\bar{a}) = 0. \quad (3.22)$$

If the initial cycle  $C$  is non-degenerate then we restrict to the case  $k = 1$  due to the projective character of the coefficients of  $C$ . Then a non-degenerate  $C$  is called *decomposable* if it is a product of lines:

$$C : (z - B)(\bar{z} - \bar{B}) = 0 \quad (3.23)$$

which means that  $m = |B|^2 = l^2 - \sigma n^2$ . A similarity preserves the class of decomposable cycles since its image is:

$$S_{a,b}(C) : (w - b + a\bar{B})(\bar{w} - \bar{b} + \bar{a}B). \quad (3.24)$$

From (3.3) it follows that a decomposable cycle has:

$$\det C_{\check{\sigma}}^s = (\check{\sigma} - \sigma)n^2. \quad (3.25)$$

#### 4. The rotation of a cycle

In this section we suppose that  $\sigma \neq 0$ . In  $A(\sigma)$  we introduce the rotation map  $R : (u, v) \rightarrow i \cdot (u, v) = (\sigma v, u)$ ; then its square is:  $R^2 = \sigma I$ . It follows that a given cycle  $C$  has an associated rotation cycle  $R(C)$  with equation:

$$R(C) : k(\sigma^2 v^2 - \sigma u^2) - 2l\sigma v - 2nu + m = 0. \quad (4.1)$$

A short computation gives a more simple form:

$$R(C) : k(u^2 - \sigma v^2) + 2(\sigma n)u + 2lv - \sigma m = 0 \quad (4.2)$$

and then we have the deformation:

$$C = (k, l, n, m) \rightarrow R(C) = (k, -\sigma n, -l, -\sigma m). \quad (4.3)$$

The general rotation of conics is treated in [4].

**Remark 4.1.** Concerning the compositions  $J \circ R$  and  $R \circ J$  we have:

$$J \circ R(C) = (-\sigma m, -\sigma n, l, k), \quad R \circ J(C) = (m, \sigma n, -l, -\sigma k) \quad (4.4)$$

and then  $J$  and  $R$  anti-commutes in the hyperbolic setting respectively  $J$  and  $R$  commutes if and only if  $l = 0$  in the complex setting:  $\sigma = -1$ .  $\square$

In terms of associated matrix we have:

$$R(C)_{\check{\sigma}}^s = \begin{pmatrix} -\sigma n - \check{i}sl & \sigma m \\ k & \sigma n - \check{i}sl \end{pmatrix}, \quad \|R(C)_{\check{\sigma}}^s\|^2 = 2(n^2 + \check{\sigma}l^2 + \sigma km). \quad (4.5)$$

Then  $R$  preserves the norm of  $C$  if and only if:

$$(\sigma + 1)km + (\check{\sigma} - 1)l^2 + (1 - \check{\sigma})n^2 = 0. \quad (4.6)$$

Also, recall from section 2 that the  $e/h$ -center of  $C$  is  $M(\frac{l}{k}, -\sigma\frac{n}{k})$  and hence its rotation is  $R(M) = (-\frac{n}{k}, \frac{l}{k})$ . But the center of  $R(C)$  is  $\bar{M} = (-\frac{\sigma n}{k}, \frac{\sigma l}{k})$  and then  $\bar{M} = \sigma R(M)$ ; these points coincide for  $\sigma = 1$ .

Concerning the orthogonality of this new cycle with the previous three cycles we have:



**Proposition 4.2.** *Let  $C$  be a cycle with  $k \neq 0$ . Then:*

*i)  $C$  is  $\check{\sigma}$ -orthogonal to its rotated cycle  $R(C)$  if and only if:*

$$(\check{\sigma} - \sigma)nl + (\sigma - 1)km = 0. \tag{4.7}$$

*ii)  $\tilde{C}$  is  $\check{\sigma}$ -orthogonal to  $R(C)$  if and only if:*

$$2(\check{\sigma} - \sigma)nl + \sigma(n^2 + km) - l^2 = 0. \tag{4.8}$$

*iii)  $C^m$  is  $\check{\sigma}$ -orthogonal to  $R(C)$  if and only if:*

$$\check{\sigma}l^2 = n^2. \tag{4.9}$$

*Proof.* A straightforward computation gives:

$$\langle C_{\check{\sigma}}^s, R(C)_{\check{\sigma}}^s \rangle = 2[(\check{\sigma} - \sigma)nl + (\sigma - 1)km], \tag{4.10}$$

$$\langle \tilde{C}_{\check{\sigma}}^s, R(C)_{\check{\sigma}}^s \rangle = 2(\check{\sigma} - \sigma)nl + \sigma(n^2 + km) - l^2, \tag{4.11}$$

$$\langle C_{\check{\sigma}}^{m,s}, R(C)_{\check{\sigma}}^s \rangle = 2\sigma(\check{\sigma}l^2 - n^2) \tag{4.12}$$

which yields the conclusion.  $\square$

**Example 4.3.** Suppose that  $\sigma = \check{\sigma} = 1$ . Then  $R(C)$  is orthogonal to  $C$  and:

a) is orthogonal to  $\tilde{C}$  if and only if:  $l^2 = n^2 + km$ ; for  $k = 1$  this means that  $C$  is decomposable,

b) is orthogonal to  $C^m$  if and only if:  $l_{\pm} = \pm n$ , which is exactly the relation (2.21).  $\square$

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