The diagonalization map as submersion, the cubic equation as immersion and Euclidean polynomials

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Abstract. In the first part of this note a restriction of the diagonalization map from $Sym(2)$ to $\mathbb{R}^2$ is studied as Riemannian submersion by using the Hermitian parameters. A strong relationship with the Hopf bundle is pointed out and the symmetric matrices with determinant $-\frac{1}{2}$ are obtained as an extremal case. The Hopf invariant is computed for some classes of examples. In the second part we prove that the solution map of the depressed cubic equation with strictly negative discriminant is an immersion. In third part we define a class of polynomials, called by us Euclidean due to the isometrical character of the solution map.

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1. Introduction

It is well-known the huge role played by submersions and immersions (as maps of maximal rank) in the geometry of manifolds. Then, the appearance of these main differential tools in various settings is a proof of the richness of some theories, especially if these maps are combined with other structures. The aim of this note is to study two maps of algebraic nature in terms of submersions and immersions.

More precisely, this paper consists of three parts. In the first one, the diagonalization map on the real linear space $Sym(2) \simeq \mathbb{R}^3$ is obtained as a Riemannian submersion with fibers as circles. For the general theory of such Riemannian submersion the book [15] is an excellent source while the Finslerian case is discussed in [6]. In order to prove this result we do not use directly the elements of a given symmetric matrix Γ but its Hermitian parameters. This approach is motivated by the simplification of some computations; but in order to compare the calculus difficulties we include also a proof using the elements of Γ. The given pair of Hermitian coefficients of Γ is interpreted as the sides of a triangle and then its area and the angle through the origin are computed. Four classes of examples are considered as well as the fixed points of the linear rational transformation interpreted as $\Gamma \in Sym(2)$. The first example concerns with a minimal value of the determinant of Γ given by the attempt to connect our setting with the classical Hopf S^1-bundle. We note that the diagonalization map for the general case of $Sym(n)$ is proved in [20] to be a submersion (called the eigenvalue map) without our restriction on distinct eigenvalues but also without concrete examples.

In the second part we prove that the solution map of the depressed cubic equation with strictly negative discriminant is an immersion. We note that several computations of this work are made by hand but some of the complicated computations are obtained by using WolframAplha. We compute the Gaussian curvature of the inverse metric of this immersion. We finish this section with a map which we call CubicHopf since it is a composition of three functions, one being the Hopf map, the second being the stereographic projection of the 2-sphere $S^2\left(\frac{1}{2}\right)$ while the last being the inverse of the cubic map.

The last section concerns with a class of degree $n \geq 2$ polynomials inspired by the approach of the third section. Namely, we start with a real polynomial having all real roots and ask for its solution map to be an Euclidean isometry. Due to this property we call *Euclidean* these polynomials. Since this class is characterized through the sphere S^{n-2} we discuss the first four n's. We finish this work with the isometry with respect to the L_4-norm but discussed only for the quadratic and cubic equation, for which also the palindromic possibilities are analyzed.

2. The digonalization on $Sym(2)$ as Riemannian submersion

Fix $M_2(\mathbb{R})$ the four-dimensional real linear space of all real 2×2 matrices and its subset $Diag(2, \mathbb{R})$ of diagonalizable matrices. It is well-known that $Diag(2, \mathbb{R})$ contains the three-dimensional subspace $Sym(2)$ of the symmetric matrices

$$\Gamma = \Gamma(a, b, c) := \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a, b, c \in \mathbb{R}. \tag{2.1}$$

Hence, there exists $R \in O(2)$ and $\lambda_1 = \lambda_1(\Gamma) \leq \lambda_2 = \lambda_2(\Gamma) \in \mathbb{R}$ such that

$$\Gamma_d := diag(\lambda_1, \lambda_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = R\Gamma R^{-1} \tag{2.2}$$

and then we have the nonlinear map

$$Diag : Sym(2) = \mathbb{R}^3 \to \mathbb{R}^2, \quad (a, b, c) \to (\lambda_1, \lambda_2). \tag{2.3}$$

The aim of this section is to prove that a natural restriction

$$Diag^r : Sym(2) \setminus \{\mathbb{R} \cdot I_2\} = \mathbb{C}^* \times \mathbb{R} \to \mathbb{R}^2_< := \{(\lambda_1, \lambda_2) \in \mathbb{R}^2; \lambda_1 < \lambda_2\} \quad (2.4)$$

is a Riemannian submersion if the Euclidean metric of the source space is scaled by a factor equal to 2; here "r" means "restricted". For this goal we use the approach of the papers [8]-[9] where the matrix $\Gamma \in Sym(2)$ is uniquely fixed not by a, b, c but by its *Hermitian parameters* (A, B)

$$2B := Tr\Gamma = a + c = \lambda_1 + \lambda_2 \in \mathbb{R}, \quad A := \frac{a-c}{2} - bi \in \mathbb{C}.$$

In the book [17, p. 56] the complex number A is denoted by $L(\Gamma)$ and is called *the Hopf invariant of* Γ. The initial parameters of the matrix Γ can be recovered through the relations

$$a = B + Re(A), \quad c = B - Re(A), \quad b = -Im(A) \qquad (2.5)$$

and the applications of this approach in the differential geometry of 2-dimensional metrics are presented in [10]. Since Γ is a symmetric matrix its Gersgorin circles reduce to intervals

$$\begin{cases} I_1 = [a - |b|, a + |b|] = [B + Re(A) - |Im(A)|, B + Re(a) + |Im(A)|], \\ I_2 = [c - |b|, c + |b|] = [B - Re(A) - |Im(A)|, B - Re(a) + |Im(A)|]. \end{cases}$$

It follows that our main hypothesis is that $A \in \mathbb{C} \setminus \{0\}$ which means that denoting also $A := x + yi$ we have $|A|^2 = x^2 + y^2 > 0$. The determinant of Γ is

$$\delta := \det \Gamma = ac - b^2 = \lambda_1 \lambda_2 = B^2 - |A|^2 < B^2 \qquad (2.6)$$

and hence the considered map is

$$\begin{cases} Diag^r : (A = x + yi, B) \in \mathbb{C}^* \times \mathbb{R} \to (\lambda_1 = B - |A| < \lambda_2 = B + |A|) \in \mathbb{R}^2_<, \\ \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |A| \\ B \end{pmatrix}, \Gamma_{diag} := \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \in Sym(2) \cap sl(2, \mathbb{R}). \end{cases} \qquad (2.7)$$

This smooth map has the differential in an arbitrary point (A, B) not depending on B i.e. independent of the trace of Γ

$$d(Diag^r)(A, B) = \begin{pmatrix} -\frac{x}{|A|} & -\frac{y}{|A|} & 1 \\ \frac{x}{|A|} & \frac{y}{|A|} & 1 \end{pmatrix} \qquad (2.8)$$

with

$$\Delta_1 = \begin{vmatrix} -\frac{x}{|A|} & 1 \\ \frac{x}{|A|} & 1 \end{vmatrix} = \frac{-2x}{|A|}, \quad \Delta_2 = \begin{vmatrix} -\frac{y}{|A|} & 1 \\ \frac{y}{|A|} & 1 \end{vmatrix} = \frac{-2y}{|A|}. \qquad (2.9)$$

From $\Delta_1^2 + \Delta_2^2 = 4 > 0$ it results that $Diag^r$ is indeed a submersion. In order to underline the usefulness of the Hermitian parameters we provide an alternative proof of:

Proposition 2.1. *$Diag^r$ is a proper surjective submersion.*

Proof. The characteristic polynomial of Γ

$$P_\Gamma(t) := \det(\Gamma - tI_2) = t^2 - (Tr\Gamma)t + \delta$$

has the discriminant $\Delta := (a-c)^2 + 4b^2 = 4|A|^2 > 0$ and the eigenvalues are

$$2\lambda_1 = a + c - \sqrt{\Delta} < 2\lambda_2 = a + c + \sqrt{\Delta}.$$

The differential of $Diag^r$ is

$$d(Diag^r)(A,B) = \frac{1}{2\sqrt{\Delta}}\begin{pmatrix} \sqrt{\Delta} - 2(a+c) & -2b & \sqrt{\Delta} + 2(a+c) \\ \sqrt{\Delta} + 2(a+c) & 2b & \sqrt{\Delta} - 2(a+c) \end{pmatrix}.$$

Let Δ_1 be the determinant corresponding to the columns C_1 and C_3 and Δ_2 that corresponding to the columns C_1 and C_2. Since

$$\Delta_1 = \frac{-2(a-c)}{\sqrt{\Delta}}, \quad \Delta_2 = \frac{b}{\sqrt{\Delta}}, \quad \Delta_1^2 + \Delta_2^2 = \frac{4(a-c)^2 + b^2}{\Delta} > 0$$

we have the first part of the conclusion. The properness of the map $Diag^r$ follows from the fact that it preserves the Euclidean norms of the source and the target linear spaces

$$\|\Gamma\|_e^2 = a^2 + 2b^2 + c^2 = \lambda_1^2 + \lambda_2^2 = 2(B^2 + |A|^2) > 2B^2 \tag{2.10}$$

which means that $Diag^r$ is also an isometry.

We point out that the fibre over an arbitrary $(\lambda_1, \lambda_2) \in \mathbb{R}^2_<$ is the nondegenerate circle

$$(Diag^r)^{-1}(\lambda_1, \lambda_2) = \mathcal{C}_{xOy}\left(O, \frac{\lambda_2 - \lambda_1}{2}\right) \times \left\{\frac{\lambda_1 + \lambda_2}{2}\right\} \tag{2.11}$$

where $\mathcal{C}_{xOy}(O, r)$ denotes the circle centered in origin of the plane $xOy \subset \mathbb{R}^3$ and having the radius $r > 0$. Concerning the property of being proper for the map $Diag^r$ we note that the inequality $0 < \|\Gamma\|_e^2 = \lambda_1^2 + \lambda_2^2 \leq K$ implies that the radius of the above circle is $r \leq \sqrt{2K}$. $\square$

Remark 2.2. We can provide a geometrical meaning of the sum $B^2 + |A|^2$ from (2.10) for the case $B \neq 0$. Namely, let us consider the triangle with vertices $O((0,0) \in \mathbb{R}^2)$, $M_A(A \in \mathbb{R}^2)$, $M_B((B,0) \in \mathbb{R}^2)$; let us call it *the Hermitian triangle of the matrix* Γ. With ψ_O the angle in O of this triangle we have

$$M_A M_B^2 = OM_A^2 + OM_B^2 - 2OM_A \cdot OM_B \cdot \cos\psi_O$$

which means

$$\cos\psi_O = \frac{|A|^2 + B^2 - \|(c,-b)\|_e^2}{2B|A|} = \frac{\|\Gamma\|_e^2 - 2(b^2 + c^2)}{4B|A|} =$$

$$= \frac{a^2 - c^2}{(a+c)\sqrt{(a-c)^2 + 4b^2}} = \frac{a-c}{\sqrt{(a-c)^2 + 4b^2}} \in [-1, 1].$$

Equivalently, $\cot\psi_O = \frac{|a-c|}{2|b|}$ and hence, modulo its sign, ψ_O is the argument of the complex number $A = A(\Gamma)$. In the first Example below we see a case when ψ_0 is the minus argument of $A \in \mathbb{C}$. The area of the Hermitian triangle is

$$\mathcal{A} = \frac{1}{4}\sqrt{(b^2 + c^2 - \lambda_1^2)(\lambda_2^2 - b^2 - c^2)}.$$

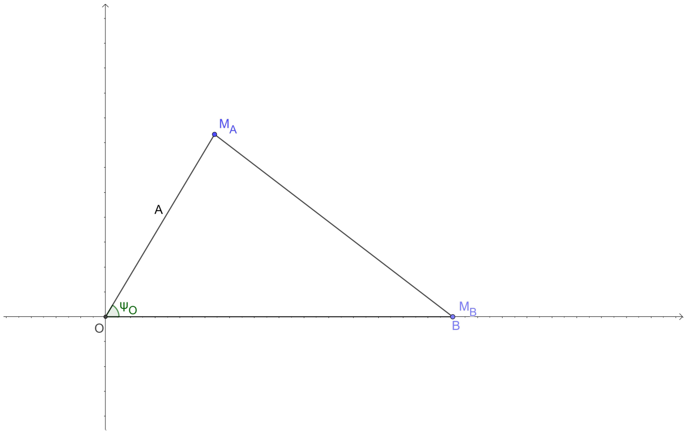

FIGURE 1. The Hermitian triangle of a symmetric matrix with $|A| \cdot B \neq 0$

If $c = 0$ then this triangle is isosceles in M_B. This triangle is an isosceles one with $OM_A = OM_B$ if and only if the first eigenvalue $\lambda_1 = 0$; equivalently $b^2 = ac$ and hence $B = |A| = \frac{\lambda_2}{2} > 0 = \delta$. In particular, the Hermitian triangle is an equilateral one if and only if $b^2 = ac$ and $3c = a$ and the corresponding matrix is

$$\begin{cases} \Gamma_{equilateral} = \Gamma(a > 0) := a \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, a \in \mathbb{R}, \\ A(a) = 2ae^{\frac{\pi i}{3}}, \quad 2B(a) = 4a = \lambda_2. \end{cases}$$

The diagonalizing matrix is

$$R = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{5\pi}{3} & -\sin\frac{5\pi}{3} \\ \sin\frac{5\pi}{3} & \cos\frac{5\pi}{3} \end{pmatrix}, \quad R^{-1} = R^t.$$

We finish this remark with the important aspect that the differential of $Diag^r$ is an *angular matrix* i.e. depends only on the angle ψ_O

$$d(Diag^r)(A, B) = \begin{pmatrix} -\cos\psi_O & -\sin\psi_O & 1 \\ \cos\psi_0 & \sin\psi_O & 1 \end{pmatrix}.$$

□

We arrive now at the main result of this section:

Theorem 2.3. *The map* $Diag^r : M := (Sym(2) \setminus \{\mathbb{R} \cdot I_2\} = \mathbb{C}^* \times \mathbb{R}, 2ds_3^2 = 2(dx^2 + dy^2 + dB^2)) \to N := (\mathbb{R}^2_<, ds_2^2 = d\lambda_1^2 + d\lambda_2^2)$ *is a Riemannian submersion with totally geodesic horizontal distribution. If* $\mathcal{H} \subset \mathfrak{X}(M)$ *is its horizontal distribution and* $\mathcal{H}^0 \subset \Omega^1(M)$ *is the associated annihilator then*

$\mathcal{H}\oplus\mathcal{H}^0$ is a Dirac structure on the big tangent bundle $T^{big}M := TM\oplus T^*M$ of M.

Proof. Let (u, v, w) be the coordinates in the tangent space $T_{(A,B)}\mathbb{R}^3$. The vertical subspace $V_{(A,B)}\mathbb{R}^3$ of $Diag^r$ is the kernel of the differential $d(Diag^r)(A, B)$ from (2.8)

$$d(Diag^r)(A,B)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\frac{x}{|A|} & -\frac{y}{|A|} & 1 \\ \frac{x}{|A|} & \frac{y}{|A|} & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.12)$$

We deduce that the vertical subspace is 1-dimensional

$$V_{(A,B)}\mathbb{R}^3 = \{\mathbb{R}\cdot A^{\perp}\}\times\{0\} = \{\mathbb{R}\cdot iA\}\times\{0\} = \{(-\alpha y, \alpha x, 0)\in\mathbb{R}^3; \alpha\in\mathbb{R}\}. \quad (2.13)$$

Then the horizontal subspace of $Diag^r$ is 2-dimensional

$$H_{(A,B)}\mathbb{R}^3 = \{\mathbb{R}\cdot A\}\times\mathbb{R} = \{(\alpha x, \alpha y, \beta)\in\mathbb{R}^3; \alpha, \beta\in\mathbb{R}\}. \quad (2.14)$$

The claimed conclusion follows from

$$\begin{cases} d(Diag^r)(A,B)\begin{pmatrix} \alpha x \\ \alpha y \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{x}{|A|} & -\frac{y}{|A|} & 1 \\ \frac{x}{|A|} & \frac{y}{|A|} & 1 \end{pmatrix}\begin{pmatrix} \alpha x \\ \alpha y \\ \beta \end{pmatrix} = \begin{pmatrix} \beta-\alpha|A| \\ \beta+\alpha|A| \end{pmatrix}, \\ (\beta-\alpha|A|)^2 + (\beta+\alpha|A|)^2 = 2(\alpha^2|A|^2+\beta^2) \geq 2\beta^2. \end{cases} \quad (2.15)$$

Let us point out that (A, B) is a horizontal tangent vector for $Diag^r$ and that

$$(\lambda_1, \lambda_2) = d(Diag^r)(A, B)(A, B). \quad (2.16)$$

Also, we note that the orthogonal projection $P_{(A,B)} : \mathbb{R}^3 \to span\{(A, B)\} = \{\rho(x, y, B)\in\mathbb{R}^3; \rho\in\mathbb{R}\}$ has the expression

$$P_{(A,B)}(u, v, w) = \frac{<(u,v,w),(x,y,B)>}{\|(A,B)\|_e^2}(A,B) = \frac{2(ux+vy+wB)}{\|\Gamma\|_e^2}(A,B)$$

with $<\cdot,\cdot>$ the Euclidean inner product.

In terms of vector fields we consider the well-known two-dimensional vector fields, namely the rotational (or angular) and the Euler (or radial) vector field respectively

$$\Gamma_{rot} := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad Euler_2 := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = A\frac{\partial}{\partial A} + \bar{A}\frac{\partial}{\partial\bar{A}}$$

where for the second expression of $Euler_2$ we use the complex partial derivatives. Then, we describe the vertical and horizontal distributions of $Diag^r$ as

$$\mathcal{V} := \Gamma(V) = span\{\Gamma_{rot}\}, \quad \mathcal{H} := \Gamma(H) = span\{Euler_2, \frac{\partial}{\partial z} = \frac{\partial}{\partial B}\}$$

and then both $\mathcal{V}$ and $\mathcal{H}$ are involutive distributions; also Γ_{rot} and $Euler_2$ commutes. If ∇^3 and R^3 are respectively the Levi-Civita connection and the curvature tensor field of $2ds_3^2$ then

$$\begin{cases} \nabla^3_{Euler_2}\Gamma_{rot} = \nabla^3_{\Gamma_{rot}} Euler_2 = \Gamma_{rot}, \\ \nabla^3_{\Gamma_{rot}}\Gamma_{rot} = -Euler_2 = -\nabla^3_{Euler_2} Euler_2, \\ 2ds_3^3(\nabla_{\Gamma_{rot}}\Gamma_{rot}, Euler_2) = -2(x^2+y^2) < 0, \\ R^3(\Gamma_{rot}, Euler_2)\Gamma_{rot} = -2Euler_2, \\ R^3(\Gamma_{rot}, Euler_2)Euler_2 = R^3(\Gamma_{rot}, Euler_2)\frac{\partial}{\partial z} = 0 \\ R^3(\Gamma_{rot}, \frac{\partial}{\partial z})\Gamma_{rot} = R^3(\Gamma_{rot}, \frac{\partial}{\partial z})\frac{\partial}{\partial z} = R^3(\Gamma_{rot}, \frac{\partial}{\partial z})Euler_2 = 0, \end{cases}$$

and then only $\mathcal{H}$ is a totally geodesic distribution while the mean curvature vector field of $\mathcal{V}$ is

$$\vec{H}_{\mathcal{V}} = -2(x^2+y^2)Euler_2.$$

We can decompose any vector field X on the source manifold as

$$X = X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial y} + X^3\frac{\partial}{\partial z} = X_{planar} + X^3\frac{\partial}{\partial z}$$

and hence its vertical and horizontal components are

$$X^V(A,B) = \frac{1}{x^2+y^2}(-yX^1 + xX^2)\Gamma_{rot} = \frac{1}{|A|^2} < iA, X_{planar} >_{\mathbb{R}^2} \Gamma_{rot},$$

$$X^H(A,B) = \frac{1}{x^2+y^2}(xX^1 + yX^2)Euler_2 + X^3\frac{\partial}{\partial z} =$$

$$= \frac{1}{|A|^2} < A, X_{planar} >_{\mathbb{R}^2} Euler_2 + X^3\frac{\partial}{\partial z}.$$

For the claimed Dirac structure we apply the Theorem 2 of the paper [7] and the annihilator $\mathcal{H}^0 \subset \Omega^1(M)$ is $span\{\omega = -ydx + xdy\}$. □

We observe now that this theorem agrees with the radii and the dimensions of vertical and horizontal distributions involved in the most known Riemannian submersion, namely the Hopf bundle $Hopf : S^3 \subset \mathbb{C}^2 \to S^2(\frac{1}{2}) \subset \mathbb{R}\times\mathbb{C}$

$$Hopf(z,w) = \left(\frac{1}{2}(|z|^2 - |w|^2), z\bar{w}\right). \tag{2.17}$$

When the general Hopf bundle is written as $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$ then the Riemannian metric on the complex projective space is induced by the negative of the Killing form of $SU(n+1)$. We can consider the subset $Sym(2,1) = \{\Gamma \in Sym(2); \|\Gamma\|_e = 1\} \subset S^3$ and hence we have the restriction of the Hopf map

$$Hopf|_{Sym(2,1)}(z = A, w = B \in \mathbb{R}) = \left(-\frac{\delta}{2}, BA\right) \in S^2\left(\frac{1}{2}\right). \tag{2.18}$$

The condition $\|\Gamma\|_e^2 = 1$ means the existence of $t = t(\Gamma) \in \mathbb{R}$ such that $\{\lambda_1, \lambda_2\} = \{\cos t, \sin t\}$ and the determinant can be expressed only in terms of B: $\delta = 2B^2 - \frac{1}{2} \geq -\frac{1}{2}$. In the Euclidean space $\mathbb{R}^3$ of coordinates (a,b,c) the equation

$$ac - b^2 = -\frac{1}{2}$$

represents an one-sheeted hyperboloid.

Example. The minimal value $-\frac{1}{2}$ of δ is characterized by the 1-parameter class of examples

$$\Gamma = \Gamma(\varphi) := \frac{1}{\sqrt{2}} S_\varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \in sl(2, \mathbb{R}), \quad \varphi \in \mathbb{R} \quad (2.19)$$

with the diagonalization matrix and the eigenvalues (these are universal for this class of examples i.e does not depend on $\varphi \in \mathbb{R}$)

$$\begin{cases} R = R^{-1} = \begin{pmatrix} -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \\ \cos\frac{\varphi}{2} & \sin\frac{\varphi}{2} \end{pmatrix} = S_{\frac{\pi}{2}+\frac{\varphi}{2}} \in O^-(2) := O(2) \setminus SO(2), \\ \lambda_1 = -\frac{1}{\sqrt{2}} = \cos\frac{3\pi}{4}, \quad \lambda_2 = +\frac{1}{\sqrt{2}} = \sin\frac{3\pi}{4}. \end{cases} \quad (2.20)$$

The diagonal expression of $\Gamma(\varphi)$ is $-\sigma_3$ with

$$\{\sigma_0 = I_2, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = diag(1, -1)\}$$

the Pauli basis of the four-dimensional real linear space $H(2)$ of 2×2 Hermitian matrices ([11]); hence $Sym(2) = span\{\sigma_0, \sigma_1, \sigma_3\}$ and Γ_{diag} from (2.7) is $\Gamma_{diag} = \sigma_1 - \sigma_4$ while $\Gamma_{equilateral} = a(2\sigma_0 - \sqrt{3}\sigma_1 + \sigma_3)$. The Hopf invariant of $\Gamma(\varphi)$ is $A(\varphi) = \frac{1}{\sqrt{2}} e^{-i\varphi}$ and $Hopf(\Gamma(\varphi)) = (\frac{1}{4}, 0, 0) \in S^2(\frac{1}{2})$ for all φ. The circle (2.11) has the radius $r = \frac{\lambda_2 - \lambda_1}{2} = \frac{1}{\sqrt{2}}$; a circle with this radius appears in the paper [3] as containing the zeros of certain weakly holomorphic modular forms. For $\varphi \neq \frac{2k+1}{2}\pi$ with $k \in \mathbb{Z}$, the angle ψ_O of the Remark 2.2 is exactly φ even if $B = 0$.

If in formula (2.19) we put $r > 0$ instead of $\frac{1}{\sqrt{2}}$ then we obtain the symmetric matrix $\Gamma = \Gamma(r, \varphi)$ with the Hopf invariant being the anti-holomorphic map

$$A : z = (r, \varphi) \in \mathbb{C}^* \to \bar{z} = (r, -\varphi) \in \mathbb{C}^*.$$

The expression of $\Gamma_{equilateral}$ in terms of Pauli basis gives the 4-dimensional vector $\bar{v}_{equilateral}(a) = a(2, -\sqrt{3}, 0, 1)$ which belongs to the sphere S^3 only for $a_\pm = \pm\frac{1}{2\sqrt{2}}$. We have the Hopf projection

$$Hopf\left(\frac{1}{2\sqrt{2}}(2, -\sqrt{3}, 0, 1)\right) = \frac{1}{8}(3, -\sqrt{3} - 2i) \in S^2\left(\frac{1}{2}\right).$$

The minimal polynomial of the complex number $\frac{1}{8}(-\sqrt{3} - 2i)$ is: $P(x) = 4096x^4 + 128x^2 + 49$. Also, it results the unit vector $\frac{1}{4}(3, -\sqrt{3}, -2) \in S^2$ which then can be written as

$$(\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha) \in S^2$$

for $\alpha = \beta = -\frac{\pi}{6}$. $\square$

To the general matrix Γ we associate the rational linear (or Möbius) transformation $f_\Gamma : \mathbb{R}P^1 = \mathbb{R} \cup \infty \to \mathbb{R}P^1$

$$f_\Gamma(t) = \frac{at + b}{bt + c} \quad (2.21)$$

which is an involution if $\Gamma \in sl(2,\mathbb{R})$; an interesting problem is the study of its fixed points. For $\Gamma(\varphi)$ from (2.19) the equation of the fixed points is

$$(\sin\varphi)t^2 - 2(\cos\varphi)t - \sin\varphi = 0 \tag{2.22}$$

with the solutions: $t_1 = -\tan\frac{\varphi}{2}$, $t_2 = \cot\frac{\varphi}{2}$. For the particular case $\varphi = \frac{3\pi}{4}$ of the Example above we have $t_1 = -\tan\frac{3\pi}{8} = -\cot\frac{\pi}{8} = -1-\sqrt{2}$, $t_2 = \cot\frac{3\pi}{8} = \tan\frac{\pi}{8} = -1+\sqrt{2}$; remark that the middle interval $[\frac{-1-\sqrt{2}}{2} = \frac{t_1}{2}, \frac{-1+\sqrt{2}}{2} = \frac{t_2}{2}]$ appears as eigenvalue-free interval for threshold graphs in [16]. For the equilateral matrix in the Remark 2.2 the corresponding fixed points are $t_1 = -\sqrt{3} < t_2 = \frac{1}{\sqrt{3}}$.

Example. Following [13] we consider the 2-parameters family of *bi-symmetric matrices*

$$\Gamma = \Gamma(a, b \neq 0) := \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \delta = a^2 - b^2, \quad \psi_O = \frac{\pi}{2} \tag{2.23}$$

characterized by $A(\Gamma) \in i\mathbb{R}^*$; for $a = B > 0$ the triangle ΔOM_AM_B is right in O. Its eigenvalues, the fibre radius and the area of the Hermitian triangle are, respectively

$$\lambda_1 = a - |b| < \lambda_2 = a + |b|, \quad r = |b| > 0, \quad \mathcal{A} = \frac{a|b|}{2}. \tag{2.24}$$

The fixed points of its associated f_Γ are universal: $t_\pm = \pm 1$. A first particular case is

$$\Gamma = \Gamma(\varphi) := \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix} \in SL(2,\mathbb{R}), \quad \varphi \in \mathbb{R} \tag{2.25}$$

and for a strictly positive φ it results the eigenvalues: $\lambda_1 = e^{-\varphi} < \lambda_2 = e^{\varphi}$ and the area of the Hermitian triangle $\mathcal{A} = \frac{\sinh 2\varphi}{4}$. Its Hermitian parameters are: $A = -i\sinh\varphi$, $B = \cosh\varphi$. We point out that the class of diagonal matrices $(e^{-\varphi}, e^{\varphi})$ are useful in the 2D dynamics of Teichmüller flows according to [21, p. 95].

A second particular case is provided by rank 2 symmetric hyperbolic Kac–Moody algebras $\mathcal{H}(a)$ with the Cartan matrices

$$C_b := \begin{pmatrix} 2 & -b \\ -b & 2 \end{pmatrix} = 2\sigma_0 - b\sigma_1, \quad A(C_b) = bi, \quad B(C_b) = 2 \tag{2.26}$$

with $b \geq 3$; these are studied in [18]. □

Example. We treat in the present setting the Fibonacci example of [13]. Fix the matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in Sym(2). \tag{2.27}$$

Its natural powers are expressed in terms of the Fibonacci sequence $\{F_n; n \in \mathbb{N}^*\}$

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \quad B_n = F_{n-1} + \frac{1}{2}F_n. \tag{2.28}$$

Its Hopf invariant:

$$A_n = F_n\left(\frac{1}{2} - i\right) \tag{2.29}$$

recalls the famous Riemann hypothesis. The eigenvalues of Q^n and hence the fibre radius are expressed in terms of the Golden mean $\phi = \frac{1+\sqrt{5}}{2}$ ([12]) as:

$$\lambda^1_n = (1-\phi)F_n + F_{n-1} < \lambda^2_n = \phi F_n + F_{n-1}, \quad r_n = (\phi - \frac{1}{2})F_n. \tag{2.30}$$

The fixed points of the associated f_{Q^n} are universal: $t_1 = 1-\phi < t_2 = \phi$. The trigonometrical functions of the angle ψ_O are: $\cos\psi_0 = \frac{1}{\sqrt{5}}$, $\sin\psi_0 = \frac{2}{\sqrt{5}}$. □

Example. In the book [14] at the page 10 to the case $\lambda_1 = B - |A| > 0$ is associated an ellipse $E(\Gamma)$ having the eccentricity

$$e = e(\Gamma) := \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \sqrt{\frac{2|A|}{B+|A|}}. \tag{2.31}$$

Hence, we call $\Gamma \in Sym(2)$ with strictly positive spectrum as being *self-complementary* it its associated ellipse is self-complementary i.e. $e = \frac{1}{\sqrt{2}}$, see [8] for a detailed study of these ellipses. It follows a class of examples

$$\Gamma = \Gamma(a > 0) := a \cdot diag(1,2) = a\left[\frac{3}{2}\sigma_0 - \frac{1}{2}\sigma_3\right], \quad r(a) = \frac{a}{2}. \tag{2.32}$$

More generally, given two matrices $\Gamma_1, \Gamma_2 \in Sym(2)$ both with strictly positive spectrum we call them as being *complementary* if the $e_1^2 + e_2^2 = 1$. □

3. The depressed cubic equation with strictly negative discriminant as immersion

In order to extend the results above to $Sym(3)$ is necessary to express the solutions of the cubic equations in terms of their coefficients similar to the expression of λ's from the first equation (2.7). Unfortunately, this approach seems extremely complicated due to the well-known *casus irreducibilis* from the theory of cubic equations. Fortunately, we can work with the depressed cubic equation but in the dual direction, namely as an immersion.

Fix the depressed cubic equation and suppose that its discriminant is strictly negative

$$y^3 + py + q = 0, \quad D = D(p,q) := \frac{p^3}{27} + \frac{q^2}{4} < 0 \tag{3.1}$$

in order to have three (and distinct) real solutions

$$y^1 = P+Q, y^2 = -\frac{1}{2}(P+Q) + \frac{i\sqrt{3}}{2}(P-Q), y^3 = -\frac{1}{2}(P+Q) - \frac{i\sqrt{3}}{2}(P-Q) \tag{3.2}$$

with the well-known formulae

$$P^3 := -\frac{q}{2} + \sqrt{D}, \quad Q^3 := -\frac{q}{2} - \sqrt{D}. \tag{3.3}$$

Hence the solution map is the function

$$\begin{cases} Cubic : \mathbb{R}^2(D<0) \to \mathbb{R}^3(distinct), \quad F(p,q) = (y^1, y^2, y^3) \\ \mathbb{R}^2(D<0) := \{(p,q) \in \mathbb{R}^2; D(p,q) < 0\}, \\ \mathbb{R}^3(distinct) := \{(x,y,z) \in \mathbb{R}^3; x \neq y \neq z \neq x\}. \end{cases} \tag{3.4}$$

Its differential is

$$dCubic(p,q) := \begin{pmatrix} y^1_p & y^1_q \\ y^2_p & y^2_q \\ y^3_p & y^3_q \end{pmatrix} \in M_{3,2}(\mathbb{R}) \tag{3.5}$$

where, as usually, the subscript denotes the partial derivative with respect to the corresponding variable. A long but straightforward computation gives:

$$\begin{cases} y^1_p = \frac{1}{6\sqrt{D}}(Q^2 - P^2), \quad y^2_p = \frac{1}{12\sqrt{D}}[(P^2 - Q^2) + i\sqrt{3}(P^2 + Q^2)], \\ y^3_p = \frac{1}{12\sqrt{D}}[(P^2 - Q^2) - i\sqrt{3}(P^2 + Q^2)], \\ y^1_q = \frac{3}{4p^2\sqrt{D}}[Q^2(q - 2\sqrt{D}) - P^2(q + 2\sqrt{D})], \\ y^2_q = -\frac{3}{8p^2\sqrt{D}}[Q^2(q - 2\sqrt{D}) - P^2(q + 2\sqrt{D})] + \\ +\frac{3i\sqrt{3}}{8p^2\sqrt{D}}[Q^2(q - 2\sqrt{D}) + P^2(q + 2\sqrt{D})], \\ y^3_q = -\frac{3}{8p^2\sqrt{D}}[Q^2(q - 2\sqrt{D}) - P^2(q + 2\sqrt{D})] - \\ -\frac{3i\sqrt{3}}{8p^2\sqrt{D}}[Q^2(q - 2\sqrt{D}) + P^2(q + 2\sqrt{D})]. \end{cases} \tag{3.6}$$

For the matrix (3.5) we consider the determinant defined by the first two lines

$$\Delta_1 = \begin{vmatrix} y^1_p & y^2_p \\ y^1_q & y^2_q \end{vmatrix} = y^1_p y^2_q - y^2_p y^1_q = \begin{vmatrix} y^1_p & y^2_p + \frac{y^1_p}{2} \\ y^1_q & y^2_q + \frac{y^1_q}{2} \end{vmatrix}$$

and again, a direct calculus yields the value

$$\Delta_1 = \frac{\sqrt{3}}{18\sqrt{-D}} \neq 0 \tag{3.7}$$

and we obtain the main result of this section:

Theorem 3.1. *The solution map Cubic of the depressed cubic equation with strictly negative discriminant is an immersion. With respect to the Euclidean norms of the source and target spaces, the map Cubic is an isometry if and only if the coefficients of the depressed cubic equation are*

$$p = -2\sin^2 t, \quad q = \sin(2t) = \sqrt{-p(p+2)}, \tag{3.8}$$

with the real parameter t in the domain provided by

$$\sin^2 t \in \left(\frac{3\sqrt{3 \cdot 59} - 27}{16}, 1 \right] \tag{3.9}$$

which means that $p \in \left[-2, \frac{27-3\sqrt{177}}{8} = -1.61405...\right)$ *and*

$$D = D(p) = \frac{p^3}{27} - \frac{p(p+2)}{4} < 0.$$

Proof. We must prove the second claim. The equality

$$y_1^2 + y_2^2 + y_3^2 = -2p = p^2 + q^2 \tag{3.10}$$

implies the existence of a real parameter t such that $p = \cos(2t) - 1$ and $q = \sin(2t)$. Replacing $p = p(t)$ and $q = q(t)$ in the expression (3.1) of the discriminant D we get the inequality:

$$27D(t) := \sin^2 t \left(27\cos^2 t - 8\sin^4 t\right) < 0. \tag{3.11}$$

Solving this quadratic equation in the unknown $\sin^2 t$ gives the claimed relation (3.9). □

Remark 3.2. The first fundamental form I of this immersion is

$$\begin{cases} g_{11} = E := \|y_p\|^2 = \frac{1}{6(-D)}(PQ)^2 = \frac{p^2}{54(-D)}, \\ g_{12} = F := <y_p, y_q> = \frac{3q}{4(-D)}\left(\frac{PQ}{p}\right)^2 = \frac{q}{12(-D)}, \\ g_{22} = G := \|y_q\|^2 = \frac{1}{2pD}(PQ)^2 = \frac{p}{18D}. \end{cases} \tag{3.12}$$

since $PQ = -\frac{p}{3}$. Here, $< \cdot, \cdot >$ and $\| \cdot \|$ are respectively the Euclidean inner product and the Euclidean norm of $\mathbb{R}^3$. The determinant of the first fundamental form is

$$\det I = EG - F^2 = -\frac{1}{36D} > 0$$

and then the inverse I^{-1} is a two-dimensional Riemannian metric with linear and quadratic components

$$I^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{pmatrix} = \begin{pmatrix} -2p & -3q \\ -3q & \frac{2}{3}p^2 \end{pmatrix}$$

and hence its Gaussian curvature is

$$K(I^{-1}) = \frac{-1}{108D^2}\left(\frac{p^3}{27} + \frac{p^2}{6} - \frac{q^2}{2}\right) = \frac{-1}{108D^2}\left(D + \frac{p^2}{6} - \frac{3q^2}{4}\right)$$

and then the depressed cubic equation with flat I^{-1} is characterized by

$$q^2 = \frac{2p^3}{27} + \frac{p^2}{3} \rightarrow D = D(p) = \frac{p^3}{18} + \frac{p^2}{12}$$

while the negativity of D implies the existence domain $(-\infty, -\frac{3}{2})$ of p. For the isometrical case (3.8) we derive:

$$K(I^{-1}) = \frac{-p}{108D^2}\left(\frac{p^2}{27} + \frac{2p}{3} + 1\right)$$

with the available zero $p = 3(\sqrt{6} - 3) = -1.651...$ as root of the quadratic equation $p^2 + 18p + 27 = 0$ belonging to intersection between the interval provided by the theorem 3.1 and the above interval. The corresponding (3.8) parameter is $t_{\pm} = \pm 65.33°$. □

If Δ_2 is the determinant built with the first and third line of the matrix (3.5) then $\Delta_2 = -\Delta_1$. Due to the general character of this framework we can consider only one proper example:

Example. Supposing that $p = -t^2 < 0$ and $q = 0$ since $(y_1, y_2, y_3) = (0, t, -t)$ we have the reduced solution map

$$Cubic^r : t \in (0, +\infty) \to (t, -t) \in \{(x > 0, y < 0) \in \mathbb{R}^2\} \tag{3.13}$$

which is obviously an open part of the second bisectrix of the plane $B_2 : y = -x$. The first fundamental form is $E(t) = \frac{1}{2t^2}$, $F(t) = 0$ and $G(t) = \frac{3}{2t^4}$ since $D = -\frac{t^6}{27}$, $P = -Q = \frac{i\frac{1}{3}t}{\sqrt{3}}$. The Gaussian curvature of I^{-1} is

$$K(I^{-1}) = \frac{2t^2 - 9}{8t^8} = \frac{-2p - 9}{8p^4}.$$

The minimal case of the Theorem 3.1 is $(p = -2, q = 0)$ and belongs to this example since it corresponds to $t = \sqrt{2}$. $\square$

Example. The semicubical parabola $C : 4p^3 + 27q^2 = 0$ has 7 lattice points given by Mathematica as

$$(p, q) \in \{(0, 0), (-3, \pm 2), (-12, \pm 16), (-27, \pm 54)\}.$$

In order to connect the contents of the first section with the present one we compute the discriminant D for $p = \lambda_1 = B - |A|$ and $q = \lambda_2 = B + |A|$. It results

$$D = D(|A|, B) := \frac{1}{27}(|A|^3 + B^3) + \frac{1}{4}(|A|^2 + B^2) + \frac{|A|B}{18}(9 + 2|A| - 2B). \tag{3.14}$$

Asking (λ_1, λ_2) be the above points we obtain three remarkable symmetric matrices given by their Hermitian parameters and determinant

$$\begin{cases} (|A_1| = r_1, B_1, \delta_1) = (\frac{5}{2}, -\frac{1}{2}, -6 = -2 \cdot 3), \\ (|A_2| = r_2, B_2, \delta_2) = (14, 2, -192 = -2^6 \cdot 3), \\ (|A_3| = r_3, B_3, \delta) = (\frac{81}{2}, \frac{27}{2}, -1458 = -2 \cdot 3^6). \end{cases} \tag{3.15}$$

$\square$

We can re-prove that the solution map *Cubic* of the depressed cubic equation is an immersion using the trigonometric approach of this equation in the case $p < 0$. Following [4, p. 18] if we introduce the angle θ provided by

$$\cos 3\theta := \frac{3\sqrt{3}q}{2p\sqrt{-p}} \tag{3.16}$$

then *Cubic* becomes:

$$Cubic^{angle}(p, \theta) = (x_1, x_2, x_3) := 2\sqrt{\frac{-p}{3}}\left(\cos\theta, \cos(\theta + \frac{2\pi}{3}), \cos(\theta + \frac{4\pi}{3})\right) \tag{3.17}$$

and hence, its differential is

$$dCubic^{angle}(p, \theta) = \begin{pmatrix} \frac{-1}{\sqrt{-3p}}\cos\theta & -2\sqrt{\frac{-p}{3}}\sin\theta \\ \frac{-1}{\sqrt{-3p}}\cos(\theta + \frac{2\pi}{3}) & -2\sqrt{\frac{-p}{3}}\sin(\theta + \frac{2\pi}{3}) \\ \frac{-1}{\sqrt{-3p}}\cos(\theta + \frac{4\pi}{3}) & -2\sqrt{\frac{-p}{3}}\sin(\theta + \frac{4\pi}{3}) \end{pmatrix}. \tag{3.18}$$

The determinant defined by the first two lines is

$$\Delta_1 = \frac{2}{3}\sin\frac{2\pi}{3} = \frac{1}{\sqrt{3}} > 0 \tag{3.19}$$

and the conclusion follows.

Example. For $t \in \left(0, \frac{\pi}{2}\right)$ the depressed cubic equation:

$$y^3 - 3y + 2\cos t = 0 \tag{3.20}$$

has $D = -\sin^2 t < 0$ and hence this last approach yields all three solutions

$$\theta = \frac{\pi - t}{3}, \quad y^k = 2\cos\frac{(2k+1)\pi - t}{3}, \quad k = 1, 2, 3 \tag{3.21}$$

as well as, in the first approach

$$P = e^{i\frac{\pi-t}{3}}, Q = e^{i\frac{5\pi+t}{3}}, \Delta_1 = \frac{\sqrt{3}}{18\sin t} > 0, K(I^{-1}) = \frac{3 - 4\sin^2 t}{216\sin^4 t} \tag{3.22}$$

with

$$I = \frac{1}{6\sin^2 t}\begin{pmatrix} 1 & \cos t \\ \cos t & 1 \end{pmatrix}. \tag{3.23}$$

□

Inspired by the search of critical points of height functions in submanifolds theory we ask for $3D$ unit vectors $\bar{v} = (v^1, v^2, v^3) \in S^2$ such that:

$$\bar{v} \cdot dCubic^{angle}(p, \theta) = \bar{0}_{\mathbb{R}^2} = (0, 0). \tag{3.24}$$

It follows a linear system not depending of p:

$$\begin{cases} v^1\cos\theta + v^2\cos(\theta + \frac{2\pi}{3}) + v^3\cos(\theta + \frac{4\pi}{3}) = 0, \\ v^1\sin\theta + v^2\sin(\theta + \frac{2\pi}{3}) + v^3\sin(\theta + \frac{4\pi}{3}) = 0. \end{cases} \tag{3.25}$$

Such a system will be discussed in the second example of the next section.

Let us finish this section by considering the stereographic projection $SP : S^2\left(\frac{1}{2}\right) \to \mathbb{R}^3$ and the open set $Cubic^{-1}(\mathbb{R}^2(D < 0)) \subset \mathbb{R}^3(distinct) \subset \mathbb{R}^3$. It follows other two open sets: $SP^{-1}(Cubic^{-1}(\mathbb{R}^2(D < 0))) \subset S^2\left(\frac{1}{2}\right)$ and $Hopf^{-1}(SP^{-1}(Cubic^{-1}(\mathbb{R}^2(D < 0)))) \subset S^3$. By denoting the last as $S^3(D < 0)$ we obtain a new function, which can be called *cubic Hopf map*

$$CubicHopf : S^3(D < 0) \to \mathbb{R}^2(D < 0), CubicHopf := Cubic^{-1} \circ SP \circ Hopf \tag{3.26}$$

corresponding to the chain: $S^3 \to S^2\left(\frac{1}{2}\right) \to \mathbb{R}^3 \to \mathbb{R}^2(D < 0)$. We hope to return to this map within a future paper.

4. Euclidean polynomials

The discussion of the previous section leads to a new notion:

Definition 4.1. Fix a monic polynomial $P \in \mathbb{R}_n[x]$ of degree $n \geq 2$:

$$P = x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n$$

having all roots $x_1, ..., x_n \in \mathbb{R}$ i.e. P is a hyperbolic polynomial; moreover P is strictly hyperbolic if all its roots are real and distinct. Then P is called *Euclidean* if its solution map is an Euclidean isometry:

$$x_1^2 + ... + x_n^2 = a_1^2 + ... + a_n^2. \tag{4.1}$$

Due to the first two Vieta's formulas we have a simple characterization:

Proposition 4.2. *P is an Euclidean polynomial if and only if we have the following unit vector which does not depend on a_1:*

$$(a_2 + 1, a_3, ..., a_n) \in S^{n-2} \tag{4.2}$$

and then the set of Euclidean hyperbolic polynomials is a subset of the product manifold $\mathbb{R} \times S^{n-2}$ of dimension $n-1$. If $n \geq 3$ then $a_2 < 0$.

Proof. The left hand side of (4.1) is $a_1^2 - 2a_2$ and then adding 1 to both sides it results

$$1 = (a_2 + 1)^2 + \sum_{i=3}^{n} a_i^2 \tag{4.3}$$

which means the first part conclusion. In the second part of conclusion the factor $\mathbb{R}$ is provided by a_1 and we remark that the condition (4.2) is invariant under the action of the group $\{1\} \times O(n-2)$ on $\mathbb{R}^{n-1}$. □

As in the first section we insists in examples.

Example. For $n = 2$ we have $a_2 \in \{0, -2\}$ since $S^0 = \{\pm 1\}$. Indeed, if $a_2 = 0$ then $x_1 = 0 = a_2$ and $x_2 = -a_1$ while if $a_2 = -2$ then

$$2x_{1,2} = -a_1 \pm \sqrt{a_1^2 + 8} \in \mathbb{R} \tag{4.4}$$

with: $4(x_1^2 + x_2^2) = 4a_1^2 + 16 = 4(a_1^2 + a_2^2)$. An important results in the theory of hyperbolic polynomials is that the pyramid Π_n of hyperbolic polynomials of a given degree n with $a_1 = 0$ and $|a_2| \leq 1$ satisfies the following Whitney condition: any two points of it can be joined inside Π_n by a path whose length exceeds the Euclidean distance between the ends of the path by not more than C times where C is independent of the given points. The subset of Euclidean elements of Π_2 reduces to a singleton containing only $P = x^2$ which is not strictly hyperbolic. □

Example. For $n = 3$ we have, as in Theorem 3.1, the existence of a real parameter t such that

$$a_2 = a_2(t) := -2\sin^2 t \leq 0, \quad a_3 = a_3(t) := \sin 2t. \tag{4.5}$$

A first example is provided by $t \in \mathbb{Z}\pi$ when $a_1 = a_3 = 0 = x_2 = x_3$ and $x_1 = -a_1$ so the relation (4.1) holds. A second example concerns with the palindromic cubic polynomials: $a_1 = a_2$ and $a_3 = 1$. The last condition yields

$a_1 = a_2 = -1$ and hence the unique palindromic Euclidean cubic polynomial is provided by $t = \frac{\pi}{4}$ in (4.5)

$$P_{palindromic} = x^3 - x^2 - x + 1 = \left(x - \frac{1}{3}\right)^3 - \frac{4}{3}\left(x - \frac{1}{3}\right) + \frac{16}{27}. \qquad (4.6)$$

Indeed, its roots are $x_1 = x_2 = 1$ and $x_3 = -1$ satisfying $x_1^2 + x_2^2 + x_3^2 = a_1^2 + a_2^2 + a_3^2 = 3$. The *multiplicity vector* of a given hyperbolic polynomial is the vector whose components are equal to the multiplicities of the distinct roots of the polynomial arranged in ascending order. Hence, the order $(-1, 1, 1)$ of the roots gives the multiplicity vector $(1, 2)$ of $P_{palindromic}$. The request $|a_2| \leq 1$ of the Whitney condition reduces the angle t from (4.5) to $t \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and then a_3 covers the whole interval $[-1, 1]$.

With $p = -\frac{4}{3}$ and $q = \frac{16}{27}$ in (3.16) it results $\Delta = 0$ and $\cos 3\theta = -1$ and then the angle of the reduced $P_{palindromic}$ is $\theta = \frac{\pi}{3}$ yielding indeed $(y_1, y_2, y_3) = (\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}) = (x_1, x_2, x_3) - \frac{1}{3}$.

Since θ is explicitly given we solve completely the linear system (3.25) and its two solutions are

$$\begin{cases} v^1 - 2v^2 + v^3 = 0, \\ v^1 - v^3 = 0, \end{cases} \rightarrow v_{\pm} = \pm\frac{1}{\sqrt{3}}(1, 1, 1).$$

The associated height functions of the $Cubic^{angle}$ are

$$f^{v_{\pm}}\left(p, \theta = \frac{\pi}{3}\right) = < v_{\pm}, Cubic^{angle}\left(p, \theta = \frac{\pi}{3}\right) >_{\mathbb{R}^3} = \pm\frac{1}{\sqrt{3}}(x^1 + x^2 + x^3)$$

which are constant functions.

Another cubic Euclidean polynomial is

$$P_3(x) = x^3 + x^2 - x - 1 = (x+1)^2(x-1) = \left(x + \frac{1}{3}\right)^3 - \frac{4}{3}\left(x + \frac{1}{3}\right) - \frac{16}{27}$$

with the same equality $x_1^2 + x_2^2 + x_3^2 = a_1^2 + a_2^2 + a_3^2 = 3$ since $x_1 = x_2 = -1 = a_2 = a_3$ and $x_3 = 1 = a_1$; its multiplicity vector is $(2, 1)$. It corresponds to the angles $t = \frac{3\pi}{4}$ and $\theta = 0$. □

Example. For $n = 4$ it follows the existence of two real parameters, θ and ψ, such that

$$\begin{cases} a_2 = a_2(\theta, \psi) := \cos\theta\cos\psi - 1, \\ a_3 = a_3(\theta, \psi) = \cos\theta\sin\psi, \\ a_4 = a_4(\theta, \psi) = \sin\theta. \end{cases} \qquad (4.7)$$

The bisectrix case $\theta = \psi := t$ gives the curve in $\mathbb{R}^3$

$$a_2(t) = -\sin^2 t \leq 0, \quad a_3(t) = \frac{1}{2}\sin 2t, \quad a_4(t) = \sin t \qquad (4.8)$$

satisfying the condition $|a_2| \leq 1$ and again $t \in \mathbb{Z}\pi$ gives the reduction case $a_2 = a_3 = a_4 = 0 = x_2 = x_3 = x_4$ and $x_1 = -a_1$. Asking again for a possible palindromic example the condition $a_4 = 1$ yields $a_1 = a_3 = 0$ and $a_2 = -1$ but the biquadratic polynomial $P = x^4 - x^2 + 1$ is not a hyperbolic one having all complex roots. Hence, there are no palindromic Euclidean polynomials of fourth degree. □

Example. For $n = 5$ it follows that we can apply the Hopf map to the pair of complex numbers $z := (a_2 + 1, a_3)$, $w := (a_4, a_5)$. We obtain

$$\begin{cases} Hopf(a_2+1, a_3, a_4, a_5) = \left(\frac{1}{2}[(a_2+1)^2 + a_3^2 - a_4^2 - a_5^2], Z\right), \\ Z = [a_4(a_2+1) + a_3a_5] + i[a_3a_4 - a_5(a_2+1)] \in \mathbb{C}. \end{cases} \tag{4.9}$$

□

The Euclidean norm is the case $p = 2$ of the usual L_p-norms. The following even case is $p = 4$ and we study the possible L_4-isometry case only for the quadratic and cubic polynomials. Starting with $P = x^2 + a_1x + a_2 \in \mathbb{R}_2[x]$ and applying again the Vieta's formulas gives

$$x_1^4 + x_2^4 = a_1^4 - 4a_1^2a_2 + 2a_2^2 \tag{4.10}$$

and by equalize with $a_1^4 + a_2^4$ it results

$$a_2^4 = 2a_2^2 - 4a_1^2a_2 \tag{4.11}$$

which splits the study into two cases. Case I) If $a_2 = 0$ then we have obviously the desired equality of L_4-norms as in the quadratic Example above. Case II) If $a_2 \neq 0$ then

$$a_1^2 = \frac{1}{4}(2a_2 - a_2^3) \geq 0 \tag{4.12}$$

and the discriminant D of P is

$$D = a_1^2 - 4a_2 = -\frac{a_2}{4}(a_2^2 + 14) > 0 \tag{4.13}$$

where the sign is imposed by the hyperbolic character of P. It follows $a_2 < 0$ which combined with the inequality (4.12) yields the domain $a_2 \in (-\infty, -\sqrt{2}]$. In the limit case $a_2 = -\sqrt{2}$ and $a_1 = 0$ we have $x_{1,2} = \pm\sqrt[4]{2}$ and the L_4-norms are

$$x_1^4 + x_2^4 = 2 + 2 = 4 = a_1^4 + a_2^4.$$

In conclusion, the result is as follows:

Proposition 4.3. *If $a_2 \in (-\infty, -\sqrt{2}]$ and a_1 is provided by the relation* (4.12) *then the solution map of the quadratic equation corresponding to P is an L_4-isometry. Since $a_2 < 1$ there are no palindromic examples.*

For the cubic (and hyperbolic) polynomial $P = x^3 + a_1x^2 + a_2x + a_3 \in \mathbb{R}_3[x]$ the L_4-isometry reads in a more complicated way

$$a_2^4 + a_3^4 = 2a_2^2 - 4a_1^2a_2 + 4a_1a_3 \tag{4.14}$$

and one simple case is again $a_2 = a_3 = 0$. The pair of conditions $(a_2 = 0, a_3 \neq 0)$ yields

$$a_1 = \frac{a_3^3}{4} \to P = x^3 + \frac{a_3^3}{4}x^2 + a_3 \tag{4.15}$$

and we must search the hyperbolic characterization of P above. Another simple case is of an initial depressed P i.e. $a_1 = 0$ and it follows the existence of $2t \in [0, \pi]$ such that

$$a_2 = p = \pm\sqrt{2}\cos t, \quad a_3 = q = \pm\sqrt{\sin 2t}. \tag{4.16}$$

We choose the minus variant and the condition $\Delta < 0$ means

$$27 \sin t < 4\sqrt{2} \cos^2 t. \tag{4.17}$$

The solution of the equation $27 \sin t = 4\sqrt{2} \cos^2 t$ is provided by WolframAlpha as $t \cong 0.1012$.

Again we search for the palindromic conditions $a_3 = 1$, $a_2 = a_1 = u$ in (3.14) and it results the equation

$$u^4 + 4u^3 - 2u^2 - 4u + 1 = 0 \tag{4.18}$$

which is a hyperbolic equation: $u_1 = -1$, $u_2 = 1$, $u_3 = -2-\sqrt{5}$, $u_4 = \sqrt{5}-2$. Then we have the following possibilities:
I) $P^1_{palindromic} = x^3 - x^2 - x + 1$ is exactly the palindromic polynomial of the second example above and indeed $x_1^4 + x_2^4 + x_3^4 = 3 = a_1^4 + a_2^4 + a_3^4$.
II) $P = x^3 + x^2 + x + 1$ is not a hyperbolic polynomial.
III) $P^2_{palindromic} = x^3 - (2 + \sqrt{5})x^2 - (2 + \sqrt{5})x + 1 = \left(x - \frac{2+\sqrt{5}}{3}\right)^3 - \left(5 + \frac{7\sqrt{5}}{3}\right)\left(x - \frac{2+\sqrt{5}}{3}\right) - \left[2 + \frac{76+70\sqrt{5}}{27}\right]$ is a hyperbolic polynomial with $x_1 = -1$ and

$$\begin{cases} x_2 = \frac{1}{2}\left(3 + \sqrt{5} - \sqrt{10 + 6\sqrt{5}}\right) = 0.1985... \\ x_3 = \frac{1}{2}\left(3 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}}\right) = 5.0375... \end{cases}, \tag{4.19}$$

and is indeed an L_4-example since

$$x_1^4 + x_2^4 + x_3^4 = a_1^4 + a_2^4 + a_3^4 = 323 + 144\sqrt{5}. \tag{4.20}$$

IV) $P = x^3 + (\sqrt{5}-2)x^2 + (\sqrt{5}-2)x + 1$ is also a non-hyperbolic polynomial since the palindromic cubic polynomial $P = x^3 + ux^2 + ux + 1$ is hyperbolic if and only if

$$u \in (-\infty, -1] \cup [3, +\infty). \tag{4.21}$$

We arrive at:

Proposition 4.4. *The only palindromic cubic polynomials which are L_4-isometries are*

$$P^1_{palindromic} = x^3 - x^2 - x + 1, P^2_{palindromic} = x^3 - (2+\sqrt{5})x^2 - (2+\sqrt{5})x + 1. \tag{4.22}$$

The data of $P^2_{palindromic}$ can be expressed in terms of the Golden mean

$$a_2 = a_3 = -(2\phi + 1), \quad x_{2,3} = \phi^2 \mp \sqrt{\phi^4 - 1} \tag{4.23}$$

and the half of the identity (4.20) *means*

$$(2\phi + 1)^4 = (2\phi^4 - 1)^2 + 4\phi^5(\phi^2 + 1). \tag{4.24}$$

We note that the plane curves corresponding to the palindromic polynomials (4.22)

$$E^1 : y^2 = x^3 - x^2 - x + 1, \quad E^2 : y^2 = x^3 - (2 + \sqrt{5})(x^2 + x) + 1 \tag{4.25}$$

have some special features: the former has the parametrization $E^1 : x(t) = t^2 - 1, y(t) = t(t^2 - 2)$, $t \in \mathbb{R}$ and the latter has 3 lattice points: $(0, \pm 1)$, $(-1, 0)$. More generally, the 1-parametric family of plane curves

$$\mathcal{C}_c : Y^2 = X^3 - 3c^2 X - 2c^3, \quad c \in \mathbb{R}$$

has the parametrization: $X_c(t) = t^2 + 2c$, $Y_c(t) = t(t^2 + 3c)$, $t \in \mathbb{R}$ and the special point $P_c(-c, 0)$.

The matrix

$$\Gamma = \begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix} \tag{4.26}$$

has $(-P^1_{palindromic})$ as characteristic polynomial with $\bar{v}_1 = (3, 5, 6)$ eigenvector corresponding to the eigenvalue $\lambda_1 = -1$ and $\bar{v}_2 = (-1, 0, 1)$, $\bar{v}_3 = (2, 1, 0)$ as linear independent eigenvectors corresponding to the double eigenvalue $\lambda_2 = \lambda_3 = 1$. Hence, with the non-orthogonal matrices

$$R = \begin{pmatrix} 3 & -1 & 2 \\ 5 & 0 & 1 \\ 6 & 1 & 0 \end{pmatrix}, R^{-1} = \begin{pmatrix} -1 & 2 & -1 \\ 6 & -12 & 7 \\ 5 & -9 & 5 \end{pmatrix}, \det R = \det R^{-1} = 1 \tag{4.27}$$

we have $R^{-1}\Gamma R = diag(-1, 1, 1)$ and this last matrix is involved in [5] in finding the Pythagorean triple preserving matrices in a projective approach.

We finish this note with five remarks concerning $P^1_{palindromic}$:
1) this polynomial is an L_{2r}-isometry for any $r \in \mathbb{N}^*$,
2) written in the complex variable z as $P_2(z) = (z-1)(z^2-1)$ it is the second *polygonal polynomial* of [2],
3) the derivative of a (strictly) hyperbolic polynomial is again a (strictly) hyperbolic one. The hyperbola $H : y^2 = (P^1_{palindromic})'(x) = 3x^2 - 2x - 1$ has four countable infinite families of lattice points

$$\begin{cases} x_n = \frac{1}{3}[(7+4\sqrt{3})^n + (7-4\sqrt{3})^n + 1], \\ y^{\pm}_n = \pm\frac{1}{\sqrt{3}}[(7+4\sqrt{3})^n - (7-4\sqrt{3})^n] \end{cases} \tag{4.28}$$

$$\begin{cases} 3X_n = (\sqrt{3}-2)(7+4\sqrt{3})^n - (\sqrt{3}+2)(7-4\sqrt{3})^n + 1, \\ 3Y^{\pm}_n = \pm[(3-2\sqrt{3})(7+4\sqrt{3})^n + (3+2\sqrt{3})(7-4\sqrt{3})^n]. \end{cases} \tag{4.29}$$

The center of this hyperbola is $C\left(\frac{1}{3}, 0\right)$, the eccentricity is $e = 2$ and the lattice point $P(1, 0) = (x_0, y^+_0)$ gives the rational parametrization of $H \setminus \{P\}$

$$H : (x, y)_{\pm}(t) = \left(\frac{t^2+1}{t^2-3}, \pm\frac{4t}{t^2-3}\right), \quad t \in \mathbb{R} \setminus \{\pm\sqrt{3}\}. \tag{4.30}$$

The derivative $(P^1_{palindromic})'(x) = 3x^2 - 2x - 1$ is strictly hyperbolic with the roots 1 and $-\frac{1}{3}$.
4) This polynomial offers a good example for the Erdös-Gallai inequality for

a special class of hyperbolic polynomials P, for which we cite [1, p. 147]: if P has among the roots the numbers ± 1 and $P(x) > 0$ for $x \in (-1, 1)$ then

$$\frac{2}{3}T \leq A \leq \frac{2}{3}R \tag{4.31}$$

with equality only for the degree two hyperbolic polynomials. Here

$$T = \frac{2P'(1)P'(-1)}{P'(1) - P'(-1)}, \quad A = \int_{-1}^{1} P(x)dx, \quad R = 2P(b)$$

with $b \in (-1, 1)$ a point of maximum for P. Our polynomial $P^1_{palindromic}$ has:

$$b = -\frac{1}{3}, \quad T = 0, \quad A = \frac{4}{3}, \quad R = \frac{2 \cdot 32}{27} \tag{4.32}$$

and hence the Erdös-Gallai strict inequality reads as: $0 < \frac{4}{3} < \frac{4}{3} \cdot \frac{32}{27}$.
5) Recall that to every polynomial $P \in \mathbb{R}_n[x]$ a Hurwitz matrix $H(P) \in M_n(\mathbb{R})$ is naturally associated. If P is monic and cubic then

$$H(P) := \begin{pmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & 0 \\ 0 & a_1 & a_3 \end{pmatrix},$$

and hence our polynomial has

$$H(P^1_{palindromic}) := \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{4.33}$$

with the eigenvalues and eigenvectors as follows

$$\lambda_1 = -2, v_1 = (-3, 3, 1), \lambda_2 = 0, v_2 = (1, 1, 1), \lambda_3 = 1, v_3 = (0, 0, 1). \tag{4.34}$$

Hence the rank of this matrix is two.

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