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The radial curvature of plane curves
Dedicated to the memory of Academician Radu Miron

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Abstract We introduce and study a new curvature function for plane curves inspired by the weighted mean curvature of M. Gromov. We call it *radial* being the difference between the usual curvature and the inner product of the normal vector field and the radial vector field. But, since the problem of vanishing of this curvature involves complicated expressions, we computed it for several examples.

Keywords plane curve · radial vector field · radial curvature · curvature

Mathematics Subject Classification (2010) 53A04 · 53A45 · 53A55

The last forty years known an intensive research in the area of geometric flows. The most simple of them is the *curve shortening flow* and already the excellent survey [2] is almost twenty years old. Recall that the main geometric tool for studying this last flow is the well-known curvature of plane curves. Hence, in order to give a re-start to this problem it seem advantageous to search for variants of the curvature or in terms of [11], deformations of the usual curvature. The goal of this short note is to propose such a deformation.

Fix $I \subseteq \mathbb{R}$ an open interval and $C \subset \mathbb{R}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)), \quad \|r'(t)\| > 0, \quad t \in I. \tag{1}$$

The ambient setting, namely $\mathbb{R}^2$, is an Euclidean vector space with respect to the canonical inner product:

$$\langle u, v\rangle = u^1v^1 + u^2v^2, \quad u = (u^1, u^2), \quad v = (v^1, v^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u\rangle. \tag{2}$$

The infinitesimal generator of the rotations in $\mathbb{R}^2$ is the linear vector field, called *angular*:

$$\xi(u) := -u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \quad \xi(u) = i \cdot u = i \cdot (u^1 + iu^2). \tag{3}$$

It is a complete vector field with integral curves the circles $\mathcal{C}(O,R)$:

$$\begin{cases}\gamma^{\xi}_{u_0}(t)=(u^1_0\cos t-u^2_0\sin t, u^1_0\sin t+u^2_0\cos t)=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\cdot\begin{pmatrix}u^1_0\\ u^2_0\end{pmatrix}=SO(2)\cdot u_0,\\ R=\|u_0\|=\|(u^1_0,u^2_0)\|, \quad t\in\mathbb{R},\end{cases} \tag{4}$$

and since the rotations are isometries of the Riemannian metric $g_{can}=dx^2+dy^2$ it follows that ξ is a Killing vector field of the Riemannian manifold $(\mathbb{R}^2, g_{can})$. The first integrals of ξ are the Gaussian functions i.e. multiples of the square norm: $f_C(x,y)=C(x^2+y^2)$, $C\in\mathbb{R}$. For an arbitrary vector field $X=A(x,y)\frac{\partial}{\partial x}+B(x,y)\frac{\partial}{\partial y}$ its Lie bracket with ξ is:

$$[X,\xi]=(yA_x-xA_y-B)\frac{\partial}{\partial x}+(A+yB_x-xB_y)\frac{\partial}{\partial y}$$

where the subscript denotes the variable corresponding to the partial derivative. For example, ξ commutes with *the radial* (or Euler) vector field: $E(x,y)=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$,which is also a complete vector field having as integral curves the homotheties $\gamma^E_{u_0}(t)=e^t u_0$ for all $t\in\mathbb{R}$. The vector field E is the basis of the 1-dimensional annihilator of the Liouville (or tautological) 1-form $\lambda=\frac{1}{2}(-ydx+xdy)$ whose exterior derivative is the area 2-form $dx\wedge dy$. We point out also that the opposite vector field $W=-E$ is exactly the wind in the Zermelo navigation problem corresponding to the Funk metric in the unit disk of $\mathbb{R}^2$, [3]. For an arbitrary Euclidean space $\mathbb{R}^n$ with $n\geq 2$ the radial vector field $E=x^i\frac{\partial}{\partial x^i}$ defines the notion of *horizontal* 1*-form* ρ as satisfying $i_E\rho=0$ with i_E the interior product.

The Frenet apparatus of the curve C is provided by:

$$\begin{cases}T(t)=\frac{r'(t)}{\|r'(t)\|}, \quad N(t)=i\cdot T(t)=\frac{1}{\|r'(t)\|}(-y'(t),x'(t))\\ k(t)=\frac{1}{\|r'(t)\|}\langle T'(t),N(t)\rangle=\frac{1}{\|r'(t)\|^3}\langle r''(t), ir'(t)\rangle=\frac{1}{\|r'(t)\|^3}[x'(t)y''(t)-y'(t)x''(t)].\end{cases} \tag{5}$$

Hence, if C is naturally parametrized (or parametrized by arc-length) i.e. $\|r'(t)\|=1$ for all $t\in I$ then $r''(t)=k(t)ir'(t)$. In a complex approach based on $z(t)=x(t)+iy(t)\in\mathbb{C}=\mathbb{R}^2$ we have $2\lambda=Im(\bar{z}dz)$ and:

$$\begin{cases}k(t)=\frac{1}{|z'(t)|^3}Im(\bar{z}'(t)\cdot z''(t))=\frac{1}{|z'(t)|}Im\left(\frac{z''(t)}{z'(t)}\right)=\frac{1}{|z'(t)|}Im\left[\frac{d}{dt}\left(\ln z'(t)\right)\right],\\ Re(\bar{z}'(t)\cdot z''(t))=\frac{1}{2}\frac{d}{dt}\|r'(t)\|^2, \quad f_C(z)=C|z|^2.\end{cases}$$

This short note defines a new curvature function for C inspired by a notion introduced by M. Gromov in [9, p. 213] and concerning with hypersurfaces M^n in a weighted Riemannian manifold $(\tilde{M}, g, f\in C^{\infty}_{+}(\tilde{M}))$. More precisely, the *weighted mean curvature* of M is the difference:

$$H^f:=H-\langle\tilde{N},\tilde{\nabla}f\rangle_g \tag{6}$$

where H is the usual mean curvature of M and $\tilde{N}$ is the unit normal to M. This curvature was studied in several papers.

The rotational field ξ is not a g_{can}-gradient vector field but E is the gradient of the Gaussian function $f_{\frac{1}{2}}$. Hence we follow this path and associated to the new area form $\frac{1}{2}(x^2+y^2)dx \wedge dy$ we consider:

Definition 1 The *radial curvature* of C is the smooth function $k_{rad} : I \to \mathbb{R}$ given by:

$$k_{rad}(t) := k(t) - \langle N(t), E(r(t))\rangle. \tag{7}$$

Before starting its study we point out that this work is dedicated to the memory of Academician Radu Miron (1927-2022). He was always interested in the geometry of curves and besides his theory of *Myller configuration* [14] he generalized also a type of curvature for space curves in [13]. Returning to our subject we note:

Proposition 2 *The expression of the radial curvature is:*

$$k_{rad}(t) = k(t) - \frac{\langle ir'(t), r(t)\rangle}{\|r'(t)\|}. \tag{8}$$

As consequences:
i) *if* $t_0 \in I$ *satisfies* $r(t_0) = O(0,0)$ *then* $k_{rad}(t_0) = k(t_0)$,
ii) *the curve* C *and its trigonometrically rotation* iC *share the same radial curvatures*,
iii) *suppose that the curvature function* k *has a constant sign, say* $k > 0$. *Then the Frenet decomposition of the position vector* $r(t)$ *is*:

$$r(t) = \frac{k'_{rad}(t) - k'(t)}{\|r'(t)\|k(t)} T(t) + [k(t) - k_{rad}(t)]N(t). \tag{9}$$

Proof We have directly:

$$\langle N(t), E(r(t))\rangle = \langle iT(t), r(t)\rangle \tag{10}$$

and the conclusion (8) follows. The first consequence derives from the inequality:

$$|k_{rad}(t) - k(t)| \leq \|r(t)\| \tag{11}$$

or alternatively, directly from the equality $E(O) = 0$. Concerning the second consequence it is obvious that the C and $iC : t \to (-y(t), x(t))$ share the same curvature k and the same second term from (8). The last consequence follows from deriving (8) and the use of the first Frenet equation.

□

Example 3 i) If C is the line $r_0 + tu, t \in \mathbb{R}$ with the vector $u \neq \bar{0} = (0,0)$ then k_{rad} is the constant:

$$k_{rad}(t) = -\frac{\langle r_0, iu\rangle}{\|u\|}. \tag{12}$$

In particular, if $O \in C$ then C is a radial-flat curve i.e. $k_{rad} \equiv 0$.
ii) If C is the circle $\mathcal{C}(O,R) : r(t) = Re^{it}$ the k_{rad} is again a constant $k_{rad} = \frac{1}{R} + R \geq \max\{2, \frac{1}{R} = k, R\}$. For the case of logarithmic spiral expressed in polar coordinates as $\rho_{R,\alpha}(t) = Re^{\alpha t}$, $R, \alpha > 0$ and $t \in \mathbb{R}$ we have the increasing function:

$$k_{rad}(t) = \frac{1}{\sqrt{\alpha^2+1}}\left(\frac{1}{Re^{\alpha t}} + Re^{\alpha t}\right) \geq \frac{2}{\sqrt{\alpha^2+1}}$$

and for $\alpha \to 0$ we re-obtain the radial curvature of the circle $\mathcal{C}(O,R)$.
The circle example, considered from the radii and curvatures, suggests the map $f_{rad} : (0,+\infty) \to (0,+\infty)$:

$$f_{rad}(x) = \frac{x}{x^2+1} = \frac{1}{2}\sin 2(\arctan x) \leq \min\left\{\frac{1}{2}, \frac{1}{x}\right\}.$$

A relationship between f_{rad} and the Gaussian potential of the vector field E appears as the main inequality of [10]:

$$f_{rad}(x) < R(x) := e^{\frac{x^2}{2}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt < \frac{1}{x}.$$

iii) Suppose that C is positively oriented in the terms of Definition 1.14 from [15, p. 17]. Suppose also that C is convex; then applying the Theorem 1.18 of page 19 from the same book it results for the usual curvature the inequality $k \geq 0$. From the inequality (11) it results that $k_{rad}(t) \geq -\|r(t)\|$, for all $t \in I$.

□

Let $J \subseteq \mathbb{R}$ be another open interval and fix the diffeomorphism $\varphi : s \in J \to t \in I$ with the smooth inverse $\varphi^{-1} : t \in I \to s \in J$. Since $r'(s) = \varphi'(s)r'(t(s))$ we restrict our study to the class $Diff_+(J,I)$ of orientation-preserving diffeomorphisms: $\varphi'(s) > 0$, for all $s \in J$. The transformation of the rotational curvature under the action of φ is:

$$k_{rad}(s) = k(t) - \langle r(\varphi^{-1}(t)), N(t)\rangle \tag{13}$$

and then:

$$k_{rad}(s) - k_{rad}(t) = \langle r(t) - r(\varphi^{-1}(t)), N(t)\rangle. \tag{14}$$

Proposition 4 *The orientation-preserving diffeomorphism φ preserves also the radial curvature of C if and only if the vector $T(t)$ is parallel to the vector $r(t) - r(\varphi^{-1}(t))$ for all $t \in I$.*

An important problem is to determine the class of curves with prescribed radial curvature. For example, if we ask the vanishing of the radial curvature for a naturally parametrized curve then it follows the characterizing equation:

$$\langle r(t), ir'(t)\rangle = \|r''(t)\| = \langle r''(t), ir'(t)\rangle \tag{15}$$

which says that the vectors $r''(t) - r(t)$, $r'(t)$ are collinear: there exists a smooth function $\lambda : I \to \mathbb{R}$ such that $r''(t) - r(t) = \lambda(t) r'(t)$. Using the formalism of [16, p. 2] if $r : S^1 \simeq [0, 2\pi) \to \mathbb{R}^2$ is naturally parametrized then there exists the smooth function $\theta : S^1 \to \mathbb{R}$, called *normal angle*, such that:

$$N(t) = e^{i\theta(t)} = (\cos\theta(t), \sin\theta(t)), \quad T(t) = -iN(t) = -ie^{i\theta(t)} = e^{i(\theta(t) - \frac{\pi}{2})} \tag{16}$$

and then the Frenet equations yields:

$$\frac{d\theta}{dt}(t) = k(t). \tag{17}$$

Suppose again that C is with $k > 0$. Then the tangential component of the identity (9) means:

$$\langle r(t), r'(t)\rangle + \frac{k'(t)}{k(t)} = 0 \tag{18}$$

which yields:

$$\lambda(t) = \frac{k'(t)}{k(t)}. \tag{19}$$

Proposition 5 *Suppose that t is a natural parameter on the curve C and θ is a strictly increasing function. Then C is radial-flat if and only if:*

$$r(t) = \theta'(t) e^{i\theta(t)} - \frac{\theta''(t)}{\theta'(t)} e^{i(\theta(t) - \frac{\pi}{2})}. \tag{20}$$

subject to the unitary condition:

$$[\theta'(t)]^2 + \left(\frac{\theta''(t)}{\theta'(t)}\right)^2 = 1, \tag{21}$$

or equivalently, in terms of $k = \theta'$:

$$k^2(t) + \left(\frac{k'(t)}{k(t)}\right)^2 = 1. \tag{22}$$

Also, the equation (18) *can be integrated to a Gaussian expression of the curvature function in terms of square distance function*:

$$k(t) = \mathcal{C} e^{-\frac{\|r(t)\|^2}{2}}, \quad \mathcal{C} > 0, \quad \theta(\|r(t)\|) = \mathcal{C}\sqrt{\frac{\pi}{2}} erf\left(\frac{\|r(t)\|^2}{\sqrt{2}}\right) + \mathcal{C}_1, \quad \mathcal{C}_1 > 0. \tag{23}$$

Remarks 6 i) The subject of plane curves with curvature depending on position is a very interesting one and a recent study is the paper [1].

ii) The function $e^{-\frac{\|r(t)\|^2}{2}}$ is considered as example in the paper [8, p. 170] in the setting of Gromov type deformation of curvature for plane curves. The

curve shortening problem associated to a density is studied in the paper [12]. The isoperimetric plane problem with respect to a radial density is solved in [7].
iii) Since the relation (17) can be rewritten as $\theta = \int k dt$ we can introduce a f-curvature k_f with respect to an arbitrary density function f through the relation:

$$\theta = \int k_f e^f dt \to k_f(t) = e^{-f(t)} \frac{d\theta}{dt}(t).$$

This new curvature function will be studied in a forthcoming paper. A generalized radial curvature of plane curves is already studied in the paper [4] while a hyperbolic curvature was introduced in the paper [5].

□

Now, we work the same vanishing problem but in the polar coordinates $(\rho, t) \in \mathbb{R}_{>0} \times [0, 2\pi)$ of $\mathbb{R}^2$; hence $E = \rho \frac{\partial}{\partial \rho} = z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}$ and the equation of C is:

$$C : \rho = \rho(t), \quad t \in [0, 2\pi).$$

Recall that the curvature of C is then:

$$k(t) = \frac{2(\rho'(t))^2 + \rho(t)^2 - \rho(t)\rho''(t)}{[\rho(t)^2 + (\rho'(t))^2]^{\frac{3}{2}}}. \tag{24}$$

Then the vanishing of k_{rad} means the equality:

$$\frac{2(\rho'(t))^2 + \rho(t)^2 - \rho(t)\rho''(t)}{[\rho(t)^2 + (\rho'(t))^2]^{\frac{3}{2}}} = \frac{-\rho^2}{[\rho(t)^2 + (\rho'(t))^2]^{\frac{1}{2}}} \tag{25}$$

which implies:

Proposition 7 *Let C be curve not containing O. Its radial curvature vanishes identically if and only if ρ is the solution of the following non-linear second-order ODE:*

$$\rho'' = \frac{2}{\rho}(\rho')^2 + \rho[1 + \rho(t)^2 + (\rho'(t))^2]. \tag{26}$$

In the following we present a couple of examples in order to remark the computational aspects of our approach.

Example 8 We study completely a curve with non-constant radial curvature. Namely, the involute of the unit circle S^1 is:

$$C : r(t) = (\cos t + t \sin t, \sin t - t \cos t) = (1 - it)e^{it}, \quad t \in (0, +\infty). \tag{27}$$

A direct computation gives:

$$r'(t) = (t \cos t, t \sin t) = te^{it}, \quad k(t) = \frac{1}{t} > 0, \quad \|r(t)\|^2 = 1 + t^2 \tag{28}$$

and then the radial curvature is:

$$k_{rad}(t) = \frac{1}{t} + t \geq \max\{2, \frac{1}{t} = k(t)\}. \tag{29}$$

This example can be treated also with respect to a natural parameter $s \in (0, +\infty)$ which is provided by $t := \sqrt{2s}$. For example, the normal angle function is $\theta(s) = \frac{\pi}{2} + \sqrt{2s}$ since $r'(s) = e^{i\sqrt{2s}}$. The curve C has a single radial-vertex i.e. a vanishing point for the derivative of k_{rad} namely $t = 1$ which provides the minimum value, namely 2, of k_{rad}.

□

Example 9 Recall that for $R > 0$ the cycloid of radius R has the equation:

$$C : r(t) = R(t - \sin t, 1 - \cos t) = R[(t, 1) - e^{i(\frac{\pi}{2} - t)}], \quad t \in \mathbb{R}. \tag{30}$$

We have immediately:

$$r'(t) = R(1 - \cos t, \sin t) = R[(1, 0) - e^{it}], k(t) = -\frac{1}{4R \sin \frac{t}{2}}, \|r'(t)\| = 2R|\sin \frac{t}{2}| \tag{31}$$

and then we restrict our definition domain to $(0, \pi)$. It follows:

$$k_{rad}(t) = -\frac{1}{4R \sin \frac{t}{2}} - R\left(2 \sin \frac{t}{2} - t \cos \frac{t}{2}\right). \tag{32}$$

Again a natural parameter s is provided by: $t = 2 \arccos\left(1 - \frac{s}{4R}\right)$.

□

Example 10 Fix a graph $C : y = f(x)$, $x \in I$ with the second derivative f'' strictly positive. With the usual parametrization $C : r(t) = (t, f(t))$ we have:

$$r'(t) = (1, f'(t)), \quad k(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}} > 0, \quad \|r(t)\|^2 = t^2 + f^2(t) \tag{33}$$

which gives that C is convex and:

$$k_{rad}(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}} - \frac{f(t) - tf'(t)}{[1 + (f'(t))^2]^{\frac{1}{2}}}. \tag{34}$$

It follows that the function f making k_{rad} constant zero is a solution of the non-autonomous differential equation:

$$f''(t) = [f(t) - tf'(t)][1 + (f'(t))^2]. \tag{35}$$

□

Example 11 The derivative curve r' from (31) is an Archimedes' spiral. The general such spiral is given in polar coordinates as:

$$A(spiral) : \rho(t) = Rt, \quad R > 0 \tag{36}$$

and hence:

$$k_{rad}(t) = \frac{2+t^2}{R(t^2+1)^{\frac{3}{2}}} + \frac{Rt}{(t^2+1)^{\frac{1}{2}}} > 0. \tag{37}$$

□

Example 12 Fix $\alpha \in \mathbb{R}^*$ and the naturally parametrized curve C. Then the *α-parallel curve* of C is the new curve:

$$C_\alpha : \tilde{r}(t) := r(t) + \alpha N(t), \quad t \in I \tag{38}$$

with:

$$\tilde{T}(t) = \frac{1-\alpha k(t)}{|1-\alpha k(t)|}T(t), \quad \tilde{N}(t) = \frac{1-\alpha k(t)}{|1-\alpha k(t)|}N(t), \quad \tilde{k}(t) = k(t). \tag{39}$$

Hence, we consider that α does not belongs to the range of the function $\frac{1}{k}$, say $1 > \alpha k(t)$, and the new radial curvature is:

$$\tilde{k}_{rad}(t) = k_{rad}(t) - \alpha. \tag{40}$$

□

We end this note with an approach in terms of Fermi-Walker derivative. Let $\mathcal{X}_\gamma$ be the set of vector fields along the curve γ. Then the Fermi-Walker derivative is the map ([6]) $\nabla^{FW}_\gamma : \mathcal{X}_\gamma \to \mathcal{X}_\gamma$:

$$\nabla^{FW}_\gamma(X) := \frac{d}{dt}X + k\|r'(\cdot)\|[\langle X, N\rangle T - \langle X, T\rangle N] = \frac{d}{dt}X + k\|r'(\cdot)\|[X^\flat(N)T - X^\flat(T)N] \tag{41}$$

with $X^\flat$ the differential 1-form dual to X with respect to the Euclidean metric. For $X = E \circ r = r$ we have:

$$\nabla^{FW}_\gamma(r) = \|r'(\cdot)\|(T + k[\langle r, N\rangle T - \langle r, T\rangle N]) \tag{42}$$

and then, using (9):

$$(\nabla^{FW}_\gamma r)(t) = \|r'(t)\|[1 + k^2(t) - k(t)k_{rad}(t)]T(t) + [k'(t) - k'_{rad}(t)]N(t). \tag{43}$$

Hence, for a radial-flat C we have:

$$(\nabla^{FW}_\gamma r)(t) = \|r'(t)\|[1 + k^2(t)]T(t) + k'(t)N(t) \tag{44}$$

which is nowhere zero vector due to the strictly positive coefficient of $T(t)$.

Acknowledgements I am very grateful to an anonymous referee for several useful corrections to an initial version of this work.

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Received: 17.V.2022 / Revised: 26.IX.2022 / Accepted: 03.XI.2022

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