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The rotational curvature of plane curves

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Dedicated to the memory of Academician Radu Miron

Abstract

We introduce and study a new curvature function for plane curves inspired by the weighted mean curvature of M. Gromov. We call it *rotational* being the difference between the usual curvature and the inner product of the normal vector field and the angular vector field. But, since the problem of vanishing of this curvature involves complicated expressions we define a second rotational curvature. Both these curvatures are computed for several examples.

The last forty years known an intensive research in the area of geometric flows. The most simple of them is the *curve shortening flow* and already the excellent survey [1] is almost twenty years old. Recall that the main geometric tool in this last flow is the well-known curvature of plane curves. Hence, to give a re-start to this problem seams to search for variants of the curvature or in terms of [7], deformations of the usual curvature. The goal of this short note is to propose such a deformation.

Fix $I \subseteq \mathbb{R}$ an open interval and $C \subset \mathbb{R}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)), \quad \|r'(t)\| > 0, \quad t \in I. \tag{1}$$

The ambient setting, namely $\mathbb{R}^2$, is an Euclidean vector space with respect to the canonical inner product:

$$< u, v >= u^1v^1 + u^2v^2, u = (u^1, u^2), v = (v^1, v^2) \in \mathbb{R}^2, \|u\|^2 =< u, u > . \tag{2}$$

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The infinitesimal generator of the rotations in $\mathbb{R}^2$ is the linear vector field, called *angular*:

$$\xi(u) := -u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \quad \xi(u) = i\cdot u = i\cdot(u^1+iu^2). \tag{3}$$

It is a complete vector field with integral curves the circles $\mathcal{C}(O,R)$:

$$\begin{cases} \gamma^{\xi}_{u_0}(t) = (u^1_0\cos t - u^2_0\sin t, u^1_0\sin t + u^2_0\cos t) = \\ = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix} = SO(2)\cdot u_0, \\ R = \|u_0\| = \|(u^1_0, u^2_0)\|, \quad t\in\mathbb{R}, \end{cases} \tag{4}$$

and since the rotations are isometries of the Riemannian metric $g_{can} = dx^2 + dy^2$ it follows that ξ is a Killing vector field of the Riemannian manifold $(\mathbb{R}^2, g_{can})$. The first integrals of ξ are the Gaussian functions i.e. multiples of the square norm: $f_C(x,y) = C(x^2+y^2)$, $C\in\mathbb{R}$. For an arbitrary vector field $X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ its Lie bracket with ξ is:

$$[X,\xi] = (yA_x - xA_y - B)\frac{\partial}{\partial x} + (A + yB_x - xB_y)\frac{\partial}{\partial y}$$

where the subscript denotes the variable corresponding to the partial derivative. For example, ξ commutes with *the radial* (or Euler) vector field: $E(x,y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

The Frenet apparatus of the curve C is provided by:

$$\begin{cases} T(t) = \frac{r'(t)}{\|r'(t)\|}, \quad N(t) = i\cdot T(t) = \frac{1}{\|r'(t)\|}(-y'(t), x'(t)) \\ k(t) = \frac{1}{\|r'(t)\|} < T'(t), N(t) >= \frac{1}{\|r'(t)\|^3} < r''(t), ir'(t) >, \\ k(t) = \frac{1}{\|r'(t)\|^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases} \tag{5}$$

Hence, if C is naturally parametrized (or parametrized by arc-length) i.e. $\|r'(t)\| = 1$ for all $t\in I$ then $r''(t) = k(t)ir'(t)$. In a complex approach based on $z(t) = x(t) + iy(t) \in \mathbb{C} = \mathbb{R}^2$ we have:

$$\begin{cases} k(t) = \frac{1}{|z'(t)|^3} Im(\bar{z}'(t)\cdot z''(t)) = \frac{1}{|z'(t)|} Im\left(\frac{z''(t)}{z'(t)}\right) = \frac{1}{|z'(t)|} Im\left[\frac{d}{dt}(\ln z'(t))\right], \\ Re(\bar{z}'(t)\cdot z''(t)) = \frac{1}{2}\frac{d}{dt}\|r'(t)\|^2, \quad f_C(z) = C|z|^2. \end{cases}$$

This short note defines a new curvature function for C inspired by a notion introduced by M. Gromov in [6, p. 213] and concerning with hypersurfaces M^n in a weighted Riemannian manifold $(\tilde{M}, g, f\in C^\infty_+(\tilde{M}))$. More precisely, the *weighted mean curvature* of M is the difference:

$$H^f := H - < \tilde{N}, \tilde{\nabla} f >_g \tag{6}$$

where H is the usual mean curvature of M and $\tilde{N}$ is the unit normal to M. This curvature was studied in several papers. Recently, we introduce two new curvatures: a first called *hyperbolic* in [2] and a second called *generalized radial* in [3].

Although ξ is not a g_{can}-gradient vector field we follow the above path and consider:

Definition 1 The *rotational curvature* of C is the smooth function $k_{rot} : I \to \mathbb{R}$ given by:

$$k_{rot}(t) := k(t) - < N(t), \xi(r(t)) > . \tag{7}$$

Before starting its study we point out that this work is dedicated to the memory of Academician Radu Miron (1927-2022). He was always interested in the geometry of curves and besides its theory of *Myller configuration* ([9]) he generalizes also a type of curvature for space curves in [8]. Returning to our subject we note:

Proposition 2 *The expression of the rotational curvature is:*

$$k_{rot}(t) = k(t) - \frac{< r'(t), r(t) >}{\|r'(t)\|} = k(t) - \frac{1}{2\|r'(t)\|}\frac{d}{dt}\left(\|r(t)\|^2\right). \tag{8}$$

As consequence, the curve C and its trigonometrically rotation iC share the same rotational curvatures.

Proof We have directly:

$$< N(t), \xi(r(t)) > = < iT(t), ir(t) > = < T(t), r(t) > \tag{9}$$

and the conclusion follows. Concerning the consequence it is obvious that C and $iC : t \to (-y(t), x(t))$ share the same curvature k and the same second term from (8). □

Example 3 i) If C is the line $r_0 + tu, t \in \mathbb{R}$ with the vector $u \neq \bar{0} = (0,0)$ then k_{rot} is the linear function:

$$k_{rot}(t) = -\|u\|t - \frac{< r_0, u >}{\|u\|}. \tag{10}$$

In particular, if $O \in C$ and u is an unit vector then $k_{rot}(t) = -t$.
ii) If C is the circle $\mathcal{C}(O, R)$ the k_{rot} reduces to $k_{circle} = \frac{1}{R} > 0$.
iii) Suppose that C is positively oriented in the terms of Definition 1.14 from [11, p. 17]. Suppose also the C is convex; then applying the Theorem 1.18 of page 19 from the same book it results for the usual curvature the inequality $k \geq 0$. Hence a convex positively oriented curve with decreasing distance function $t \to \|r(t)\|^2$ has everywhere nonnegative rotational curvature. □

Let $J \subseteq \mathbb{R}$ be another open interval and fix the diffeomorphism $\varphi : s \in J \to t \in I$ with the smooth inverse $\varphi^{-1} : t \in I \to s \in J$. Since $r'(s) = \varphi'(s)r'(t(s))$ we restrict our study to the class $Diff_+(J, I)$ of orientation-preserving diffeomorphisms: $\varphi'(s) > 0$, for all $s \in J$. The transformation of the rotational curvature under the action of φ is:

$$k_{rot}(s) = k(t) - \frac{< r(\varphi^{-1}(t)), r'(t) >}{\|r'(t)\|} \tag{11}$$

and then:

$$k_{rot}(s) - k_{rot}(t) = \frac{< r(t) - r(\varphi^{-1}(t)), r'(t) >}{\|r'(t)\|}. \tag{12}$$

Proposition 4 *The orientation-preserving diffeomorphism φ preserves also the rotational curvature of C if and only if the vector $r'(t)$ is parallel to the vector $i(r(t) - r(\varphi^{-1}(t)))$ for all $t \in I$. Equivalently, on J one have*:

$$r' \circ \varphi \parallel i(r \circ \varphi - r). \tag{13}$$

An important problem is the class of curves with prescribed rotational curvature. For example, if we ask the vanishing of the rotational curvature for a naturally parametrized curve then it follows that the square distance to the origin $O(0,0) \in \mathbb{R}^2$ is the double of the total curvature:

$$\|r(t)\|^2 = 2\int_{t_0}^{t} k(s)ds, \quad t_0 \in I \tag{14}$$

since the characterizing equation reads:

$$< r(t), r'(t) >= \|r''(t)\|. \tag{15}$$

In the right-hand side of (15) we have the norm of *the curvature vector field* $r''(t)$ while if C is a closed convex curve the integral of (14) is *the normal angle function*. Indeed, using the formalism of [12, p. 2] if $r : S^1 \simeq [0, 2\pi) \to \mathbb{R}^2$ is naturally parametrized then there exists the smooth function $\theta : S^1 \to \mathbb{R}$, called *normal angle*, such that:

$$N(t) = e^{i\theta(t)} = (\cos\theta(t), \sin\theta(t)), \quad T(t) = -iN(t) = -ie^{i\theta(t)} = e^{i(\theta(t) - \frac{\pi}{2})}$$

and then the Frenet equations yields:

$$\frac{d\theta}{dt}(t) = k(t).$$

In conclusion, the vanishing of the rotational curvature of a closed convex curve is equivalent to $0 \leq \|r(t)\|^2 = 2\theta(t)$ for all $t \in S^1$.

Now, we work the same vanishing problem but in the polar coordinates $(\rho, t) \in \mathbb{R}_{>0} \times [0, 2\pi)$ of $\mathbb{R}^2$; hence the equation of C is:

$$C : \rho = \rho(t), \quad t \in [0, 2\pi).$$

Recall that the curvature of C is then:

$$k(t) = \frac{2(\rho'(t))^2 + \rho(t)^2 - \rho(t)\rho''(t)}{[\rho(t)^2 + (\rho'(t))^2]^{\frac{3}{2}}}. \tag{16}$$

Then the vanishing of k_{rot} means the equality:

$$\frac{2(\rho'(t))^2 + \rho(t)^2 - \rho(t)\rho''(t)}{[\rho(t)^2 + (\rho'(t))^2]^{\frac{3}{2}}} = \frac{\rho\rho'}{[\rho(t)^2 + (\rho'(t))^2]^{\frac{1}{2}}} \tag{17}$$

which implies:

Proposition 5 *Let C be curve not containing O. Its rotational curvature is constant zero if and only if ρ is the solution of the following non-linear second-order ODE*:

$$\rho'' = -(\rho')^3 + \frac{2}{\rho}(\rho')^2 - \rho^2\rho' + \rho. \tag{18}$$

Remarks 6 i) Although this differential equation is autonomous the Wolfram Alpha site can not solves it explicitly.
ii) A method to eliminate the trouble from equation (18) is to define from the beginning:

$$k_{rot}^{second}(t) := k(t) - \frac{< N(t), \xi(r(t)) >}{\|r'(t)\|^2} \tag{19}$$

and therefore we have the new differential equation:

$$\rho'' = \frac{2}{\rho}(\rho')^2 + \rho - \rho'. \tag{20}$$

Its linear part:

$$f'' = f - f' \tag{21}$$

is a Sturm-Liouville equation:

$$\frac{d}{dt}[e^t f'(t)] = e^t f(t) \tag{22}$$

with solution:

$$f(t) = C_1 e^{-\frac{\sqrt{5}+1}{2}t} + C_2 e^{\frac{\sqrt{5}-1}{2}t}. \tag{23}$$

We remark also that (22) is the Euler-Lagrange equation of the time-dependent Lagrangian (although for us t means an angle):

$$L(t, f, f') = \frac{e^t}{2}\left[(f')^2 + f^2\right]. \tag{24}$$

We have arrived now to the second great interest of Academician Radu Miron, namely the geometrization of Lagrangians and recall the importance of the book [10] in this area of research.

We point out also that $\Phi = \frac{\sqrt{5}+1}{2}$ is the well-known *golden ratio* which is studied from the point of view of differential geometry in [5]. For lines through O and circles the curvature k_{rot}^{second} reduces to k_{rot}. $\square$

Returning to the vanishing rotational curvature in Cartesian coordinates we have:

Proposition 7 *Suppose that t is a natural parameter on the curve C. Then C is rotational-flat if and only if there exists a smooth function $\lambda : I \to \mathbb{R}$ such that the functions defining C satisfy the differential system:*

$$\begin{cases} y'' - x = \lambda y', \\ x'' + y = \lambda x' \end{cases} \tag{25}$$

in addition to $(x')^2 + (y')^2 = 1$. In the matrix form, the above differential system is expressed as:

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Proof Supposing the vanishing of k_{rot} it results that C is not a line. From the first equality of (8) it holds:

$$k(t) = < r(t), T(t) > = - < N'(t), T(t) > \tag{26}$$

and hence:

$$< r(t) + N'(t), T(t) > = 0 \tag{27}$$

which means the existence of a function $\lambda = \lambda(t)$ such that:

$$r(t) + N'(t) = \lambda(t) i T(t). \tag{28}$$

The expression on components of this equation is exactly the claimed system while the function λ is provided by:

$$\lambda(t) = x'(t)y(t) - y'(t)x(t) = < T(t), -ir(t) > .$$

Also, from the system (25) we can express the parametrization $r(\cdot)$ of C in terms of the functions λ, k and normal angle θ:

$$\begin{cases} x(t) = \lambda(t)\cos\theta(t) + k(t)\sin\theta(t), \\ y(t) = k(t)\cos\theta(t) - \lambda(t)\sin\theta(t), k(t) = \frac{d\theta}{dt}(t). \end{cases} \tag{29}$$

□

In the following we present a couple of examples in order to remark the computational aspects of our approach.

Example 8 We study completely a curve with non-constant rotational curve. Namely, the involute of the unit circle S^1 is:

$$C : r(t) = (\cos t + t\sin t, \sin t - t\cos t) = (1 - it)e^{it}, \quad t \in (0, +\infty). \tag{30}$$

A direct computation gives:

$$r'(t) = (t\cos t, t\sin t) = te^{it}, \quad k(t) = \frac{1}{t} > 0, \quad \|r(t)\|^2 = 1 + t^2 \tag{31}$$

and then the rotational curvature is:

$$k_{rot}(t) = \frac{1}{t} - 1, \quad k^{second}_{rot}(t) = \frac{1}{t} - \frac{1}{t^2}. \tag{32}$$

Hence, for $t \in (0, 1)$ we have $k_{rot}(t) > 0 > k^{second}_{rot}(t)$ while for $t \in (1, +\infty)$ we have $k^{second}_{rot}(t) > 0 > k_{rot}(t)$. The point $r(1) = (\cos 1 + \sin 1, \sin 1 - \cos 1) = (1.3817..., 0.3011....) \in C$ is a common zero for both rotational curvatures. We remark that both $\cos 1+\sin 1$ and $\sin 1-\cos 1$ are transcendental numbers. This example can be treated also with respect to a natural parameter $s \in (0, +\infty)$ which is provided by $t := \sqrt{2s}$. For example, the normal angle function is $\theta(s) = \frac{\pi}{2} + \sqrt{2s}$ since $r'(s) = e^{i\sqrt{2s}}$. □

Example 9 Recall that for $R > 0$ the cycloid of radius R has the equation:

$$C : r(t) = R(t - \sin t, 1 - \cos t) = R[(t, 1) - e^{i(\frac{\pi}{2} - t)}], \quad t \in \mathbb{R}. \tag{33}$$

We have immediately:

$$r'(t) = R(1-\cos t, \sin t) = R[(1, 0)-e^{it}], k(t) = -\frac{1}{4R\sin\frac{t}{2}}, \|r'(t)\| = 2R|\sin\frac{t}{2}| \tag{34}$$

and then we restrict our definition domain to $(0, \pi)$. It follows:

$$k_{rot}(t) = -\frac{1}{4R\sin\frac{t}{2}} - Rt\sin\frac{t}{2} < 0, \quad k^{second}_{rot}(t) = -\frac{1 + t}{4R\sin\frac{t}{2}} < 0. \tag{35}$$

Again a natural parameter s is provided by: $t = 2\arccos\left(1 - \frac{s}{4R}\right)$. □

Example 10 Fix a graph $C : y = f(x)$, $x \in I$ with the second derivative f'' strictly positive. With the usual parametrization $C : r(t) = (t, f(t))$ we have:

$$r'(t) = (1, f'(t)), \quad k(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}} > 0, \quad \|r(t)\|^2 = t^2 + f^2(t) \tag{36}$$

which gives that C is convex and:

$$k_{rot}(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}} - \frac{t + f(t)f'(t)}{[1 + (f'(t))^2]^{\frac{1}{2}}}, k_{rot}^{second}(t) = \frac{f''(t) - t - f(t)f'(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}}. \tag{37}$$

It follows that the function f making k_{rot}^{second} constant zero is a solution of the Riccati equation:

$$2f'(t) = f^2(t) + t^2 \geq 0 \tag{38}$$

and the explicit solution is an increasing function which involves the Bessel function of the first kind. □

Example 11 The derivate curve r' from (31) is an Archimedes' spiral. The general such spiral is given in polar coordinates as:

$$A(spiral) : \rho(t) = Rt, \quad R > 0 \tag{39}$$

and hence:

$$k_{rot}(t) = \frac{2 + t^2}{R(t^2 + 1)^{\frac{3}{2}}} - \frac{Rt}{(t^2 + 1)^{\frac{1}{2}}}, \quad k_{rot}^{second}(t) = \frac{t^2 - t + 2}{R(t^2 + 1)^{\frac{3}{2}}} > 0. \tag{40}$$

□

Example 12 Fix $\alpha \in \mathbb{R}^*$ and the naturally parametrized curve C. Then the *α-parallel curve* of C is the new curve:

$$C_\alpha : \tilde{r}(t) := r(t) + \alpha N(t), \quad t \in I \tag{41}$$

with:

$$\tilde{T}(t) = \frac{1 - \alpha k(t)}{|1 - \alpha k(t)|} T(t), \quad \tilde{N}(t) = \frac{1 - \alpha k(t)}{|1 - \alpha k(t)|} N(t), \quad \tilde{k}(t) = k(t). \tag{42}$$

Hence, we consider that α does not belongs to the range of the function $\frac{1}{k}$ and the new rotational curvature is:

$$\tilde{k}_{rot}(t) = k(t) - \frac{1}{|1 - \alpha k(t)|}[k_{rot}(t) - k(t) - \alpha < r(t), T(t) >]. \tag{43}$$

□

We end this note with an expression of the rotational curvature in terms of Fermi-Walker derivative. Let $\mathcal{X}_\gamma$ be the set of vector fields along the curve γ. Then the Fermi-Walker derivative is the map ([4]) $\nabla^{FW}_\gamma : \mathcal{X}_\gamma \to \mathcal{X}_\gamma$:

$$\nabla^{FW}_\gamma(X) := \frac{d}{dt}X + k[< X, N > T - < X, T > N] =$$
$$= \frac{d}{dt}X + k[X^\flat(N)T - X^\flat(T)N] \tag{44}$$

with $X^\flat$ the differential 1-form dual to X with respect to the Euclidean metric. For $X = \xi \circ r = \xi|_\gamma$ we have:

$$\nabla^{FW}_\gamma(\xi \circ r) = N + k[< \xi \circ r, N > T - < \xi \circ r, T > N] \tag{45}$$

and then we consider only the tangent component:

$$k < \xi \circ r, N >= [\nabla^{FW}_\gamma(\xi \circ r)]^T =< \nabla^{FW}_\gamma(\xi \circ r), T > . \tag{46}$$

Hence, if C is not a line then its rotational curvature is:

$$k_{rot} = k - \frac{1}{k} < \nabla^{FW}_\gamma(\xi \circ r), T > . \tag{47}$$

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