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# Conics from the Cartan Decomposition of $SO(2, 1)$

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**Abstract:** The aim of this paper is to introduce and study the class of conics provided by the symmetric matrices of the Cartan decomposition of the Lie group  $SO(2, 1)$ . This class depends on two real parameters as components of the cylinder  $S^1 \times \mathbb{R}$  and we use a deformation inspired by Finsler indicatrices in order to obtain proper ellipses. A complex approach is also included.

**Keywords:** conic; Cartan decomposition of  $SO(2, 1)$ ; complex variable

**MSC:** 11D09; 51N20; 30C10; 22E47

## 1. Introduction

After more than two thousand years, conics continue to be a versatile object of mathematics and the very recent book, [1], is veritable proof of this fact. A lot of techniques, from analytical to projective, have been developed to handle these remarkable curves.

The starting point of note is the article [2], where symmetric Pythagorean triple preserving (PTP) matrices are used to generate conics. Hence, we continue this line of research with the following other classes of symmetric matrices of order 3: (i) those produced by the adjoint representation of the 3-dimensional matrix Lie group  $SU(2) = S^3$  in [3]; and (ii) the magic matrices in [4].

A fourth class of remarkable matrices are provided by the Cartan decomposition of the Lie group  $SO(2, 1)$ , and the present paper studies the associated conics. More precisely, we obtain only degenerated conics and, hence, we perform a translation inspired by the notion of indicatrix from Finsler geometry. In Section 1, ellipses indexed by the product  $S^1 \times \mathbb{R}$ , but having a rotational symmetry, more precisely, having a canonical form depending only on the parameter  $\beta \in \mathbb{R}$  are presented. Their eccentricity depends bijectively on  $\beta$ , and, hence, we can express the canonical form only in terms of eccentricity.

Two classes of examples are discussed in Section 2, namely self-complementary ellipses, i.e., with eccentricity  $\frac{1}{\sqrt{2}}$ , and symmetric ellipses, i.e., with a common coefficient for  $x^2$  and  $y^2$ . A canonical conic function is also computed for our eccentricity, depending on  $\beta$ . We finish the section 2 with the expression of the fixed points of the linear fractional (or Möbius) function associated to the two by two symmetric  $SO(2, 1)$ -matrices. These fixed points do not depend on  $\beta$ .

In the third section we discuss our conic in terms of its three Hermitian coefficients and another complex number, called *affix*. Another remarkable class of examples appears when its two complex Hermitian coefficients are pure real. We note that some hard computations were performed with WolframAlpha.

## 2. Conics Provided by the Cartan Decomposition of $SO(2, 1)$

In the setting of two-dimensional Euclidean space ( $\mathbb{R}^2, g_{can} = \text{diag}(1, 1)$ ) let us consider the conic  $\Gamma$  implicitly defined by  $f \in C^\infty(\mathbb{R}^2)$  as:  $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid f_\Gamma(x, y) = 0\}$  where  $f_\Gamma$  is a quadratic function of the form  $f_\Gamma(x, y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$  with  $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$ .



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The study of  $\Gamma$  is based on the symmetric matrices ( $e$  means extended):

$$\Gamma := \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix} \in Sym(2), \quad \Gamma^e := \begin{pmatrix} r_{11} & r_{12} & r_{10} \\ r_{12} & r_{22} & r_{20} \\ r_{10} & r_{20} & r_{00} \end{pmatrix} \in Sym(3). \tag{1}$$

In fact, the algebraic invariants associated to  $\Gamma$  are:

$$I := r_{11} + r_{22} = Tr\Gamma, \delta := \det \Gamma, \Delta := \det \Gamma^e, D := \delta + r_{11}r_{00} - r_{10}^2 + r_{22}r_{00} - r_{20}^2. \tag{2}$$

The necessity to search for remarkable symmetric matrices of order three follows.

A very useful three-dimensional matrix Lie group in both mathematics and physics is  $SO(2, 1)$  with its Lie algebra  $so(2, 1) \simeq su(1, 1)$ . Recall that  $O(2, 1)$  is the matrix group preserving the 3D Minkowski-Lorentz norm  $\|\vec{v} = (x, y, z)\|^2 = x^2 + y^2 - z^2$  and  $SO(2, 1)$  is its subgroup with determinant 1. An important property of  $SO(2, 1)$  is that it is semisimple. On the Lie algebra level we have the isomorphisms:  $so(2, 1) \simeq sp(2, \mathbb{R}) \simeq sl(2, \mathbb{R}) \simeq su(1, 1)$ . More precisely, with  $\gamma = diag(1, 1, -1)$  we have:

$$SO(2, 1) = \{S \in SL(3, \mathbb{R}); S \cdot \gamma \cdot S^t = \gamma\}, \quad so(2, 1) = \{\Gamma \in gl(3, \mathbb{R}); \Gamma^t = -\gamma \cdot \Gamma \cdot \gamma\}.$$

For  $SO(2, 1)$  the Cartan decomposition  $SO(2, 1) = K \cdot A \cdot K$  is well known, where the groups  $K \simeq SO(2) \times \{1\}$  and  $A \subset Sym(3)$  are given, respectively, by:

$$K(\alpha) := \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K, \quad A(\beta) := \begin{pmatrix} \cosh \beta & 0 & \sinh \beta \\ 0 & 1 & 0 \\ \sinh \beta & 0 & \cosh \beta \end{pmatrix} \in A. \tag{3}$$

Let  $H^+$  be the positive sheet of the hyperboloid  $\mathcal{H} : \|\vec{v}\|^2 = -1$  i.e.,  $z(v) > 0$ . Then  $H^+$  is exactly the homogeneous space  $SO(2, 1)/K$  since  $K$  is precisely the stabilizer of the point  $(0, 0, 1) \in H^+$ . The group  $K$  is isomorphic to the unit circle group  $(S^1, \cdot)$ , which is the 1-dimensional torus.

The  $KAK$  decomposition of  $SO(2, 1)$  means that every matrix  $X \in SO(2, 1)$  is a product:

$$X := X(\alpha_1, \beta, \alpha_2) = K(\alpha_1) \cdot A(\beta) \cdot K(\alpha_2), \quad \alpha_1, \alpha_2, \beta \in \mathbb{R} \tag{4}$$

and a straightforward computation gives the  $SO(2, 1)$ -matrix:

$$X(\alpha_1, \beta, \alpha_2) = \begin{pmatrix} \cos \alpha_1 \cos \alpha_2 \cosh \beta - \sin \alpha_1 \sin \alpha_2 & -\cos \alpha_1 \sin \alpha_2 \cosh \beta - \sin \alpha_1 \cos \alpha_2 & \cos \alpha_1 \sinh \beta \\ \sin \alpha_1 \cos \alpha_2 \cosh \beta + \cos \alpha_1 \sin \alpha_2 & -\sin \alpha_1 \sin \alpha_2 \cosh \beta + \cos \alpha_1 \cos \alpha_2 & \sin \alpha_1 \sinh \beta \\ \cos \alpha_2 \sinh \beta & -\sin \alpha_2 \sinh \beta & \cosh \beta \end{pmatrix}. \tag{5}$$

The generic matrix  $X$  is symmetric if, and only if,  $\alpha_2 = -\alpha_1$  and then we arrive at the symmetric  $SO(2, 1)$ -matrix:

$$X(\alpha, \beta) := X(\alpha, \beta, -\alpha) = \begin{pmatrix} \cos^2 \alpha \cosh \beta + \sin^2 \alpha & \cos \alpha \sin \alpha (\cosh \beta - 1) & \cos \alpha \sinh \beta \\ \sin \alpha \cos \alpha (\cosh \beta - 1) & \sin^2 \alpha \cosh \beta + \cos^2 \alpha & \sin \alpha \sinh \beta \\ \cos \alpha \sinh \beta & \sin \alpha \sinh \beta & \cosh \beta \end{pmatrix} \tag{6}$$

which has the trace  $TrX = 1 + 2 \cosh \beta \geq 3$  and the eigenvalues  $\lambda_1 = 1, \lambda_2 = \cosh \beta - \sinh \beta = e^{-\beta}, \lambda_3 = \cosh \beta + \sinh \beta = e^{\beta}$ . For the sake of comparison, we mention that the set  $\{Y \in SO(3) \cap Sym(3); TrY = -1\}$  is described by:

$$Y(\alpha, \beta) := \begin{pmatrix} \cos^2 \alpha \cos 2\beta - \sin^2 \alpha & \cos^2 \alpha \sin 2\beta & \sin 2\alpha \cos \beta \\ \cos^2 \alpha \sin 2\beta & -\cos^2 \alpha \cos 2\beta - \sin^2 \alpha & \sin 2\alpha \sin \beta \\ \sin 2\alpha \cos \beta & \sin 2\alpha \sin \beta & -\cos 2\alpha \end{pmatrix}, \lambda_1 = \lambda_2 = -1, \lambda_3 = 1.$$

Let us consider the universal (i.e., independent of  $\alpha$  and  $\beta$ ) not special and not symmetrical orthogonal matrix:

$$S = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \in SO(3), \quad S^{-1} = S^t = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and then  $S^{-1} \cdot X(\alpha, \beta) \cdot S$  is:

$$\begin{pmatrix} \sin^2 \alpha \cosh \beta & \frac{\sin \alpha}{\sqrt{2}} [\cos \alpha (\cosh \beta - 1) - \sinh \beta] & \frac{\sin \alpha}{\sqrt{2}} [\cos \alpha (\cosh \beta - 1) + \sinh \beta] \\ \frac{\sin \alpha}{\sqrt{2}} [\cos \alpha (\cosh \beta - 1) - \sinh \beta] & \frac{1}{2} [(\cos^2 \alpha + 1) \cosh \beta + \sin^2 \alpha] & -\frac{\sin^2 \alpha}{2} (\cosh \beta - 1) \\ \frac{\sin \alpha}{\sqrt{2}} [\cos \alpha (\cosh \beta - 1) + \sinh \beta] & -\frac{\sin^2 \alpha}{2} (\cosh \beta - 1) & \frac{1}{2} [(\cos^2 \alpha + 1) \cosh \beta + \sin^2 \alpha] \end{pmatrix} + \cos \alpha \cdot \text{diag}(\cos \alpha, -\sinh \beta, \sinh \beta).$$

Since  $X(0, \beta) = A(\beta)$ ,  $S$  gives the diagonal form of  $A(\beta)$ :  $S^t \cdot A(\beta) \cdot S = \text{diag}(1, e^{-\beta}, e^\beta) = \text{Exp}(\text{diag}(0, -\beta, \beta))$ . The associated spectral curve is:

$$R(\beta, \mu) = (\mu - e^{-\beta})(\mu - 1)(\mu - e^\beta)$$

is studied as Example 9.3 in ([5], p. 93).  $X(\alpha, \beta)$  yields the following class of conics:

**Definition 1.** A  $SO(2, 1)$ -conic is a conic depending on Cayley–Klein parameters  $(\alpha, \beta) \in S^1 \times \mathbb{R}$  in the form:

$$\Gamma(\alpha, \beta) : [\cos^2 \alpha (\cosh \beta - 1) + 1]x^2 + \sin 2\alpha (\cosh \beta - 1)xy + [\sin^2 \alpha (\cosh \beta - 1) + 1]y^2 + 2 \sinh \beta [\cos \alpha x + \sin \alpha y] + \cosh \beta = 0. \tag{7}$$

As functions we have  $f_{\Gamma(\alpha, \beta)}(x, y) = f_{\Gamma(\frac{\pi}{2} - \alpha, \beta)}(y, x)$ .

We immediately have:

**Proposition 1.** All invariants of  $\Gamma(\alpha, \beta)$  depend only on  $\beta$ :

$$\delta = \delta(\beta) = \cosh \beta \geq 1, I = I(\beta) = \cosh \beta + 1 \geq 2, D = D(\beta) = 2 \cosh \beta + 1 \geq 3, \Delta = 1. \tag{8}$$

Hence, any  $SO(2, 1)$ -conic  $\Gamma$  is an imaginary ellipse with eccentricity  $e_\Gamma$  depending, again, only on  $\beta$ :

$$e_\Gamma^2(\beta) = 1 - \cosh \beta \leq 0. \tag{9}$$

The only  $SO(2, 1)$ -circle is the void circle  $x^2 + y^2 + 1 = 0$  characterized by  $\beta = 0$  and  $X(\alpha, 0) = I_3$ .

This negative result inspired us to translate the given conic in the following way: if  $\Gamma : f(x, y) = 0$  is the initial conic, then it is natural to study its associated indicatrix  $f(x, y) = 1$ . The conic figuratrix of Finslerian functions were studied in [6]

### 3. The Translated $SO(2, 1)$ -Conics

Following the discussion above, we introduce:

**Definition 2.** A translated  $SO(2, 1)$ -conic is a conic depending on  $(\alpha, \beta) \in S^1 \times \mathbb{R}$  in the form:

$$\tilde{\Gamma}(\alpha, \beta) : [\cos^2 \alpha (\cosh \beta - 1) + 1]x^2 + \sin 2\alpha (\cosh \beta - 1)xy + [\sin^2 \alpha (\cosh \beta - 1) + 1]y^2 + 2 \sinh \beta [\cos \alpha x + \sin \alpha y] + \cosh \beta - 1 = 0. \tag{10}$$

We immediately have:

**Proposition 2.** All invariants of  $\tilde{\Gamma}(\alpha, \beta)$  depend only on  $\beta$ :

$$\tilde{\delta}(\beta) = \cosh \beta \geq 1, \quad \tilde{I}(\beta) = \cosh \beta + 1 \geq 2, \quad \tilde{D}(\beta) = -\tilde{\Delta}(\beta) = \cosh \beta - 1 \geq 0. \quad (11)$$

Hence, any translated  $SO(2, 1)$ -conic  $\tilde{\Gamma}$  is an ellipse with eccentricity  $\tilde{e} = e_{\tilde{\Gamma}}$  depending, again, only on  $\beta$ :

$$\tilde{e}(\beta) = \sqrt{1 - \frac{1}{\cosh \beta}} = \frac{\sqrt{2} \sinh \frac{\beta}{2}}{\sqrt{\cosh \beta}}. \quad (12)$$

The only translated  $SO(2, 1)$ -circle is the only degenerated one of this type, namely the double point

$O(x = 0, y = 0)$ , characterized by  $\beta = 0$ .

In fact, excepting this double point, the trigonometrical rotation of angle  $\frac{\pi}{2} - \alpha$  with the inverse is:

$$x = (\sin \alpha)\tilde{x} + (\cos \alpha)\tilde{y}, \quad y = (-\cos \alpha)\tilde{x} + (\sin \alpha)\tilde{y} \quad (13)$$

which gives the canonical form, independent of  $\alpha$ , and symmetric with respect to the  $O\tilde{y}$ -axis i.e., invariant with respect to the map  $\tilde{x} \rightarrow -\tilde{x}$ :

$$\tilde{\Gamma}(\beta) : \frac{\cosh \beta}{\cosh \beta - 1} \tilde{x}^2 + \frac{\cosh^2 \beta}{\cosh \beta - 1} (\tilde{y} + \tanh \beta)^2 - 1 = 0. \quad (14)$$

**Remark 1.** (i) The function  $\beta \rightarrow \tilde{e}(\beta)$ , given by (12), is a bijective one, so, then, we can express the canonical form (14) entirely in terms of  $\tilde{e}$ :

$$\tilde{\Gamma}(\tilde{e}) : \frac{\tilde{x}^2}{\tilde{e}^2} + \frac{(\tilde{y} + \tilde{e}\sqrt{2 - \tilde{e}^2})^2}{\tilde{e}^2(1 - \tilde{e}^2)} - 1 = 0. \quad (15)$$

Furthermore:  $\tilde{\delta}(\tilde{e}) = \frac{1}{1 - \tilde{e}^2}$ ,  $\tilde{I}(\tilde{e}) = \frac{2 - \tilde{e}^2}{1 - \tilde{e}^2}$ ,  $\tilde{D}(\tilde{e}) = -\tilde{\Delta}(\tilde{e}) = \frac{\tilde{e}^2}{1 - \tilde{e}^2}$ . Two proper ellipses of canonical equation:

$$\tilde{\Gamma}(\tilde{e}) : \frac{X^2}{\tilde{e}^2} + \frac{Y^2}{\tilde{e}^2(1 - \tilde{e}^2)} - 1 = 0$$

cannot be confocal since the two conditions:

$$\tilde{e}_2^2 - \tilde{e}_1^2 = \lambda, \quad \tilde{e}_2^2(1 - \tilde{e}_2^2) - \tilde{e}_1^2(1 - \tilde{e}_1^2) = \lambda$$

yield an impossible relation  $\tilde{e}_1^4 + \tilde{e}_2^4 = 0$ . The area enclosed by the ellipse  $\tilde{\Gamma}(\tilde{e})$  is:

$$\mathcal{A}(\tilde{e}) = \pi \tilde{a} \tilde{b} = \pi \tilde{e}^2 \sqrt{1 - \tilde{e}^2} = \frac{\pi(\cosh \beta - 1)}{(\cosh \beta)^{\frac{3}{2}}}.$$

(ii) In ([7], p. 360) a function is defined, called the canonical conic function, on a set of ellipses with the same eccentricity  $e$  as:

$$C_{\text{ellipse}}(e) := \frac{1}{8(1 - e^2)^2} \left[ \pi \sqrt{1 - e^2} - 2e + 2e^3 - 2\sqrt{1 - e^2} \arcsin e \right]. \quad (16)$$

For our translated  $SO(2, 1)$ -ellipse  $\tilde{\Gamma}$  we obtain:

$$C_{\text{ellipse}}(\tilde{e}(\beta)) = \frac{(\cosh \beta)^{\frac{3}{2}}}{8} \left[ \pi - \frac{2\tilde{e}(\beta)}{\sqrt{\cosh \beta}} - 2 \arcsin \sqrt{1 - \frac{1}{\cosh \beta}} \right]. \quad (17)$$

(iii) In [8] a quaternion-inspired (but non-internal) product is considered on the set  $(0, 1]$ :

$$u_1 \odot_c u_2 = \sqrt{\frac{(1 - u_1 u_2)^2}{u_1^2 + u_2^2}}. \tag{18}$$

The  $\odot_c$ -square of our eccentricity is:

$$\tilde{e}_{\odot_c}^2 = \sqrt{\frac{2 \cosh \beta - 1}{2 \cosh \beta (\cosh \beta - 1)}}. \tag{19}$$

(iv) A notion specific to hyperbolic geometry is Lobachevsky’s angle of parallelism function  $\Pi$  defined by ([4], p. 141):  $\sin \Pi(x) = \frac{1}{\cosh x}$ . Then, the eccentricity is expressed in terms of  $\Pi$ , as:

$$\tilde{e}(\beta) = \left| \cos \frac{\Pi(\beta)}{2} - \sin \frac{\Pi(\beta)}{2} \right|. \tag{20}$$

(v) Recall that, in ([9], p. 16), the motion in Newton’s gravitational potential is governed by the effective potential:

$$V_{eff}(r) := V(r) + \frac{L^2}{2r^2} = -\frac{M}{r} + \frac{L^2}{2r^2}$$

and that the bounded motions are ellipses with the eccentricity  $e = \frac{kL^2}{M}$  for a constant  $k \geq 0$ . For  $L > 0$  and  $k > 0$  let us call Kepler ellipses these bounded trajectories. Hence, for a translated  $SO(2, 1)$ -Kepler ellipse we have  $M = \frac{kL^2 \sqrt{\cosh \beta}}{\sqrt{\cosh \beta - 1}}$  and the effective potential is:

$$V_{eff}(r) = \frac{L^2}{2r^2} \left( 1 - \frac{2kr \sqrt{\cosh \beta}}{\sqrt{\cosh \beta - 1}} \right).$$

(vi) It is well known that the locus of points with orthogonal tangents to the conic  $\Gamma$  is the Monge (or director) circle with general equation:

$$\mathcal{C}(\Gamma) : \delta(x^2 + y^2) - 2 \begin{vmatrix} r_{12} & r_{10} \\ r_{22} & r_{20} \end{vmatrix} x + 2 \begin{vmatrix} r_{11} & r_{10} \\ r_{12} & r_{20} \end{vmatrix} y + \begin{vmatrix} r_{12} & r_{10} \\ r_{10} & r_{00} \end{vmatrix} + \begin{vmatrix} r_{22} & r_{20} \\ r_{20} & r_{00} \end{vmatrix} = 0.$$

For our translated  $SO(2, 1)$ -conic it results in the circle:

$$\mathcal{C}(\Gamma) : x^2 + y^2 + 2 \tanh \beta (x \cos \alpha + y \sin \alpha) + \frac{(\cosh \beta - 1)^2}{\cosh \beta} (\sin \alpha \cos \alpha + \sin^2 \alpha) = \cosh \beta - 1.$$

(vii) The Joukowski map  $J : \mathbb{C}^* \rightarrow \mathbb{C}$ ,  $J(z) := z + \frac{1}{z}$  transforms the circle of radius  $r > 1$  into the ellipse  $E(r)$  with  $a = r + \frac{1}{r}$  and  $b = r - \frac{1}{r}$ ; hence, the eccentricity of  $E(r)$  is  $\frac{2r}{r^2+1}$ . The last ratio is equal to  $\tilde{e}(\beta)$  if, and only if:

$$r \in \left\{ \frac{\sqrt{\cosh \beta} - 1}{\sqrt{\cosh \beta - 1}}, \frac{\sqrt{\cosh \beta} + 1}{\sqrt{\cosh \beta - 1}} \right\}$$

but only the second value is greater than 1. Hence:

$$r = r(\beta) = \frac{\sqrt{\cosh \beta} + 1}{\sqrt{\cosh \beta - 1}} > 1, a = a(\beta) := \frac{2\sqrt{\cosh \beta}}{\sqrt{\cosh \beta - 1}} = \frac{2}{\tilde{e}(\beta)}, b = b(\beta) := \frac{2}{\sqrt{\cosh \beta - 1}}.$$

**Example 1.** A special class of ellipses is called self-complementary and given by  $\tilde{e} = \frac{1}{\sqrt{2}} = 0.7071 \dots$ ; see details in [4]. It follows  $\cosh \beta = 2$ ,  $\sinh \beta = \sqrt{3}$  which means  $\beta = \ln(2 + \sqrt{3})$ ,  $\Pi(\beta) = \frac{\pi}{6}$ ,  $r = \sqrt{2} + 1 > 1$  and the self-complementary ellipse depends on  $\alpha$ :

$$\begin{cases} \tilde{\Gamma}^s(\alpha) : (\cos^2 \alpha + 1)x^2 + \sin 2\alpha xy + (\sin^2 \alpha + 1)y^2 + 2\sqrt{3}[(\cos \alpha)x + (\sin \alpha)y] + 1 = 0, \\ \tilde{\delta} = 2, \tilde{I} = 3, \tilde{D} = -\tilde{\Delta} = 1, \mathcal{C} : x^2 + y^2 + \sqrt{3}(x \cos \alpha + y \sin \alpha) + \frac{\sin \alpha \cos \alpha + \sin^2 \alpha}{2} - 1 = 0. \end{cases} \tag{21}$$

In particular:

$$\tilde{\Gamma}^s(0) : 2x^2 + y^2 + 2\sqrt{3}x + 1 = 0, \quad \tilde{\Gamma}^s(\pi) : 2x^2 + y^2 - 2\sqrt{3}x + 1 = 0. \tag{22}$$

Another form of the general ellipse  $\tilde{\Gamma}^s$  is:

$$\tilde{\Gamma}^s(\alpha) : x^2 + y^2 + [(\cos \alpha)x + (\sin \alpha)y + \sqrt{3}]^2 - 2 = 0. \tag{23}$$

and the canonical form (5), equivalently (6), is:

$$\tilde{\Gamma}^s(\alpha) : 2\tilde{x}^2 + 4\left(\tilde{y} + \frac{\sqrt{3}}{2}\right)^2 - 1 = 0 \tag{24}$$

with center  $\tilde{O}(0, -\frac{\sqrt{3}}{2})$  in  $(\tilde{x}, \tilde{y})$ -coordinates. In the Equation (2.14) we observe the circle  $\mathcal{C}(O, r = \sqrt{2})$  and the double line  $d_\alpha : (\cos \alpha)x + (\sin \alpha)y + \sqrt{3} = 0$ , so, in terms of [10], the conic  $\tilde{\Gamma}^s(\alpha)$  belongs to a bitangent pencil. We point out that a tube of radius  $\beta = \ln(2 + \sqrt{3})$  appears in the classification result of [11].

**Remark 2.** For  $\beta \neq 0$  we can divide the Equation (1) to  $\cosh \beta - 1$  and get the limit  $\beta \rightarrow +\infty$ . The double line follows:

$$\tilde{\Gamma}(\alpha, +\infty) : (\cos \alpha)x + (\sin \alpha)y + 1 = 0. \tag{25}$$

The rotation (13) yields the horizontal line  $\tilde{\Gamma}(\alpha, +\infty) : \tilde{y} = -1$ .

**Example 2.** The general conic  $\Gamma$  is called symmetric if  $r_{11} = r_{22}$ . For (1) this means  $\alpha \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$  and then we have two symmetrically-translated  $SO(2, 1)$ -ellipses:

$$\begin{cases} \tilde{\Gamma}(\frac{\pi}{4}, \beta) : (\cosh \beta + 1)(x^2 + y^2) + 2(\cosh \beta - 1)(xy + 1) + 2\sqrt{2} \sinh \beta(x + y) = 0, \\ \tilde{\Gamma}(\frac{3\pi}{4}, \beta) : (\cosh \beta + 1)(x^2 + y^2) + 2(\cosh \beta - 1)(xy + 1) + 2\sqrt{2} \sinh \beta(x - y) = 0. \end{cases} \tag{26}$$

or in the form of (6):

$$\begin{cases} \tilde{\Gamma}(\frac{\pi}{4}, \tilde{e}) : (2 - \tilde{e}^2)(x^2 + y^2) + 2\tilde{e}^2(xy + 1) + 2\tilde{e}\sqrt{2(2 - \tilde{e}^2)}(x + y) = 0, \\ \tilde{\Gamma}(\frac{3\pi}{4}, \tilde{e}) : (2 - \tilde{e}^2)(x^2 + y^2) + 2\tilde{e}^2(xy + 1) + 2\tilde{e}\sqrt{2(2 - \tilde{e}^2)}(x - y) = 0. \end{cases} \tag{27}$$

In particular, we have the self-complementary symmetrically-translated  $SO(2, 1)$ -ellipses:

$$\begin{cases} \tilde{\Gamma}(\frac{\pi}{4}, \tilde{e} = \frac{1}{\sqrt{2}}) : 3(x^2 + y^2) + 2(xy + 1) + 2\sqrt{6}(x + y) = 0, \\ \tilde{\Gamma}(\frac{3\pi}{4}, \tilde{e} = \frac{1}{\sqrt{2}}) : 3(x^2 + y^2) + 2(xy + 1) + 2\sqrt{6}(x - y) = 0. \end{cases} \tag{28}$$

The first ellipse (26) is symmetric with respect to the first bisectrix, due to invariance with respect to the linear transform  $(x, y) \rightarrow (y, x)$ .

To the  $2 \times 2$  matrix  $\Gamma \in \text{Sym}(2)$  we associate the linear fractional (or Möbius) function  $f_\Gamma : \mathbb{R}P^1 = \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R}P^1$ :

$$f_\Gamma(t) := \frac{r_{11}t + r_{12}}{r_{12}t + r_{22}} \tag{29}$$

which is an involution if, and only if,  $\Gamma \in sl(2, \mathbb{R})$ . For our  $SO(2, 1)$ -matrix we obtain:

$$f_{\Gamma}(t) = \frac{[\cos^2 \alpha (\cosh \beta - 1) + 1]t + \sin \alpha \cos \alpha (\cosh \beta - 1)}{\sin \alpha \cos \alpha (\cosh \beta - 1)t + [\sin^2 \alpha (\cosh \beta - 1) + 1]} \tag{30}$$

and, since  $I = Tr\Gamma = \cosh \beta + 1 \geq 2$ , it follows that  $f_{\Gamma}$  is not an involution for  $\beta \neq 0$ . Indeed, a direct computation gives the square:

$$\Gamma^2 = \begin{pmatrix} 1 + \cos^2 \alpha (\cosh^2 \beta - 1) & \sin \alpha \cos \alpha \sinh^2 \beta \\ \sin \alpha \cos \alpha \sinh^2 \beta & 1 + \sin^2 \alpha (\cosh^2 \beta - 1) \end{pmatrix}$$

with  $Tr\Gamma^2 = \cosh^2 \beta + 1$  and the eigenvalues  $\lambda_1 = 1, \lambda_2 = \cosh^2 \beta$ .

Another important issue involves determining the fixed points of  $f_{\Gamma}$ , which are the solutions of the equation:

$$\sin \alpha \cos \alpha (\cosh \beta - 1)t^2 - \cos 2\alpha (\cosh \beta - 1)t - \sin \alpha \cos \alpha (\cosh \beta - 1) = 0. \tag{31}$$

Supposing that  $\beta \neq 0$  we must consider the equation:

$$\sin \alpha \cos \alpha t^2 - \cos 2\alpha t - \sin \alpha \cos \alpha = 0 \tag{32}$$

having the universal (i.e., not dependent on  $\alpha$ ) discriminant:

$$\Delta_{f_{\Gamma}} = 1. \tag{33}$$

In conclusion, for  $\alpha \in (0, \frac{\pi}{2})$  the function  $f_{\Gamma}$  has exactly two fixed points:

$$t_1 = -\tan \alpha \neq t_2 = \cot \alpha.$$

#### 4. A Complex Approach to Translated $SO(2, 1)$ -Conics

The aim of this section is to study the translated  $SO(2, 1)$ -conic  $\tilde{\Gamma}$  by using the complex structure of the plane. More precisely, with the usual notation  $z = x + iy \in \mathbb{C}$  we derive the complex expression of a general conic  $\Gamma$ :

$$\tilde{\Gamma} : F(z, \bar{z}) := Az^2 + Bz\bar{z} + \bar{A}\bar{z}^2 + Cz + \bar{C}\bar{z} + r_{00} = 0 \tag{34}$$

with:

$$A = \frac{r_{11} - r_{22}}{4} - \frac{r_{12}}{2}i \in \mathbb{C}, \quad 2B = r_{11} + r_{22} = I \in \mathbb{R}, \quad C = r_{10} - r_{20}i \in \mathbb{C}. \tag{35}$$

It follows that the usual rotation performed to eliminate the mixed term  $xy$  means to reduce/rotate  $A$  in the real line, while the translation which eliminates the term  $y$  has a similar meaning with respect to  $C$ . The inverse relationship between  $f$  and  $F$  is:

$$r_{11} = B + 2\Re A, \quad r_{22} = B - 2\Re A, \quad r_{12} = -2\Im A, \quad r_{10} = \Re C, \quad r_{20} = -\Im C \tag{36}$$

with  $\Re$  and  $\Im$  being, respectively, the real and imaginary parts. The conic is symmetric if, and only if,  $A$  is pure imaginary.

The linear invariant  $I$  and the quadratic invariant  $\delta$  are the traces which, respectively, are the determinants of the Hermitian matrix:

$$\Gamma^c = \begin{pmatrix} B & 2\bar{A} \\ 2A & B \end{pmatrix} \tag{37}$$

which is a special one, the entries of the main diagonal being equal; hence, their set is the three-dimensional subspace  $Sym(2)$  of the four-dimensional real linear space of  $2 \times 2$  Hermitian matrices. The square of the eccentricity is:

$$e^2 = \frac{4|A|}{2|A| - B \operatorname{sign}(\Delta)} \tag{38}$$

where  $\operatorname{sign}$  means the signum function and  $|z|$  is the modulus of the complex number  $z$ .

For our translated  $SO(2, 1)$ -conic (10) we have the new coefficients, which we call *Hermitian*:

$$\begin{cases} A(\alpha, \beta) = \frac{\cosh \beta - 1}{4} e^{-2\alpha i}, & B = B(\beta) = \frac{\cosh \beta + 1}{2} \geq 1, & C(\alpha, \beta) = \sinh \beta e^{-i\alpha}, \\ A(\alpha, \tilde{e}) = \frac{\tilde{e}^2}{4(1-\tilde{e}^2)} e^{-2\alpha i}, & B = B(\tilde{e}) = \frac{2-\tilde{e}^2}{2(1-\tilde{e}^2)}, & C(\alpha, \tilde{e}) = \frac{\tilde{e}\sqrt{2-\tilde{e}^2}}{1-\tilde{e}^2} e^{-i\alpha}. \end{cases} \tag{39}$$

which satisfy the quadratic relation:

$$8AB = C^2. \tag{40}$$

Hence, a multiplication with  $8B$  of the Equation (10) gives a new relation for the translated  $SO(2, 1)$ -conic, expressed only in  $B$  and  $C$ :

$$\tilde{\Gamma} : (Cz)^2 + 2(2B|z|)^2 + (\bar{C}\bar{z})^2 + 8B(Cz + \bar{C}\bar{z} + 2B - 2) = 0. \tag{41}$$

**Example 3.** Both  $A$  and  $C$  from (6) are real if, and only if,  $\alpha \in \mathbb{Z}\pi$  and, then, we obtain the real-translated  $SO(2, 1)$ -conic:

$$\begin{cases} \tilde{\Gamma}^r = \tilde{\Gamma}(k\pi, \beta) : (\cosh \beta)x^2 + y^2 + 2[(-1)^k \sinh \beta]x + (\cosh \beta - 1) = 0, & k \in \{0, 1\} \\ A^r = A^r(\beta) = \frac{\cosh \beta - 1}{4}, & C^r = C^r(k, \beta) = (-1)^k \sinh \beta. \end{cases} \tag{42}$$

The ellipse  $\tilde{\Gamma}^r$  is symmetric with respect to the  $Ox$ -axis due to the invariance with respect to the symmetry  $(x, y) \rightarrow (x, -y)$ . We have two families:

$$\begin{cases} \tilde{\Gamma}_0^r(\beta) = \tilde{\Gamma}^r(k = 0, \beta) : (\cosh \beta)x^2 + y^2 + 2 \sinh \beta x + (\cosh \beta - 1) = 0, \\ \tilde{\Gamma}_1^r(\beta) = \tilde{\Gamma}^r(k = 1, \beta) : (\cosh \beta)x^2 + y^2 - 2 \sinh \beta x + (\cosh \beta - 1) = 0. \end{cases} \tag{43}$$

With  $\cosh \beta = 2$  and  $\sinh \beta = \sqrt{3}$  we re-obtain the translated  $SO(2, 1)$ -self-complementary ellipses of the previous section:  $\tilde{\Gamma}_0^r(\text{self}) = \tilde{\Gamma}^s(0)$ ,  $\tilde{\Gamma}_1^r(\text{self}) = \tilde{\Gamma}^s(\pi)$ .

The triple  $(x, y, z) = (A^r, B, C^r)(\beta)$  is a curve on the infinite elliptic cone:

$$C : 8xy = z^2. \tag{44}$$

Considering  $(x, y, z)$  as homogeneous coordinates corresponding to  $X = \frac{x}{z}$ ,  $Y = \frac{y}{z}$ , it follows that the equilateral hyperbola is:  $XY = \frac{1}{8}$ .

**Example 4.** Inspired by the expression of  $A$  and  $C$  let us call the unit complex number:

$$z_{\tilde{\Gamma}} := e^{-\alpha i} \in S^1 \tag{45}$$

as being the affix of  $\tilde{\Gamma}$ . Then  $z_{\tilde{\Gamma}}$  belongs to  $\tilde{\Gamma}$  if, and only if,:

$$(\cosh \beta - 1)(\cos 2\alpha)^2 + 2 \sinh \beta \cos 2\alpha + \cosh \beta = 0 \tag{46}$$

with the unique solution:

$$\cos 2\alpha = \frac{\sqrt{\cosh \beta - 1} - \sinh \beta}{\cosh \beta - 1} = \frac{\sqrt{1 - \tilde{e}^2} - \sqrt{2 - \tilde{e}^2}}{\tilde{e}} < 0. \tag{47}$$



For the example of self-complementary ellipses we obtain:

$$\alpha = \frac{1}{2} \arccos(1 - \sqrt{3}) = 68.55^\circ. \quad (48)$$

To any  $\hat{z} \in \mathbb{C}^*$  we can associate two binary quadratic forms with null determinant:

$$F_{\hat{z}}^\pm(x, y) := (|\hat{z}| + \Re\hat{z})x^2 \pm 2(\Im\hat{z})xy + (|\hat{z}| - \Re\hat{z})y^2. \quad (49)$$

For our affix we have:

$$F_{\hat{z}}^\pm(x, y) = F^\pm(\alpha)(x, y) = (1 + \cos \alpha)x^2 \pm 2 \sin \alpha xy + (1 - \cos \alpha)y^2 = 2 \left( \cos \frac{\alpha}{2} x \pm \sin \frac{\alpha}{2} y \right)^2. \quad (50)$$

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