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Hyperbolic and Weak Euclidean Polynomials from Wronskian and Leibniz Maps

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Abstract: A real univariate polynomial is called hyperbolic or stable if all its roots are real. We search for hyperbolic polynomials of two and three degrees by using the Wronskian map W and a dual map to W called Leibniz, since it involves the classical Leibniz rule for the derivative of a product of functions. In addition to hyperbolicity, we use these two methods to search for a class of polynomials introduced by the first author and now called weak Euclidean.

Keywords: hyperbolic polynomial; (weak) Euclidean polynomial; Wronskian; Leibniz map

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1. Introduction

We recently introduced, in the paper in [1], the notion of a *Euclidean* polynomial. We now refine this concept as follows.

Definition 1. *The polynomial*

$$P(X) = X^n + a_1X^{n-1} + \ldots + a_{n-1}X + a_n \in \mathbb{R}_n[X], \quad n \geq 2 \tag{1}$$

with complex roots $x_1, \ldots x_n$ is called weak Euclidean if the following equality holds:

$$x_1^2 + \ldots + x_n^2 = a_1^2 + \ldots + a_n^2. \tag{2}$$

For such a polynomial P, we call the positive number $E(P) = a_1^2 + \ldots + a_n^2 \geq 0$ its Euclidean norm. In addition, if P is hyperbolic, that is, all its roots are real (see [2]), then P will be called Euclidean.

Regarding this concept, we note the following:

Remark 1. ***(1)*** *When P, as defined above, runs through the set $\mathbb{R}_n^e[X]$ of Euclidean polynomials of degree n, the map $(a_1, \ldots a_n) \in \mathbb{R}^n \mapsto (x_1, \ldots x_n) \in \mathbb{R}^n$ gives an Euclidean correspondence between the relevant subsets of $\mathbb{R}^n$.*

(2) *For any fixed pair $(n, a) \in \mathbb{Z}_{\geq 2} \times \mathbb{R}_+$, there is at least one Euclidean polynomial $P_a \in \mathbb{R}_n[X]$ with the given norm $E(P_a) = a$. One such example is*

$$P_a(X) = X^{n-1}(X - \sqrt{a}). \tag{3}$$

(3) *In [1], the following geometric characterization involving the standard unit $(n-2)$-sphere S^{n-2} in $\mathbb{R}^{n-1}$ for weak Euclidean polynomials is given. The polynomial $P \in \mathbb{R}_n[X]$ is weak Euclidean if and only if*

$$(a_2 + 1, a_3, \ldots, a_n) \in S^{n-2}. \tag{4}$$

We point out that this characterization implies that $a_2 \leq 0$ and it is independent of the coefficient a_1, the negative of the sum of roots of P. In particular, for $P_a(X) = X^{n-1}(X - \sqrt{a})$, one observes that $(1, 0, \ldots, 0) \in S^{n-2}$.

(4) *More generally, for the polynomial P in* (1), *we define its Euclidean defect as*

$$ED(P) := (a_1^2 + \ldots + a_n^2) - (x_1^2 + \ldots + x_n^2) \in \mathbb{R}. \tag{5}$$

Hence, P is weakly Euclidean if and only if $ED(P) = 0$. *From the inequality between the arithmetic and geometric means, we have that*

$$ED(P) \leq a_1^2 + \ldots + a_n^2 - n\sqrt[n]{a_n^2}. \tag{6}$$

This gives the following necessary condition: If $a_1^2 + \ldots + a_n^2 \geq n(a_n)^{\frac{2}{n}}$, *then P is weakly Euclidean.*

(5) *It is well known that the characteristic polynomial of a symmetric matrix* Γ *with real entries is real-rooted. So, we can call* Γ *a (weak) Euclidean if its characteristic polynomial is so.*

(6) *The hyperbolic polynomials and their multivariate generalization (called Garding hyperbolic; see, for example,* [3]*) appear in a natural way in various mathematical settings from real algebraic geometry and discrete mathematics to PDEs and (polynomial) optimization. So, there is an increasing scientific interest in producing and studying some classes of hyperbolic polynomials.*

In an effort to investigate these classes of polynomials, one can start studying methods to obtain remarkable elements in the set $\mathbb{R}_n^h[X]$ of hyperbolic polynomials; recall after [4] that this set is a basic semialgebraic set. The aim of the present note is to find hyperbolic polynomials of low degrees by using two maps, the Wronksian and a dual one, which we call Leibniz. Recall that given two C^1-maps $f, g : I \subset \mathbb{R} \to \mathbb{R}$, their Wronskian is the map $W(f, g) : I \to \mathbb{R}$, given by

$$W(f, g) := \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g. \tag{7}$$

So, the dual map will be $L(f, g) : I \to \mathbb{R}$, given by $L(f, g) := fg' + f'g = (fg)'$. We point out that the idea to use the Wronskian map was inspired by the excellent book [5], where the Schapira Theorem concerning the Wronskian of a set of polynomials is discussed; see also page 338 of [6].

Hence, since all polynomials of one degree are already Euclidean, we search for hyperbolic and weak Euclidean polynomials of two and three degrees generated by these maps applied to polynomials of these degrees. We point out that our work is oriented mainly toward examples, and hence, sometimes the class of palindromic or reciprocal polynomials is involved in our computations.

The contents of this paper are as follows. Section 2 studies the pairs $(P_1 \in \mathbb{R}_2[X], P_2 \in \mathbb{R}_1[X])$. Some of the examples from this section are connected with the splitting problem, with an example from the Schapira Theorem and with the cubic palindrome. Section 3 has two subsections according to the cases $(P_1 \in \mathbb{R}_3[X], P_2 \in \mathbb{R}_1[X])$, respectively, both P_1 and $P_2 \in \mathbb{R}_2[X]$. Since this section deals with cubic polynomials, special attention is oriented to the depressed expressions. The fourth section concerns the palindromic cubic case obtaining the polynomial $(x + 1)^3$ as a fixed point for both partial Wronskian and Leibniz maps. We also note that in sections two and three, some partial Wronskian and Leibniz maps are described as affine maps, and moreover, for the cubic cases, an associated elliptic curve is given with its lattice points. In the next section, we introduce a particular sequence $\{P_n \in \mathbb{R}_n[X]; n \in \mathbb{N}^*\}$ of polynomials, which we call Rodrigues, since all polynomials are generated by a given quadratic polynomial G through a Rodrigues-type formula; the class of Legendre polynomials is our basic example. The computations of the previous sections are applied for P_1 and P_2. The last section concerns the conclusions and presents an open problem: the interesting question raised by one of the referees of whether some of the studied polynomials can be characterized through differential equations.

2. The Second Degree

The second degree is provided in our approach by the polynomials $P_1(X) = X^2 + aX + b \in \mathbb{R}_2[X]$ and $P_2(X) = X + c \in \mathbb{R}_1[X]$, and we directly obtain

Proposition 1. *The negative Wronskian of P_1 and P_2 is*

$$-W(P_1, P_2)(X) = X^2 + 2cX + (ac - b), \quad E(-W(P_1, P_2)) = 4c^2 + (ac - b)^2 \tag{8}$$

which is a hyperbolic polynomial if and only if $c^2 - ac + b \geq 0$. The equality case of this condition means that the dual polynomial of P_1, namely $P_1^*(X) = X^2 - aX + b$, is a hyperbolic one and c is one of its roots. If $ac = b$, then the polynomial $-W(P_1, P_2)$ is Euclidean.

Example 1. *(1) If $c = 0$, then the polynomial $-W(P_1, X)$ is hyperbolic if and only if $b \geq 0$. In particular, if $a = c = 0$ and $b = u^2 > 0$, then the non-hyperbolic $P_1(X) = X^2 + u^2$ is transformed through the map $-W(\cdot, X)$ into the hyperbolic polynomial $(X - u)(X + u)$.*
The weak Euclidean polynomials of two degrees are given by $b \in \{0, -2\}$, and hence, the weak Euclidean $P_1(X) = X^2 + aX - 2$ is transformed by $-W(\cdot, X)$ into the non-hyperbolic $X^2 + 2$. In conclusion, the map $-W(\cdot, X)$ does not preserve the set $\mathbb{R}_2^h[X]$, nor the complementary set $\mathbb{R}_2[X] \setminus \mathbb{R}_2^h[X]$.

(2) If $a = 0$, then the hyperbolicity reduces to $c^2 + b \geq 0$, and then the trivial hyperbolic (and Euclidean) $P_1(X) = X^2$ is transformed by the map $-W(\cdot, X + c)$ into the hyperbolic (and Euclidean) polynomial $X(X + 2c)$.

(3) If the initial polynomial P_1 is a hyperbolic one with the roots $\alpha, \beta \in \mathbb{R}$, then the hyperblicity condition for $-W(P_1, P_2)$ reads $c^2 + c(\alpha + \beta) + \alpha\beta \geq 0$.
An important source of strictly hyperbolic polynomials (i.e., having all real and distinct roots) is the splitting problem for a pair (a given prime number p, a monic polynomial $f \in \mathbb{Z}_n[X]$), which we present after [7]. Reducing the coefficients of f modulo p gives a new polynomial f_p, which may be reducible. Then, f admits a p-splitting if f_p is the product of distinct linear factors. For example, let $1 \leq q \leq p - 1$ and $f^q(X) = X^2 + q$. Then, f^q admits a p-splitting if and only if there exists $u, v \in \mathbb{Z}$ such that the polynomial $f^{q,u,v}(X) = X^2 + (-up)X + (vp + q)$ is strictly hyperbolic with the discriminant $\Delta = w^2$ for $w \in \mathbb{N}_+ := \{1, 2, 3, ...\}$.

Example 1.1.1. *of the paper in [7] concerns $q = 1$, and the list of available p begins with* $5, 13, 17, ...$ *Indeed, for $p = 5$, we have the data $(u = 1, v = 1 = w, \alpha = 2, \beta = 3)$, while for $p = 13$, we have the data $(u = 1, v = 3 = w, \alpha = 5, \beta = 8)$.*

Concerning the second map, we have

Proposition 2. *The Leibniz map of P_1 and P_2 satisfies*

$$\frac{1}{3}L(P_1, P_2)(X) = X^2 + \frac{2}{3}(a + c)X + \frac{1}{3}(ac + b), \tag{9}$$

which is a hyperbolic polynomial if and only if $(a + c)^2 - 3(ac + b) \geq 0$ and weak Euclidean if and only if $ac + b \in \{0, -6\}$. Hence, if $ac + b = 0$, then $\frac{1}{3}L(P_1, P_2)$ is an Euclidean polynomial.

Example 2. *If the initial P_1 is a hyperbolic one with $b \geq 0$, then the polynomial $\frac{1}{3}L(P_1, X)$ is also hyperbolic, since then $a^2 \geq 4b \geq 3b$. In the Euclidean particular case of $b = 0$, we obtain an Euclidean $\frac{1}{3}L(P_1, X)$ if and only if $ac \in \{0, -6\}$.*

Example 3. *As we already mentioned in the Introduction, the Schapiro Theorem is discussed and illustrated by an example on page 338 of the book in [6]. The example is as follows: The initial complex polynomials $\tilde{P}_1(z) = (1 + i)z^2 + (1 - i)z + 2$, $\tilde{P}_2(z) = (1 - i)z^2 + (1 + i)z + 2$ have the Wronskian $-4i(z^2 + 2z - 1)$ with the real roots $-1 \pm \sqrt{2}$. Hence, the linear subspace $span\{\tilde{P}_1, \tilde{P}_2\}$ of $\mathbb{C}_2[z]$ is also generated by the polynomials $Q_1(z) = z^2 + 1$, $Q_2(z) = z + 1$ having* *all real coefficients. If we transfer Q_1 and Q_2 into our P_1 (which is non-hyperbolic) and P_2, it results the coefficients $a = 0$, $b = c = 1$, and then*

$$-W(P_1, P_2) = X^2 + X - 1 \in \mathbb{R}_2^h[X], \quad \frac{1}{3}L(P_1, P_2)(X) = X^2 + \frac{2}{3}X + \frac{1}{3} \notin \mathbb{R}_2^h[X]. \tag{10}$$

It is amazing that the dual $*$ of the first polynomial is exactly $X^2 - X - 1$, having as positive root the Golden ratio ϕ; see ([1], p. 65).

An interesting class of hyperbolic polynomials consists in polynomials with their roots lying in an interval of length ≤ 4. The above roots $-1 \pm \sqrt{2}$, although for a complex polynomial, satisfy this condition, since $2\sqrt{2} < 4$.

Example 4. *The palindromic cubic polynomial $P(X) = uX^3 + vX^2 + vX + u$, $u \neq 0$ has the real root -1, and then it has the decomposition $P(X) = (X+1)[uX^2 + (v-u)X + u]$ and, dividing by u, it follows our polynomial P_1 with $a = \frac{v}{u} - 1$, $b = 1$. The initial cubic polynomial P is hyperbolic if and only if $\Delta = (v - 3u)(v + u) \geq 0$, which means that $a \in [-\infty, -2] \cup [2, +\infty]$. The limit case $a = \pm 2$ corresponds to the hyperbolic cubic $P(X) = u(X \pm 1)^3$.*

Example 5. *Fix the polynomial $P(X) = X^2 + aX + b \in \mathbb{R}_2[X]$ and its derivative $P'(X) = 2X + a \in \mathbb{R}_1[X]$. The Wronskian of P is, by definition, the Wronskian $W(P, P')$, and we obtain*

$$-\frac{1}{2}W(P)(X) = X^2 + aX + \left(\frac{a^2}{2} - b\right) \tag{11}$$

which is hyperbolic if and only if $4b \geq a^2$, and hence, a necessary condition for hyperbolicity of $W(P)$ is $b \geq 0$. Also, we have that P and $W(P)$ are simultaneously hyperbolic if and only if P is the square $P(X) = \left(X + \frac{a}{2}\right)^2 = -\frac{1}{2}W(P)(X)$. The polynomial $-\frac{1}{2}W(P)$ is weak Euclidean if and only if $a^2 - 2b \in \{0, -4\}$.

Remark 2. *The transformations $-W(\cdot, P_2)$, $\frac{1}{3}L(\cdot, P_2)$ can be viewed as affine maps as follows:*

$$-W(\cdot, P_2) : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ c & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 2c \\ 0 \end{pmatrix}, \tag{12}$$

respectively,

$$\frac{1}{3}L(\cdot, P_2) : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} 2/3 & 0 \\ c/3 & 1/3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 2c/3 \\ 0 \end{pmatrix}. \tag{13}$$

3. The Third Degree

The case of the third degree can be obtained in two ways: firstly from the pair $(P_1(X) = X^3 + aX^2 + bX + c \in \mathbb{R}_3[X], P_2(X) = X + d \in \mathbb{R}_1[X])$ and secondly from the pair $(P_1(X) = X^2 + aX + b \in \mathbb{R}_2[X], P_2(X) = X^2 + cX + d \in \mathbb{R}_2[X])$. We recall the discussion of *the depressed cubic equation*:

$$y^3 + py + q = 0, \quad D = D(p, q) := \frac{p^3}{27} + \frac{q^2}{4}. \tag{14}$$

Namely, if *the determinant* $D < 0$, the cubic equation has three distinct real solutions, while if $D = 0$, then the equation has three real solutions, out of which two are equal; for details, see [8].

3.1. The First Pair (P_1, P_2)

We begin with the Wronskian map.

Proposition 3. *For the Wronskian of the first pair, we have*

$$-\frac{1}{2}W_1(P_1, P_2)(X) = X^3 + \frac{1}{2}(a+3d)X^2 + adX + \frac{1}{2}(bd-c). \tag{15}$$

The translation $X = Y - \frac{a+3d}{6}$ *gives the depressed cubic polynomial*

$$\begin{cases} -\frac{1}{2}W_1(P_1, P_2)(Y) = Y^3 + pY + q, \quad p = -\frac{(a-3d)^2}{12} \leq 0, \\ q = \frac{(a+3d)^3}{3 \cdot 6^2} - \frac{ad(a+3d)}{6} + \frac{bd-c}{2}. \end{cases} \tag{16}$$

As an affine map, the transformation $-\frac{1}{2}W_1(\cdot, P_2)$ *is*

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \to \begin{pmatrix} 1/2 & 0 & 0 \\ d & 0 & 0 \\ 0 & d/2 & -1/2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 3d/2 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3,$$

and we point out that the 3×3 *matrix belongs to the Lie algebra* $sl(3, \mathbb{R})$.

Example 6. *If* $d = 0$, *then the coefficients and the determinant of the depressed cubic are*

$$p = -\frac{a^2}{12} \leq 0, \quad q = \frac{a^3}{108} - \frac{c}{2}, \quad 16D = c\left(c - \frac{a^3}{27}\right), \tag{17}$$

and the above map reduces to a linear one. There are two cases of a vanishing determinant.

Case 1: $c = 0$. *We have the hyperbolic polynomial:*

$$-\frac{1}{2}W_1(P_1, X)(X) = X^2\left(X + \frac{a}{2}\right) = Y^3 - \frac{a^2}{12}Y + \frac{a^3}{108}. \tag{18}$$

For $a = 1$, *we can associate the following singular cubic curve:*

$$y^2 = -54W(P_1, X)(x) = 108x^3 - 9x + 1$$

which has 12 *lattice points* $(0, \pm 1)$, $(1, \pm 10)$, $(5, \pm 116)$, $(8, \pm 235)$, $(16, 665)$, $(21, \pm 1000)$.

Case 2: $c = u^3 > 0$ *and* $a = 3u$. *We have the following polynomials:*

$$P_1(X) = X^3 + 3uX^2 + bX + u^3, -\frac{1}{2}W_1(P_1, X)(X) = X^3 + \frac{3u}{2}X^2 - \frac{u^3}{2} = Y^3 - \frac{3u^2}{4}Y - \frac{u^3}{4}. \tag{19}$$

The Y*-solutions of the second polynomial are* $u, -\frac{u}{2}, -\frac{u}{2}$. *With* $u = 1$, *we associate the following singular cubic curve:*

$$y^2 = -2W(P_1, X)(x) = 4x^3 - 3x - 1$$

with 11 *lattice points:* $(1, 0)$, $(2, \pm 5)$, $(5, \pm 22)$, $(10, \pm 63)$, $(17, \pm 140)$, $(26, \pm 265)$.

Example 7. *If* $a = 0$, *i.e., the initial* P_1 *is depressed, then the coefficients and the determinant of the resulted cubic are*

$$p = -\frac{3d^2}{4} \leq 0, \quad q = \frac{bd-c}{2} + \frac{d^3}{4}, \quad 16D = (bd-c)(bd-c+d^3). \tag{20}$$

The case $c = bd$ *of a vanishing* D *corresponds to the following hyperbolic cubic:*

$$-\frac{1}{2}W_1(X^3 + bX + bd, X + d)(X) = X^2\left(X + \frac{3d}{2}\right) = Y^3 - \frac{3d^2}{4}Y + \frac{d^3}{4}. \tag{21}$$

and Y*-solutions of the last depressed cubic are* $-d, \frac{d}{2}, \frac{d}{2}$.

We now turn to the Leibniz map.

Proposition 4. *For the Leibniz map of the first pair, we have*

$$\frac{1}{4}L_1(P_1,P_2)(X) = X^3 + \frac{3}{4}(a+d)X^2 + \frac{1}{2}(b+ad)X + \frac{1}{4}(bd+c). \tag{22}$$

The translation $X = Y - \frac{a+d}{4}$ gives the depressed cubic polynomial $\frac{1}{4}L_1(P_1,P_2)(Y) = Y^3 + pY + q$ with

$$16p = 8(b+ad) - 3(a+d)^2, \quad 64q = 16(bd+c) + 11(a+d)^2 - 8(a+d)(b+ad). \tag{23}$$

As an affine map, the transformation $\frac{1}{4}L_1(\cdot, P_2)$ is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \to \begin{pmatrix} 3/4 & 0 & 0 \\ d/2 & 1/2 & 0 \\ 0 & d/4 & 1/4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 3d/4 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3.$$

Example 8. *$d = 0$ reduces the above map to a linear one and gives $16p = 8b - 3a^2$, $64q = 16c + 11a - 8ab$. In particular, if $b = 0$, then $16p = -3a^2 \leq 0$.*

3.2. The Second Pair (P_1, P_2)

The Wronskian map of the second pair yields a polynomial of two degrees, and hence, we postpone this method to the end of this subsection.

Proposition 5. *For the Leibniz map of the second pair, we have*

$$\frac{1}{4}L_2(P_1,P_2)(X) = X^3 + \frac{3}{4}(a+c)X^2 + \frac{1}{2}(b+d+ac)X + \frac{1}{4}(ad+bc). \tag{24}$$

The translation $X = Y - \frac{a+c}{4}$ gives the depressed cubic polynomial $\frac{1}{4}L_2(P_1,P_2)(Y) = Y^3 + pY + q$ with

$$16p = 8(b+d+ac) - 3(a+c)^2, 32q = (a+c)^3 + 8(ad+bc) - 4(a+c)(b+d+ac). \tag{25}$$

Example 9. *Suppose that both P_1 and P_2 are Euclidean with $b = d = 0$. Then,*

$$16p = -3a^2 + 2ac - 3c^2 \leq 0, \quad 32q = (a+c)^3 - 4ac(a+c) = (a+c)(a-c)^2. \tag{26}$$

Hence, there are two cases of a vanishing q and therefore a hyperbolic $\frac{1}{4}L_2(P_1,P_2)$:
Case 1: If $a = c$, i.e., $P_1 = P_2$, we obtain $p = -\frac{a^2}{4} \leq 0$.
Case 2: If $a = -c$, then $p = -\frac{a^2}{2} \leq 0$.

We return now to the Wronskian:

Proposition 6. *The Wronskian map of the second pair is*

$$W_2(P_1,P_2)(X) = (a-c)X^2 + 2(b-d)X + (bc-ad) \tag{27}$$

which is a hyperbolic polynomial if and only if

$$(b-d)^2 \geq (a-c)(bc-ad). \tag{28}$$

This condition holds for $a = c$ when $W_2(P_1,P_2)$ is one degree.

Example 10. *Suppose that $a \neq c$ and P_1, P_2 are Euclidean with $b = -2$ and $d = 0$. The resulting Wronskian is*

$$W_2(P_1, P_2)(X) = (a - c)X^2 - 4X - 2c \tag{29}$$

which is hyperbolic if and only if $c(a - c) \geq -2$. For example, if P_2 is exactly the dual P_1^*, then $c = -a$, and we obtain the following palindromic:

$$W_2(P_1, P_2)(X) = 2(aX^2 - 2X + a) \tag{30}$$

which is hyperbolic if and only if $a \in [-1, 1]$; irrespective of the value of a, the roots of the polynomial (30) are located in an interval of length < 4.

Example 11. *Suppose that both P_1 and P_2 are hyperbolic with the roots (α, β) and (γ, δ), respectively. Then, we have all the previous computations with*

$$a = -(\alpha + \beta), \quad b = \alpha\beta, \quad c = -(\gamma + \delta), \quad d = \gamma\delta. \tag{31}$$

4. The Palindromic Cubic Revisited

We now apply the computations of the previous section to the palindromic $P_p(X; a) = X^3 + aX^2 + aX + 1 \in \mathbb{R}_3[X]$. Proposition 3 gives

$$-\frac{1}{2}W_1(P_p, P_2)(X) = X^3 + \frac{1}{2}(a + 3d)X^2 + adX + \frac{1}{2}(ad - 1), \tag{32}$$

which is also palindromic if and only if $a = 3$ and $d = 1$. Hence, the following:

Proposition 7. *The palindromic hyperbolic polynomial $P_p(X) = X^3 + 3X^2 + 3X + 1 = (X + 1)^3$ is a fixed point of the map $-\frac{1}{2}W_1(\cdot, X + 1)$.*

Proposition 4 gives

$$\frac{1}{4}L_1(P_p, P_2)(X) = X^3 + \frac{3}{4}(a + d)X^2 + \frac{a}{2}(d + 1)X + \frac{1}{4}(ad + 1) \tag{33}$$

and the following:

Proposition 8. *The palindromic hyperbolic polynomial $P_p(X) = X^3 + 3X^2 + 3X + 1 = (X + 1)^3$ is a fixed point of the map $\frac{1}{4}L_1(\cdot, X + 1)$.*

The singular cubic curve $y^2 = (x + 1)^3$ again has 11 lattice points: $(-1, 0)$, $(0, \pm 1)$, $(3, \pm 8)$, $(8, \pm 27)$, $(15, \pm 64)$, $(24, \pm 125)$. With the translation $X = x + 1$, we have the semicubical parabola $y^2 = X^3$.

Example 12. *In the paper in [1], the case $a = -1$ is an example of a Euclidean cubic polynomial. From (32) and (33), we have the following polynomials:*

$$\begin{cases} W_1(d)(X) = X^3 + \frac{3d-1}{2}X^2 - dX - \frac{d+1}{2}, \\ L_1(d)(X) = X^3 + \frac{3(d-1)}{4}X^2 - \frac{d+1}{2}X + \frac{1-d}{4} \end{cases} \tag{34}$$

but both these polynomials are nonpalindromic. With the characterization in (4), we have that $W_1(d)$ is weak Euclidean only for $d_\pm = \frac{3 \pm \sqrt{19}}{5}$, while $L_1(d)$ is weak Euclidean only for $d_\pm = 1 \pm \frac{4}{\sqrt{5}}$.

5. Rodrigues Sequences of Polynomials

Fix a quadratic polynomial $G(X) = X^2 + \alpha X + \beta \in \mathbb{R}_2[X]$ and a sequence of polynomials $\mathcal{P} := \{P_n \in \mathbb{R}_n[X]; n \in \mathbb{N}^*\}$. Inspired by the theory of classical orthogonal polynomials ([9]), we introduce the following:

Definition 2. *The sequence $\mathcal{P}$ is called Rodrigues with respect to G if a Rodrigues-type formula holds the following:*

$$P_n(X) = \frac{1}{a_n} \frac{d^n}{dX^n} [G(X)]^n, \quad a_n = \frac{1}{(n+1) \cdot \ldots \cdot (2n)} = \frac{n!}{(2n)!}. \tag{35}$$

The coefficient a_n is chosen such that the coefficient of X^n in P_n is 1 and G can be called the generator of $\mathcal{P}$.

Example 13. *The classical Legendre polynomials $\mathcal{L} := \{L_n \in \mathbb{R}_n[X]; n \in \mathbb{N}^*\}$ are provided by the Rodrigues formula:*

$$L_n(X) = \frac{1}{2^n n!} \frac{d^n}{dX^n} [X^2 - 1]^n, \quad X \in [-1, 1]. \tag{36}$$

Hence, choosing $G(X) = X^2 - 1$ in the definition above, we obtain the modified Legendre polynomials $L_n^m = \frac{2^n n!}{a_n} L_n$.

Remark 3. *Returning to the general Rodrigues sequence, the first two polynomials in $\mathcal{P}$ are*

$$P_1(X) = X + \frac{\alpha}{2}, \quad P_2(X) = \frac{1}{6} L(G, G')(X) = X^2 + \alpha X + \frac{\alpha^2 + 2\beta}{6} \tag{37}$$

and then $P_2 = G$ if and only if $4\beta = \alpha^2$, which means that $G(X) = P_2(X) = \left(X + \frac{\alpha}{2}\right)^2$.

The aim of this section is to discuss the results of Section 2 on our P_1 and P_2.

(I) Applying Proposition 1, we obtain that $W(P_1, P_2)$ is hyperbolic if and only if

$$c^2 - ac + b^2 = \left(\frac{\alpha}{2}\right)^2 - \frac{\alpha^2}{2} + \frac{\alpha^2 + 2\beta}{6} \geq 0 \to 4\beta \geq \alpha^2. \tag{38}$$

Hence, $W(P_1, P_2)$ and G are simultaneously hyperbolic if and only if $4\beta = \alpha^2$, i.e., $G(X) = P_2(X) = \left(X + \frac{\alpha}{2}\right)^2$. If $\beta = \alpha^2$, the $W(P_1, P_2)$ is a Euclidean polynomial. $W(L_1^m, L_2^m)$ is not hyperbolic nor Euclidean.

(II) Applying Proposition 2, we have that $\frac{1}{3} L(P_1, P_2)$ is hyperbolic if and only if

$$(a + c)^2 - 3(ac + b) = \frac{\alpha^2 - 4\beta}{4} \geq 0. \tag{39}$$

Therefore, $\frac{1}{3} L(P_1, P_2)$ is hyperbolic if and only if the generator G is hyperbolic. If $\beta = -2\alpha^2$, then $\frac{1}{3} L(P_1, P_2)$ is a Euclidean polynomial, and $\frac{1}{3} L(L_1^m, L_2^m)$ is a non-Euclidean hyperbolic polynomial.

6. Conclusions and Future Works

Testing whether a given real monic polynomial is of hyperbolic type is a complicated job in general; see, for example, [10]. The main idea of the present study is that classes of hyperbolic and weak Euclidean polynomials of two and three degrees are generated in a unitary way through the Wronskian map and the Leibniz map. Special attention is given to polynomials having additional properties, such as the palindromic property.

The present work is only the first step in a series of papers that propose methods to obtain hyperbolic and weak Euclidean polynomials. One first future direction of study is to increase the degrees of the involved polynomials. A second variant is to fix $P_2(X) = X + a$ and to generate hyperbolic and weak Euclidean polynomials by using the Leibniz map $L(P_1, P_2) = (P_1 \cdot P_2)'$ from the Introduction. A third subject of interest is to search if the Nuij theorem ([4]) still works in the setting of weak Euclidean polynomials or possible variants of this famous result. Finally, since this paper only addresses the mathematical aspects of the proposed methods, an interesting point of view is to study the computational/algorithmic complexity of these methods.

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