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The flow-geodesic curvature and the flow-evolute of spherical curves

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Abstract: We introduce and study a deformation of the geodesic curvature for a given spherical curve γ. Also, we define a new type of evolute and two Fermi-Walker type derivatives for γ. Some concrete examples are detailed with a special attention towards space curves with a constant torsion.

Key words: Spherical curve, flow-geodesic curvature, flow-evolute

1. Introduction

The subject of curves on a given Euclidean surface $S \subset \mathbb{R}^3$ is a classical one but still preserves the flavor of a charming framework. Even more so if the given surface is a remarkable one, e.g. the unit sphere; recently, the curve shortening flow was studied on S^2 in [6]. The purpose of this work is to contribute to this setting with a deformation of the well-known geodesic curvature, somehow in the spirit of [9].

Recall that the geodesic curvature k_g of a curve $\gamma \subset S \subset \mathbb{R}^3$ is provided by an orthonormal frame $\mathcal{F}(\gamma, S)$ adapted to both γ and S; for S^2 we denote by $\mathcal{F}_s$ with s from "spherical". Our idea is to rotate this frame in the normal-radial bundle $\gamma^\perp := \{(x, v) \in (Im\ \gamma) \times S^2; v \perp x = \gamma(t) \in \mathbb{R}^3, t \in I\}$ (the notations are explained below) by an angle equal exactly to the parameter t of γ. Then we call this new one flow-frame and since we use an orthogonal transformation, i.e. a matrix from $SO(3)$, this frame yields a new curvature called flow-geodesic curvature; for the case of plane curves, this notion is already studied in [4]. In turn, this new function gives a new evolute for the given curve. As new tools in studying spherical curves we introduce a spherical, as well as a flow-spherical Fermi-Walker derivative, and both these derivatives are computed for our main vector fields along γ.

The contents of the paper is as follows. The first section is a short survey on spherical curves and we point out the relationship between k_g and the pair (curvature, torsion) of the given spherical curve γ considered a space curve. The second section gives the new curvature and the new evolute; a main result establishes the computational expression of the new curvature. The third section is concerned with several examples and some related remarks; a special attention is devoted to find flow-flat curves, i.e. spherical curves having a vanishing flow-geodesic curvature. Our study is connected through two examples with the subject of space curves having a constant torsion, a theme of great interest in contemporary differential geometry of curves.

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2. Preliminaries: spherical curves

The setting of this section is provided by the space $\mathbb{R}^3$ which is an Euclidean vector space with respect to the canonical inner product:

$$\langle u,v\rangle = u^1v^1+u^2v^2+u^3v^3, u=(u^1,u^2,u^3), v=(v^1,v^2,v^3)\in\mathbb{R}^3, 0\leq \|u\|^2=\langle u,u\rangle. \tag{2.1}$$

Let $S^2=SO(3)/SO(2)=SO(4)/U(2)$ be the unit sphere of $\mathbb{E}^3:=(\mathbb{R}^3,\langle\cdot,\cdot\rangle)$ and fix a smooth regular space curve which is a spherical one, $\gamma: I\subseteq\mathbb{R}\to S^2\subset\mathbb{R}^3$. Its spherical Frenet frame is:

$$\mathcal{F}_s := \begin{pmatrix}\gamma\\ \mathbf{t}\\ \mathbf{n}\end{pmatrix}, \quad \mathbf{t}(t):=\frac{\gamma'(t)}{\|\gamma'(t)\|}, \quad \mathbf{n}(t):=\gamma(t)\times\mathbf{t}(t), \tag{2.2}$$

and the corresponding spherical Frenet equation is provided by [11, p. 338]:

$$\frac{d}{dt}\mathcal{F}_s(t)=\|\gamma'(t)\|\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & k_g(t)\\ 0 & -k_g(t) & 0\end{pmatrix}\mathcal{F}_s(t). \tag{2.3}$$

The smooth real function $k_g: I\to\mathbb{R}$ is called the geodesic curvature of γ and its computational formula is:

$$k_g(t):=\frac{\langle\mathbf{t}'(t),\mathbf{n}(t)\rangle}{\|\gamma'(t)\|}=\frac{\det(\gamma(t),\gamma'(t),\gamma''(t))}{\|\gamma'(t)\|^3}. \tag{2.4}$$

Moreover, the usual curvature k of γ as a space curve is $k=\sqrt{k_g^2+1}\geq 1$ and the torsion of γ is $\tau=\frac{k'_g}{k_g^2+1}$. Recall also that γ is convex if $k_g>0$ and if $k_g=0$, we say that γ is a spherical-flat curve. Sometimes, another adapted frame is used, namely the Darboux-Ribaucour frame, which is connected to $\mathcal{F}_s$ through a cubic root of the unit matrix I_3:

$$\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \gamma\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\mathcal{F}_s(t), \quad R=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\in SO(3), \quad R^3=I_3.$$

The rotation matrix R is denoted by $\hat{q}$ at page 276 of [1]. The evolute of γ is a new spherical curve:

$$Ev(\gamma)(t):=\frac{k_g(t)}{k(t)}\gamma(t)+\frac{1}{k(t)}\mathbf{n}(t)\in S^2. \tag{2.5}$$

An important tool in dynamics along curves is the Fermi-Walker derivative. Let $\mathcal{X}_\gamma$ be the set of vector fields along the curve γ. Then the Fermi-Walker derivative is the map ([5]) $\nabla^{FW}_C:\mathcal{X}_\gamma\to\mathcal{X}_\gamma$:

$$\nabla^{FW}_\gamma(X):=\frac{d}{dt}X+\|\gamma'(\cdot)\|k[\langle X,N\rangle\mathbf{t}-\langle X,\mathbf{t}\rangle N] \tag{2.6}$$

for $(\mathbf{t},N,B)$ the usual Frenet frame of γ. Inspired by this expression, we introduce a spherical Fermi-Walker derivative:

$$\nabla^s_\gamma(X):=\frac{d}{dt}X+\|\gamma'(\cdot)\|k_g[\langle X,\mathbf{n}\rangle\mathbf{t}-\langle X,\mathbf{t}\rangle\mathbf{n}]. \tag{2.7}$$

The Fermi-Walker derivative for our main vector fields is:

$$\begin{cases} \nabla^{FW}_{\gamma}(\gamma)(t) = \gamma'(t), \quad \nabla^{FW}_{\gamma}(\mathbf{t}) = 0, \quad \nabla^{FW}_{\gamma}(N)(t) = \|\gamma'(t)\|\tau(t)B(t), \\ \nabla^{FW}_{\gamma}(Ev(\gamma))(t) = \frac{d}{dt}\left(\frac{k_g(t)}{k(t)}\right)\gamma(t) + \|\gamma'(t)\|k(t)\langle \mathbf{n}(t), N(t)\rangle \mathbf{t}(t) + \frac{d}{dt}\left(\frac{1}{k(t)}\right)\mathbf{n}(t). \end{cases} \tag{2.8}$$

Also, the spherical Fermi-Walker derivative of our main vector fields is:

$$\begin{cases} \nabla^{s}_{\gamma}(\gamma)(t) = \gamma'(t), \quad \nabla^{s}_{\gamma}(\mathbf{t})(t) = -\|\gamma'(t)\|\gamma(t), \quad \nabla^{s}_{\gamma}(\mathbf{n}) = 0, \\ \nabla^{s}_{\gamma}(Ev(\gamma))(t) = \frac{d}{dt}\left(\frac{k_g(t)}{k(t)}\right)\gamma(t) + \|\gamma'(t)\|\frac{k_g(t)}{k(t)}\mathbf{t}(t) + \frac{d}{dt}\left(\frac{1}{k(t)}\right)\mathbf{n}(t). \end{cases} \tag{2.9}$$

3. The flow-geodesic curvature and the flow-evolute of a spherical curve

The aim of this short note is to introduce a new curvature in order to find possible new features of spherical curves; our model is the case of plane curves studied in [4]. More precisely, we first introduce a new frame along γ, denoted by $\mathcal{F}^f$ and called the flow-spherical frame through:

$$\mathcal{F}^f_s(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \mathcal{F}_s(t) = \begin{pmatrix} \gamma \\ E^f_1 \\ E^f_2 \end{pmatrix}(t) \tag{3.1}$$

and the 3×3 matrix above being an element of the subgroup $\{1\}\times SO(2)$ of the special orthonormal group $SO(3)$, we have that $\mathcal{F}^f_s$ is also a positive oriented frame for γ. It follows that its moving equation:

$$\frac{d}{dt}\mathcal{F}^f_s(t) = \|\gamma'(t)\| \begin{pmatrix} 0 & \cos t & \sin t \\ -\cos t & 0 & k^f_g(t) \\ -\sin t & -k^f_g(t) & 0 \end{pmatrix} \mathcal{F}^f_s(t) \tag{3.2}$$

defines a new smooth function $k^f_g : I \to \mathbb{R}$ which we call the flow-geodesic curvature of γ and then if $k^f_g = 0$, we say that γ is a flow-flat spherical curve. We introduce the flow-evolute of γ as another spherical curve:

$$Ev^f(\gamma)(t) := \frac{k^f_g(t)}{\sqrt{[k^f_g(t)]^2+1}}\gamma(t) + \frac{1}{\sqrt{[k^f_g(t)]^2+1}}E^f_2(t) \in S^2. \tag{3.3}$$

We point out that $\mathbf{t}$, $\mathbf{n}$, E^f_1, and E^f_2 are also spherical curves.

A straightforward computation yields:

Theorem 3.1 *The expression of the flow-geodesic curvature is:*

$$k^f_g(t) = k_g(t) - \frac{1}{\|\gamma'(t)\|} = \frac{\det(\gamma(t), \gamma'(t), \gamma''(t)) - \|\gamma'(t)\|^2}{\|\gamma'(t)\|^3} < k_g(t). \tag{3.4}$$

Therefore, γ is a flow-flat spherical curve if and only if:

$$\det(\gamma(t), \mathbf{t}(t), \gamma''(t)) = \|\gamma'(t)\|. \tag{3.5}$$

In particular, if γ is parametrized by the arc-length s and is flow-flat, then we have the conservation law: $\det(\gamma(s), \mathbf{t}(s) = \gamma'(s), \gamma''(s)) = constant = 1$.

A setting where flow-flat curves may appear interesting is as follows: fix a remarkable map $\varphi : M^n \to S^2$ from a smooth n-dimensional manifold M^n, and a smooth curve $\Gamma : I \to M$. Then we call Γ as being φ-flow-flat if its imagine through φ is a flow-flat spherical curve. For example, any harmonic map from a simply connected Riemann surface Σ to S^2 gives rise to a spherical surface with singularities, called spherical frontals; here spherical surface means a surface in $\mathbb{R}^3$ with constant and positive Gaussian curvature, see the excellent survey in [3].

Example 3.2 *Another remarkable example of a map with the 2-sphere as target is the Hopf map, $H : \mathbb{C}^2 \setminus \{0\} \to S^2 \subset \mathbb{C} \times \mathbb{R}$:*

$$H(u,v) = \left(\frac{2u\bar{v}}{|u|^2+|v|^2}, \frac{|u|^2-|v|^2}{|u|^2+|v|^2}\right) \tag{3.6}$$

and then a curve in $\mathbb{C}^2 \setminus \{0\}$ will be Hopf-flow-flat if its image through H is a flow-flat spherical curve. □

Following the approach of the first section, we define now a flow-spherical Fermi-Walker derivative:

$$\nabla^{fs}_{\gamma}(X) := \frac{d}{dt}X + \|\gamma'(\cdot)\| k^f_g [\langle X, \mathbf{n}\rangle \mathbf{t} - \langle X, \mathbf{t}\rangle \mathbf{n}]. \tag{3.7}$$

The flow-spherical Fermi-Walker derivative of our main vector fields is:

$$\begin{cases} \nabla^{fs}_{\gamma}(\gamma)(t) = \gamma'(t), \quad \nabla^{fs}_{\gamma}(\mathbf{t})(t) = -\|\gamma'(t)\|\gamma(t) + \mathbf{n}(t), \quad \nabla^{fs}_{\gamma}(\mathbf{n})(t) = -\mathbf{t}(t), \\ \nabla^{fs}_{\gamma}(Ev(\gamma))(t) = \frac{d}{dt}\left(\frac{k_g(t)}{k(t)}\right)\gamma(t) + \|\gamma'(t)\| \frac{k^f_g(t)}{k(t)}\mathbf{t}(t) + \frac{d}{dt}\left(\frac{1}{k(t)}\right)\mathbf{n}(t). \end{cases} \tag{3.8}$$

and then the flow-spherical Fermi-Walker derivative for the elements of the flow-spherical frame is:

$$\nabla^{fs}_{\gamma}(E^f_1)(t) = -(\|\gamma'(t)\|\cos t)\gamma(t), \quad \nabla^{fs}_{\gamma}(E^f_2)(t) = -(\|\gamma'(t)\|\sin t)\gamma(t). \tag{3.9}$$

4. Examples and remarks

In what follows we are interested in computing this new function for some remarkable spherical curves.

Example 4.1 Recall the spherical coordinates $(u,v) \in [0,2\pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ giving the well-known parametrization of S^2:

$$S^2 : \bar{r}(u,v) = (\cos u \cos v, \sin u \cos v, \sin v). \tag{4.1}$$

Fix $m \in \mathbb{R}$ and the corresponding Clelia curve [7, p. 60]:

$$\gamma_m(t) = (\cos t \cos(mt), \sin t \cos(mt), \sin(mt)) = \bar{r}(u = t, v = mt), \quad t \in \mathbb{R}. \tag{4.2}$$

Then:

$$\begin{cases} \gamma'_m(t) = (-\sin t \cos(mt) - m \cos t \sin(mt), \cos t \cos(mt) - m \sin t \sin(mt), m \cos(mt)), \\ \|\gamma'_m(t)\| = \sqrt{m^2 + \cos^2(mt)} \geq \max\{|m|, 1\} > 0 \end{cases} \tag{4.3}$$

which says that γ_m is a regular curve. It follows:

$$\begin{cases} \mathbf{n}(t) = \frac{1}{\sqrt{m^2+\cos^2(mt)}}\left(m \sin t - \frac{1}{2}\cos t \sin(2mt), -m\cos t - \frac{1}{2}\sin t \sin(2mt), \cos^2(mt)\right), \\ k_g(t) = \frac{\sin(mt)[2m^2+\cos^2(mt)]}{(m^2+\cos^2(mt))^{\frac{3}{2}}} \end{cases} \tag{4.4}$$

and hence the arc $t \in \left(0, \frac{\pi}{m}\right)$ is convex. The flow-geodesic curvature is:

$$k_g^f(t) = \frac{\sin(mt)[2m^2 + \cos^2(mt)]}{(m^2 + \cos^2(mt))^{\frac{3}{2}}} - \frac{1}{(m^2 + \cos^2(mt))^{\frac{1}{2}}}. \tag{4.5}$$

□

Example 4.2 A spherical curve with prescribed constant geodesic curvature $k_g = K$ and parametrized by the arc-length s is:

$$\gamma_K(s) = \frac{1}{\sqrt{1+K^2}}\left(\cos(\sqrt{1+K^2}s), \sin(\sqrt{1+K^2}s), K\right) = \frac{1}{k}\left(\cos(ks), \sin(ks), K\right), s \in \mathbb{R} \tag{4.6}$$

with the spherical coordinates $(u = u(s) = ks, v = constant = \arcsin\left(\frac{K}{k}\right))$; its evolute is the constant unit vector $Ev(\gamma_K) = (0,0,1) = \bar{k}$ and its binormal is also constant $B = k(0,0,1)$. Then the flow-geodesic curvature of γ_K is the constant $k_g^f = K - 1$. It follows that γ_1 is a flow-flat convex spherical curve, $\gamma_1(s) = \frac{1}{\sqrt{2}}(\cos(\sqrt{2}s), \sin(\sqrt{2}s), 1)$, having the flow-evolute:

$$Ev^f(\gamma_1)(s) = E_2^f(s) = (\sin s) \cdot (-\sin(\sqrt{2}s), \cos(\sqrt{2}s), 0) - \frac{\cos s}{\sqrt{2}} \cdot (\cos(\sqrt{2}s), \sin(\sqrt{2}s), -1). \tag{4.7}$$

The stereographic projection from the North Pole $N(0,0,1)$ (respectively from the South Pole $S(0,0,-1)$) of the parallel $\gamma_1 \in S^2$ is the plane circle centered in the origin $(0,0)$ and having the radius $2+\sqrt{2}$ (respectively the radius $2-\sqrt{2}$). Concerning the example 2.2 the hypercone $H^{-1}(\gamma_1)$ of $\mathbb{C}^2 \setminus \{0\}$ is given by $|u| = (\sqrt{2}+1)|v|$ and then any curve in this hypersurface will be a Hopf-flow-flat curve. □

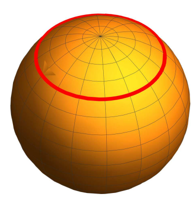

Figure 1. The flow-flat curve γ_1 of the example 4.2

Example 4.3 The tangent indicatrix of the given γ is exactly the map $t \in I \to \mathbf{t}(t) \in S^2$. Its spherical Frenet frame is:

$$\mathcal{F}_s^t := \begin{pmatrix} \mathbf{t} \\ \mathbf{t}^t \\ \mathbf{n}^t \end{pmatrix}, \mathbf{t}^t(t) := \frac{1}{k(t)}[-\gamma(t) + k_g(t)\mathbf{n}(t)], \quad \mathbf{n}^t(t) := \frac{1}{k(t)}[k_g(t)\gamma(t) + \mathbf{n}(t)]. \tag{4.8}$$

Since $\|\mathbf{t}'(t)\| = k(t)\|\gamma'(t)\|$, we get the geodesic curvature of this new spherical curve:

$$k_g^t(t) = \frac{k_g'(t)}{\|\gamma'(t)\| k^3(t)} \tag{4.9}$$

and then the flow-geodesic curvature of the tangent indicatrix is:

$$k_g^{tf}(t) = \frac{k_g'(t) - (1 + k_g^2(t))}{\|\gamma'(t)\| k^3(t)}. \tag{4.10}$$

Suppose now that γ is parametrized by arc-length. Then the tangent indicatrix is a flow-flat curve if and only if γ has the geodesic curvature $k_g(s) = \tan s$, equivalently the curvature $k(s) = \frac{-1}{\cos s}$ for $s \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$; it follows the evolute $Ev(\gamma)(t) = -[\sin t\gamma(t) + \cos t\mathbf{n}(t)]$. However, this curvature corresponds exactly to the expression (5) of [8, p. 363] for the constant torsion $\tau = 1$ and an explicit formula for γ involving hypergeometric functions is provided by the cited paper. The functions total curvature and total flow-geodesic curvature (on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$) of γ are:

$$\int k(t)dt = -\ln \frac{\cos\frac{t}{2} + \sin\frac{t}{2}}{\cos\frac{t}{2} - \sin\frac{t}{2}}, \quad \int k_g(t)dt = -\ln(-\cos t). \tag{4.11}$$

□

Remark 4.4 In the paper [10], the Delaunay variational problem defined by the arc-length functional acting on the space of curves with constant torsion $\tau = 1$ is studied. A main characterization is that a biregular curve is a critical point of the Delaunay functional if and only if the associated binormal curve γ is an elastic spherical curve, i.e. there exists $\lambda \in \mathbb{R}$ such that:

$$(k_g)_{ss} + \frac{3}{2}k_g^3 + (1 - \lambda)k_g = 0. \tag{4.12}$$

Then we define the λ-*elastic curvature* k_e^λ of the spherical curve γ through the left-hand-side of the equation above. For our example 4.3 with $k_g(s) = \frac{\cos s}{\sin s}$, we have:

$$k_g^f(s) = \frac{\cos s}{\sin s} - 1, \quad k_e^\lambda(s) = \frac{\cos s(4 - 3\cos^2 s)}{2\sin^3 s} + (1 - \lambda)\frac{\cos s}{\sin s} \tag{4.13}$$

and then a zero of k_g^f is provided by the angle $\frac{3\pi}{4}$. □

Remark 4.5 Since we arrive at the subject of curves with constant torsion, we connect our study with the proposition 1.1 from [2, p. 216]. Fix γ a spherical curve parametrized by arc-length and a constant $\tau \neq 0$. Using the pair (γ, τ), a new space curve is considered:

$$\Gamma(\gamma, \tau) := \frac{1}{\tau}\int \gamma \times \gamma' ds \tag{4.14}$$

and the cited theorem gives that the curvature k_Γ and torsion $\tau_\Gamma = \tau$ are related to the geodesic curvature of γ through: $k_g = k_\Gamma \cdot \tau$; then the flow-geodesic curvature of γ is $k_g^f = k_\Gamma \cdot \tau - 1$. Hence, we return to the curve γ_K of the previous example and the corresponding Γ is:

$$\Gamma_K(s) = \frac{1}{k\tau}\left(-\frac{K}{k}\sin(ks), \frac{K}{k}\cos(ks), s\right) \tag{4.15}$$

satisfying then $k_{\Gamma_K} = \frac{K}{\tau}$ and $\|\Gamma_K'(s)\| = \frac{1}{|\tau|}$=constant. Having both curvature and torsion as constants Γ_K is a helix lying on the cylinder $C : x^2 + y^2 = \frac{K^2}{k^2|\tau|}$. Its arc-length parametrization is:

$$\Gamma_K(u) = \frac{1}{k\tau}\left(-\frac{K}{k}\sin(k\tau u), \frac{K}{k}\cos(k\tau u), \tau u\right). \tag{4.16}$$

□

Example 4.6 The spherical nephroid is presented in [11, p. 353] as:

$$\gamma(t) = \left(\frac{3}{4}\cos t - \frac{1}{4}\cos 3t, \frac{3}{4}\sin t - \frac{1}{4}\sin 3t, \frac{\sqrt{3}}{2}\cos t\right). \tag{4.17}$$

Its geodesic curvature is:

$$k_g(t) = \frac{\cos t}{|\sin t|} \tag{4.18}$$

and then we restrict the parameter to $t \in (0, \pi)$; it results: $k(t) = \frac{1}{\sin t}$, $\tau = -1$. The flow-curvature of γ is:

$$k_g^f(t) = \frac{\cos t}{\sin t} - \frac{1}{\sqrt{3}\sin t} \tag{4.19}$$

and hence a zero t_0 of k_g^f is exactly *the magic angle* $t_0 = \arccos\left(\frac{1}{\sqrt{3}}\right) \simeq 0.955$. The total flow-geodesic curvature function is:

$$\int k_g^f(t)dt = \ln(\sin t) + \frac{1}{\sqrt{3}}\ln\cot\frac{t}{2}. \tag{4.20}$$

□

Conflict of interest

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