Transformations of generalized Lagrange metrics

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Abstract

In this paper we extend to the framework of generalized Lagrange metrics some results obtained by Kentaro Yano for the Riemannian case. The example of conformal transformations is discussed.

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Introduction

Let M be a smooth, n-dimensional manifold and $N = \left(N^i_j(x,y)\right)$ a fixed *nonlinear connection*([1, p. 107]), where $i, j, k, \ldots$ run over $1, \ldots n$. Let $g = (g_{ij}(x,y))$ be a *generalized Lagrange metric*([1, p. 181]), briefly *GL-metric*. Applying the formulae of [1, p. 124] there exists an unique symmetric N-*linear connection*, denoted $C\Gamma(N)$, such that

$$(0.1) \quad g_{ij\,|\,k} = 0, \qquad g_{ij}\,|_k = 0$$

where "$_|$" and " $|$ " are the h- and v-*covariant derivatives with respect to* $C\Gamma(N)$. *The generalized Christoffel symbols* of $C\Gamma(N) = \left(L^i_{jk}, C^i_{jk}\right)$ are given by

$$(0.2a) \quad L^i_{jk} = \frac{1}{2} g^{ia} \left(\frac{\delta g_{ak}}{\delta x^j} + \frac{\delta g_{ja}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^a} \right)$$

$$(0.2b) \quad C^i_{jk} = \frac{1}{2} g^{ia} \left(\frac{\partial g_{ak}}{\partial y^j} + \frac{\partial g_{ja}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^a} \right)$$

where $\left(\frac{\delta}{\delta x^i}\right)$ is *the adapted basis* given by([1, p. 108])

$$(0.3) \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}.$$

Particular case. Let $g = (g_{ij}(x))$ be a Riemannian metric and N the nonlinear connection given by *the kinetic energy of* g([1, p. 182])

$$(0.4) \quad \mathcal{E}(g) = \frac{1}{2} g_{ij}(x) y^i y^j$$

which is a *regular Lagrangian.* In this case L^i_{jk} are the usual Christoffel symbols Γ^i_{jk} and $C^i_{jk} = 0$ so that $C\Gamma(N)$ is exactly *the Levi-Civita connection of* g.

In this paper we study two types of transformations of GL-metrics, namely *infinitesimal* and *finite* transformations, following the approach of K. Yano([2]). Propositions 1.3, 1.5, 1.6 and 2.2 are obtained in this way. Another result is provided by the proposition 2.4 which is based on the remark that if g and $\tilde{g}$ are two Riemannian metrics with the same kinetic energy $\mathcal{E}(g) = \mathcal{E}(\tilde{g})$ then $g = \tilde{g}$. The example of infinitesimal conformal transformations is given. Some cases in which a curvature tensor field is preserved are obtained.

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1 Infinitesimal transformations of GL-metrics

Suppose that there is given a one-parameter family $(g_{ij}(x, y))_t$ of GL-metrics and let:

$$(1.1) \quad g_{ij} = (g_{ij})_0 .$$

The transformation

$$g_{ij} \to (g_{ij})_t$$

will be called *an infinitesimal transformation of GL-metrics.* Let:

$$(1.2) \quad a_{ij} = \frac{d}{dt} (g_{ij})_t |_{t=0}$$

and call (a_{ij}) *the variation tensor*. Denoting by $\left(L^i_{jk}\right)_t$ the first symbols of $(C\Gamma(N))_t$ where $(C\Gamma(N))_t$ is given by N and $(g_{ij})_t$. Also let:

$$(1.3) \quad U^i_{jk} = \frac{d}{dt}\left(L^i_{jk}\right)_t |_{t=0}$$

Following K. Yano([2]) we give:

Definition 1.1 The infinitesimal transformation of GL-metrics

$$g_{ij} \to (g_{ij})_t$$

is called:
(i) *isometric* if the variation tensor is null:

$$(1.4) \quad a_{ij} = 0 \quad i, j = 1, \ldots, n$$

(ii) *horizontal almost isometric* if:

$$(1.5) \quad g^{ij}U^k_{ij} = 0 \quad k = 1, \ldots, n$$

(iii) *horizontal affine* if :

$$(1.6) \quad U^i_{jk} = 0 \quad i, j, k = 1, \ldots, n$$

(iv) *volume preserving* if:

$$(1.7) \quad a_{ij}g^{ij} = 0.$$

Remark 1.2 It is obvious that in definition 1.1

$$(1.8) \quad (i) \Rightarrow (iv)$$

$$(1.9) \quad (iii) \Rightarrow (ii).$$

Proposition 1.3 *An infinitesimal transformation of GL-metrics is horizontal affine if and only if the variation tensor is h-covariant constant with respect to $C\Gamma(N)$ where $C\Gamma(N)$ is defined by g and N.*

Proof We have
$2U^i_{jk} = \frac{\partial}{\partial t}\left(g^{is}_t\left(\frac{\delta}{\delta x^j}(g_{sk})_t + \frac{\delta}{\delta x^k}(g_{js})_t - \frac{\delta}{\delta x^s}(g_{jk})_t\right)\right)|_{t=0}=$
$= a^{is}\left(\frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s}\right) + g^{is}\left(\frac{\delta a_{sk}}{\delta x^j} + \frac{\delta a_{js}}{\delta x^k} - \frac{\delta a_{jk}}{\delta x^s}\right).$
For any d-tensor (ω_{ij}) we may write:

$$\frac{\delta \omega_{ij}}{\delta x^s} = \omega_{ij\,|\,s} + L^u_{si}\omega_{uj} + L^u_{sj}\omega_{iu}$$

which yields:

$$2U^i_{jk} = 2a^{iu}L^s_{jk}g_{su} + 2g^{iu}L^s_{jk}a_{us} + g^{iu}\left(a_{uk\ |\ j} + a_{ju\ |\ k} - a_{jk\ |\ u}\right).$$

Derivation the equality $g^{iu}_t\left(g_{us}\right)_t = \delta^i_s$ with respect to t and making $t \to 0$ it results:

$$a^{iu}g_{us} + g^{iu}a_{us} = 0$$

and then:

$$(1.10)\quad 2U^i_{jk} = g^{iu}\left(a_{uk\ |\ j} + a_{ju\ |\ k} - a_{jk\ |\ u}\right)$$

Therefore we have the part "*if*" and $U^i_{jk} = 0$ for all i, j, k yield:

$$(1.11)\quad a_{uk\ |\ j} + a_{ju\ |\ k} - a_{jk\ |\ u} = 0$$

We add to the last relation a similar one:

$$a_{ju\ |\ k} + a_{kj\ |\ u} - a_{ku\ |\ j} = 0$$

and then we obtain $a_{ju\ |\ k} = 0$ for all u, j, k. $\square$

Definition 1.4 A GL-metric (g_{ij}) is called *horizontal irreducible with respect to* N if a relation

$$(1.12)\quad \omega_{ij\ |\ k} = 0 \quad i, j, k = 1, \ldots, n$$

imply

$$(1.13)\quad \omega_{ij} = cg_{ij} \quad i, j = 1, \ldots, n$$

with c a positive constant.

Proposition 1.5 *Consider a horizontal affine and volume preserving infinitesimal transformation of GL-metric with* (g_{ij}) *a GL-metric horizontal irreducible with respect to* N. *Then the infinitesimal transformation is isometric.*

Proof By Proposition 1.3 $a_{ij\ |\ k} = 0$ and so $a_{ij} = cg_{ij}$. The transformation being volume preserving it results:

$$(1.14)\quad g^{ij}a_{ij} = nc = 0$$

Consequently $c = 0$ and then $a_{ij} = 0$ for all i, j. $\square$

Putting:

$$(1.15)\quad U_{ijk} = U^s_{ij}g_{sk}$$

we have by (1.10):

$$(1.16) \quad 2U_{ijk} = a_{jk\ |\ i} + a_{ik\ |\ j} - a_{ij\ |\ k}.$$

Proposition 1.6 *A volume preserving infinitesimal transformation of GL-metric for which the tensor* (U_{ijk}) *is symmetric in all the indices is horizontal almost isometric.*

Proof The equality

$$(1.17) \quad a_{jk\ |\ i} + a_{ik\ |\ j} - a_{ij\ |\ k} = a_{ki\ |\ j} + a_{ji\ |\ k} - a_{jk\ |\ i}$$

imply $a_{jk\ |\ i} = a_{ij\ |\ k}$ and then

$$(1.18) \quad U^i_{jk} = \frac{1}{2} g^{iu} a_{jk\ |\ u}$$

which yields:

$$g^{jk} U^i_{jk} = \frac{1}{2} g^{jk} g^{iu} a_{jk\ |\ u} = \frac{1}{2} g^{iu} \left(g^{jk} a_{jk} \right)_{|\ u} = 0. \quad \square$$

Example 1.7(Conformal infinitesimal transformations) Fix g a GL-metric and let $(f_t)_t \in C^\infty(TM)$ be a 1-parameter family of functions on TM with $f_0 = 0$. Then let the conformal infinitesimal transformation of GL-metrics

$$(1.19) \quad g \to \exp(2f_t)\, g.$$

We have

$$(1.20) \quad a = 2\rho g$$

where

$$(1.21) \quad \rho = \frac{d}{dt} f_t \mid_{t=0}$$

and

$$(1.22) \quad U^i_{jk} = \delta^i_k \rho_{|\ j} + \delta^i_j \rho_{|\ k} - g^{is} g_{jk} \rho_{|\ s}$$

which give

Proposition 1.8 *For the conformal infinitesimal transformation* (1.19) *the following are equivalents*
(i) *is isometric*
(ii) *is volume preserving*
(iii) $\rho = 0$.

Proof We have (i)$\Longrightarrow$(ii). Since (1.20) hold it follows

$$(1.23) \quad g^{ij}a_{ij} = 2n\rho$$

and we have (ii)$\Longrightarrow$(iii). Also, (1.20) give (iii)$\Longrightarrow$(i). $\square$

Proposition 1.9 *The conformal infinitesimal transformation* (1.19) *is horizontal affine if and only if*

$$(1.24) \quad \delta^i_j \rho_{|\ k} + \delta^i_k \rho_{|\ j} = g^{is} g_{jk} \rho_{|\ s}, \qquad i, j, k = 1, \ldots, n.$$

From (1.22) we obtain

$$(1.25) \quad g^{jk} U^i_{jk} = (2-n)\, g^{is} \rho_{|\ s}$$

and then

Proposition 1.10

(i) *If* $\dim M = 2$, *that is* M *is a surface, then the conformal infinitesimal transformation* (1.19) *is horizontal almost isometric.*

(ii) *If* $\dim M \neq 2$ *then the conformal infinitesimal transformation* (1.19) *is horizontal almost isometric if and only if*

$$(1.26) \quad g^{is} \rho_{|\ s} = 0, \qquad i = 1, \ldots, n.$$

Corollary 1.11 *If* ρ *is h-covariant constant then the conformal infinitesimal transformation is horizontal affine, particularly horizontal almost isometric.*

Corollary 1.12 *If* f_t *is function only of* t, *that is* $f(q, \dot{q}, t) = f(t)$, *then the conformal infinitesimal transformation is horizontal affine, particularly horizontal almost isometric.*

Recall that $C\Gamma(N)$ has three curvature tensor fields([1, p. 48]) one of which given by([1, p. 48])

$$(1.27) \quad R^i_{jkh} = \frac{\delta L^i_{jk}}{\delta x^h} - \frac{\delta L^i_{jh}}{\delta x^k} + L^s_{jk} L^i_{sh} - L^s_{jh} L^i_{sk} + C^i_{js} R^s_{kh}$$

where $\left(R^i_{jk}\right)$ is given by([1, p. 31])

$$(1.28) \quad R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}.$$

For the Riemannian case R^i_{jkh} is exactly the usual Riemann-Christoffel curvature tensor field.

Suppose that we have one of the following situations
(I) $\frac{d}{dt}\left(C^i_{jk}\right)_t |_{t=0}= 0$ (particularly the Riemannian case)
(II) N is integrable, that is([1, p. 32])

$$(1.29)\qquad R^i_{jk} = 0.$$

Then a straightforward computation give

$$(1.30)\qquad \frac{d}{dt}\left(R^i_{jkh}\right)_t |_{t=0}= U^i_{jk\ |\ h} - U^i_{jh\ |\ k}.$$

Proposition 1.13 *If the infinitesimal transformation satisfy the case* I *or(and)* II *then this transformation preserves the curvature* $\left(R^i_{jkh}\right)$ *if and only if*

$$(1.31)\qquad U^i_{jk\ |\ h} = U^i_{jh\ |\ k}, \qquad i, j, k, h = 1, \ldots, n.$$

Corollary 1.14 *If the infinitesimal transformation satisfy the case* I *or(and)* II *and the tensor field* $\left(U^i_{jk}\right)$ *is h-covariant constant with respect to* $C\Gamma(N)$ *then this transformation preserves the curvature tensor field* $\left(R^i_{jkh}\right)$.

Suppose, in addition, that U_{ijk} is symmetric in all the indices. Applying (1.18) we find

$$(1.32)\qquad \frac{d}{dt}\left(R^i_{jkh}\right)_t |_{t=0}= \frac{1}{2}g^{iu}\left(a_{uj\ |\ k\ |\ h} - a_{uj\ |\ h\ |\ k}\right).$$

Proposition 1.15 *If the infinitesimal transformation satisfy the case* I *or(and)* II *and the tensor field* (U_{ijk}) *is symmetric in all the indices then this transformation preserves the curvature* $\left(R^i_{jkh}\right)$ *if and only if*

$$(1.33)\qquad a_{uj\ |\ k\ |\ h} = a_{uj\ |\ h\ |\ k}, \qquad u, j, k, h = 1, \ldots, n.$$

Corollary 1.16 *If the infinitesimal transformation satisfy the case* I *or(and)* II*, the tensor field* (U_{ijk}) *is symmetric in all the indices and* (a_{ij}) *is h-covariant constant with respect to* $C\Gamma(N)$ *then this transformation preserves the curvature* $\left(R^i_{jkh}\right)$.

For the particular case 1.7 if we have the case I or(and) II then

$$(1.34)\quad \frac{d}{dt}\left(R^i_{jkh}\right)_t |_{t=0}= \delta^i_k\rho_{|\ j\ |\ h} - g^{is}g_{jk}\rho_{|\ s\ |\ h} - \delta^i_h\rho_{|\ j\ |\ k} + g^{is}g_{jh}\rho_{|\ s\ |\ k}.$$

Proposition 1.17 *If the conformal infinitesimal transformation satisfy the case* I *or(and)* II *and ρ is h-covariant constant with respect to $C\Gamma(N)$ then this transformation preserves the curvature $\left(R^i_{jkh}\right)$.*

Corollary 1.18 *If the conformal infinitesimal transformation satisfy the case* I *or(and)* II *and f_t depends only of t then this transformation preserves the curvature tensor $\left(R^i_{jkh}\right)$.*

Recall that in Riemannian geometry two very important geometrical objects are:

1. *the Ricci tensor field* (R_{jk}) where

$$R_{jk} = R^i_{jik} \tag{1.35}$$

2. *the scalar curvature* R where

$$R = g^{ij} R_{ij}. \tag{1.36}$$

Applying relation (1.30) we get

Proposition 1.19 *If the infinitesimal transformation satisfy the case* I *or(and)* II *then*

$$\frac{d}{dt}\left(R_{jk}\right)_t|_{t=0} = U^i_{ji\ |\ k} - U^i_{jk\ |\ i} \tag{1.37}$$

$$\frac{d}{dt}\left(R\right)_t|_{t=0} = a^{jk} R_{jk} + g^{jk} U^i_{ji\ |\ k} - \left(g^{jk} U^i_{jk}\right)_{|\ i} \tag{1.38}$$

where is (R_{jk}) the Ricci tensor field for g.

For the conformal infinitesimal transformation we obtain

Corollary 1.20 *If the infinitesimal conformal transformation satisfy the case* I *or(and)* II *then*

$$\frac{d}{dt}\left(R_{jk}\right)_t|_{t=0} = (n-2)\,\rho_{|\ j\ |\ k} + g^{is} g_{jk} \rho_{|\ s\ |\ i} = 2\,(n-1)\,\rho_{|\ j\ |\ k} \tag{1.39}$$

$$\frac{d}{dt}\left(R\right)_t|_{t=0} = -2\rho R + 2\,(n-1)\,g^{jk} \rho_{|\ j\ |\ k} \tag{1.40}$$

where R is the scalar curvature of g.

Corollary 1.21 *If the conformal infinitesimal transformation satisfy the case* I *or(and)* II *then the scalar curvature is preserving if and only if*

$$(n-1)\,g^{jk} \rho_{|\ j\ |\ k} = \rho R. \tag{1.41}$$

Corollary 1.22 *If the conformal infinitesimal transformation satisfy the case* I *or(and)* II *and* ρ *is h-covariant constant with respect to* $C\Gamma(N)$ *(particularly* f_t *depends only of* t*) then*
(i) *the Ricci tensor field is preserving*
(ii) *the scalar curvature is preserving if and only if* $\rho = 0$ *or(and)* $R = 0$.

One consider *the Einstein tensor*

$$(1.42) \quad E_{ij} = R_{ij} - \frac{1}{2} R g_{ij}.$$

For the example of conformal infinitesimal transformations we obtain

$$(1.43) \quad \frac{d}{dt}(E_{ij})_t \mid_{t=0} = -(n-1)(n-2)\, g_{ij} g^{ks} \rho_{\mid \ k \ \mid \ s}$$

and then we get

Proposition 1.23 *If the conformal infinitesimal transformation satisfy the case* I *or(and)* II *and one of the following condition holds*
(i) $n = 1$ *or* $n = 2$
(ii) ρ *is h-covariant constant with respect to* $C\Gamma(N)$
then the Einstein tensor is preserving.

2 Finite transformations of GL-metrics

Let (g_{ij}) and $(\tilde{g}_{ij})$ be two GL-metrics and $\left(L^i_{jk}\right)$, $\left(\tilde{L}^i_{jk}\right)$ the first symbols of $C\Gamma(N)$ for g, $\tilde{g}$. Recall that N is a fixed nonlinear connection.

Definition 2.1 The finite transformation of GL-metrics $(g_{ij}) \rightarrow (\tilde{g}_{ij})$ is called:
(i) *volume preserving* if:

$$(2.1) \quad \det(g_{ij}) = \det(\tilde{g}_{ij})$$

(ii) *horizontal affine* if:

$$(2.2) \quad L^i_{jk} = \tilde{L}^i_{jk} \quad i, j, k = 1, \ldots, n.$$

Proposition 2.2 *Let* $(g_{ij}) \rightarrow (\tilde{g}_{ij})$ *be a volume preserving and horizontal affine transformation of GL-metrics with* (g_{ij}) *horizontal irreducible with respect to* N*. Then*

$$(2.3) \quad g_{ij} = \tilde{g}_{ij} foralli, j$$

Proof The equality $L^i_{jk} = \tilde{L}^i_{jk}$ imply: $0 = \tilde{g}_{ij\ \tilde{|}\ k} = \tilde{g}_{ij\ |\ k}$
where "$\tilde{|}$" denotes the h-covariant derivative with respect to $(\tilde{g}_{ij})$. Then $\tilde{g}_{ij} = cg_{ij}$ with c a positive constant. The last definition imply $c = 1$. □

Remark 2.3 The Riemannian version of propositions 1.3, 1.5, 1.6 and 2.2 was obtained by K. Yano in [2]. In his paper, K. Yano used the word "change" but we preferred to use "transformation".

Proposition 2.4 *Let $g = (g_{ij}) \to \tilde{g} = (\tilde{g}_{ij})$ be a horizontal affine transformation of GL-metrics with g horizontal irreducible with respect to N. If g and $\tilde{g}$ have the same nonvanishing kinetic energy*

$$\mathcal{E}(g) = \mathcal{E}(\tilde{g}) \neq 0 \tag{2.4}$$

then $g = \tilde{g}$.

Proof With the same arguments like in last proof we have $\tilde{g} = cg$. Relation (2.4) imply $c = 1$. □

References

[1] Miron, R., Anastasiei, M., *The Geometry of Lagrange Spaces (Theory and Applications)*, Fundamental Theories of Physics, no. 59, Kluwer Academic Publishers, 1994.

[2] Yano, K, *Notes on isometries*, Colloquium Mathematicum, XXVI(1972), 1-7.

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